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PARAMETRIZATION OF BENDING MOMENTS IN CREEP ANALYSIS OF CIRCULARLY SYMMETRIC SIMPLY SUPPORTED PLATES

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This work presents the method of the solution, based on the parametrization of the bending moments, for annular simple-supported plates. As a result of this parametrization, the solution of the boundary problem is reduced to the solution of three ordinary differential equations of the first order. The set of equations separated with respect to the derivative of unknown functions makes it possible to apply the standard Runge–Kutta integral procedures. The time of the first cracks will be calculated using the Kachanov–Rabotnov damage law.

1. Introduction

The solutions, found in literature, of the steady creep problem of circularly symmetric plates, in the case of the application of the nonlinear viscoelastic Odqvist–Norton model of the material [1], are based in general on approximate methods. In particular, the boundary problem reduced to the differential plate equation (expressed by an unknown function of deflection) is solved with the aid of: a method of successive approximation [2], variation methods [3, 4, 5] and certain methods using the discretization of the integral region under the assumption that the stress intensity is based upon the hypothesis of a maximum shear stress [6]. The numerical methods applied in the solutions of circularly symmetric plates made from elastic--plastic material with power strain-hardening [7] are also adopted [6].

The method of the solution based on parametrization of the bending moments will be presented below. As a result of this parametrization the solution of the boundary problem is reduced to the solution of three ordinary differential equations of the first order. The set of equations separated with respect to the derivative of unknown functions makes it possible to apply the standard Runge–Kutta integral procedures.

Additionally, the problem of calculation of the forming time of the first cracks, under the assumption that stresses and strain velocities are constant, is considered. That time has been calculated with the application of the Kachanov–Rabotnov damage law [8, 9].

The governing set of equations will be formulated in dimensionless variables. The algorithm which enables the numerical analysis of the creep process has been illustrated by the solution for an annular plate simply supported along its external edge.

2. Formulation of problem

It follows from the assumption of circular symmetry that all inquired magnitudes described in a cylindrical system of coordinates $\{r, \theta, z\}$ will be the functions of the radius r, coordinate z (the midspan of the plate is determined by the coordinate $z=0$) and time t. The solutions of the steady creep process with the analysis of forming of the first cracks will be integrals of the following set of equations:

$$\sigma_{r,r}+\tau_{rz,z}+\frac{1}{r}(\sigma_r-\sigma_\theta)=0, \tag{2.1}$$

$$\tau_{rz,r}+\sigma_{z,z}+\frac{1}{r}\tau_{rz}=0, \tag{2.2}$$

$$-z\,w_{,rr}=\frac{1}{2}\left(\frac{\sigma_e}{\sigma_c}\right)^{n-1}\frac{2\sigma_r-\sigma_\theta}{\sigma_c}, \tag{2.3}$$

$$-\frac{z}{r}\,w_{,r}=\frac{1}{2}\left(\frac{\sigma_e}{\sigma_c}\right)^{n-1}\frac{2\sigma_\theta-\sigma_r}{\sigma_c}, \tag{2.4}$$

$$\max_{r_0}\left\{A(n_0+1)\int_0^{t^*}[\sigma_1(r_0, z=\pm h, t)]^{n_0}\,dt\right\}=1, \tag{2.5}$$

where

$$\sigma_e^2=\sigma_r^2-\sigma_r\,\sigma_\theta+\sigma_\theta^2. \tag{2.6}$$

The symbols σ_r, σ_θ, σ_z, τ_{rz} and w denote the normal stresses related to r, θ, z directions, shear stress and the midspan rate deflection. The magnitudes A, n_0, n, σ_c are connected with the material property description. The function σ_1 denotes the maximum value of the principal tensile stress in particles of the upper $z=-h$ or lower $z=h$ surface of the radius $r=r_0$.

According to the assumptions adopted in the theory of the thin plates, the constitutive relation (Norton–Odqvist creep) (2.3) and (2.4) is written as in plane stress state. The radial and circumferential strains rate (the left-hand sides of Eqs. (2.3) and (2.4)) are expressed basing on the assumption of conservation of normals to the middle-surface.

Due to the fact that the statical boundary conditions can be satisfied only in integral form, the equilibrium equations (2.1) and (2.2) will be taken as follows [1]:

$$M_{r,r}+\frac{1}{r}(M_r-M_\theta)-Q_r=0, \tag{2.7}$$

$$rQ_{r,r}+Q_r-q(r)\,r=0, \tag{2.8}$$

where $q(r)$ is the load function of the upper plate surface

$$\sigma_z(r, z)|_{z=-h}\equiv q(r). \tag{2.9}$$

The definitions of the bending moments M_r, M_θ and transverse force Q_r are given by

(2.10) $$M_r = \int_{-h}^{h} \sigma_r \, z dz, \qquad M_\theta = \int_{-h}^{h} \sigma_\theta \, z dz, \qquad Q_r = \int_{-h}^{h} \tau_{rz} \, dz.$$

Introducing for dimensional purposes the stresses

(2.11) $$s_\rho = \frac{\sigma_r}{\sigma_c} t_0^{\frac{1}{n}}, \qquad s_\theta = \frac{\sigma_\theta}{\sigma_c} t_0^{\frac{1}{n}}, \qquad s_{\rho\zeta} = \frac{\tau_{rz}}{\sigma_c} t_0^{\frac{1}{n}},$$

the bending moments M_r, M_θ and transverse force Q_r can be written as

(2.12) $$m_\rho = \int_{-1}^{1} s_\rho \, \zeta \, d\zeta, \qquad m_\theta = \int_{-1}^{1} s_\theta \, \zeta \, d\zeta, \qquad q_\rho = \int_{-1}^{1} s_{\rho\zeta} \, d\zeta,$$

where the dimensionless coordinates ρ and ζ are expressed by

(2.13) $$\rho = \frac{r}{h}, \qquad \zeta = \frac{z}{h}.$$

Besides the material constants σ_c and n, a constant t_0 (which has time dimension) is simultaneously used for dimensional purposes.

The equilibrium equations (2.7) and (2.8) after substitution of Eqs. (2.10)–(2.13) will take the form

(2.14) $$m_{\rho,\rho} + \frac{1}{\rho}(m_\rho - m_\theta) = q_\rho,$$

(2.15) $$\rho q_{\rho,\rho} + q_\rho = \rho \bar{q},$$

where $\bar{q}$ denotes the dimensionless load function

(2.16) $$\bar{q}(\rho) = \frac{q(\rho)}{\sigma_c} t_0^{\frac{1}{n}}.$$

The components of the strain rate tensor (the left-hand sides of Eqs. (2.3) and (2.4))

(2.17) $$\epsilon_r = -z\, w_{,rr}, \qquad \varepsilon_\theta = -\frac{z}{r} w_{,r}$$

can be expressed by the rate of variability of the angle between the normal to the midspan

(2.18) $$\varphi(r) = -w_{,r},$$

Taking into account

(2.19) $$\bar{\varphi}(\rho) = \varphi(\rho)\, t_0, \qquad \bar{w}(\rho) = \frac{1}{h} w(\rho)\, t_0$$

the expression (2.18) can be written in the dimensionless form

(2.20) $$\bar{\varphi}(\rho) = -\bar{w}_{,\rho}.$$

The constitutive equations (2.3) and (2.4) as a result of substitution of Eq. (2.18) with simultaneous use of Eqs. (2.11)–(2.13) and (2.19) can be given by

$$\zeta\bar{\varphi}_{,\rho}=\frac{1}{2}s_e^{n-1}(2s_\rho-s_\theta), \tag{2.21}$$

$$\frac{\zeta}{\rho}\bar{\varphi}=\frac{1}{2}s_e^{n-1}(2s_\theta-s_\rho), \tag{2.22}$$

where s_e denotes dimensionless stress intensity

$$s_e^2=s_\rho^2-s_\rho s_\theta+s_\theta^2. \tag{2.23}$$

Similarly, introducing dimensionless magnitudes of the constant A and the principal tensile stress σ_I

$$\bar{A}=A\sigma_c^{n_0}t_0^{1-\frac{n_0}{n}},\qquad s_I=\frac{\sigma_I}{\sigma_c}t_0^{\frac{1}{n}} \tag{2.24}$$

the Kachanov rupture law (2.5) can be written in the following form:

$$\max_{\rho_0}\left\{\bar{A}(n_0+1)\int\limits_0^{\tau^*}[s_I(\rho_0,\zeta=\pm1,\tau)]^{n_0}d\tau\right\}=1. \tag{2.25}$$

Dimensionless time τ and in detail dimensionless time of the first cracks τ^* are here expressed by

$$\tau=\frac{t}{t_0},\qquad \tau^*=\frac{t}{t_0}. \tag{2.26}$$

The solution of the brittle creep rupture of circularly symmetric plates is reduced to the solution of the governing set of equations (2.14)–(2.15), (2.18), (2.21)–(2.22) and (2.25) with respect to the dimensionless functions of the bending moments m_ρ and m_θ, transverse force q_ρ, rate of deflection $\bar{w}$ and angle of deflection $\bar{\varphi}$, and to the dimensionless time of the first cracks τ^*. The method of solution will be based on the parametrization of the radial s_ρ and circumferential s_θ stress functions. By means of the relations (2.12) the parametrization will be transferred to the functions of the bending moments m_ρ and m_θ.

3. Parametrization of bending moments

The basic idea of parametrization consists in reducing the number of equations in the governing set as a result of identical satisfaction of the constitutive relation (2.22). In this order the stress state components s_ρ and s_θ are expressed by the parametrizing function $\psi(\rho,\tau)$ [10, 11]

$$\left.\begin{matrix}s_\rho\\ s_\theta\end{matrix}\right\}=\left(\frac{4}{3}\right)^{\frac{n-1}{2n}}\left[\frac{4\bar{\varphi}\zeta}{\rho(\sqrt{3}\cos\psi-3\sin\psi)}\right]^{\frac{1}{n}}\cos\left(\psi\mp\frac{\pi}{6}\right). \tag{3.1}$$

As a result of integration defined by the relations (2.12), the parametrization of the bending moments takes the form

$$\left.\begin{matrix} m_\rho \\ m_\theta \end{matrix}\right\} = \left(\frac{4}{3}\right)^{\frac{n-1}{2n}} \frac{2n}{2n+1} \left[\frac{4\bar{\varphi}}{\rho\,(\sqrt{3}\cos\psi - 3\sin\psi)}\right]^{\frac{1}{n}} \cos\left(\psi \mp \frac{\pi}{6}\right). \tag{3.2}$$

From the comparison of the expressions (3.1) and (3.2) the simple relations between the normal stresses and the bending moments can be written as

$$\left.\begin{matrix} s_\rho \\ s_\theta \end{matrix}\right\} = \left.\begin{matrix} m_\rho \\ m_\theta \end{matrix}\right\} \cdot \left(1 + \frac{1}{2n}\right) \zeta^{\frac{1}{n}}. \tag{3.3}$$

The constitutive equations (2.21) and (2.22) after substitution of Eq. (3.3) will be expressed by the functions of bending moments

$$\bar{\varphi}_{,\rho} = \frac{1}{2}\left(1 + \frac{1}{2n}\right)^n m_e^{n-1} (2m_\rho - m_\theta), \tag{3.4}$$

$$\frac{1}{\rho}\bar{\varphi} = \frac{1}{2}\left(1 + \frac{1}{2n}\right)^n m_e^{n-1} (2m_\theta - m_\rho), \tag{3.5}$$

where, by analogy to the stress intensity, the bending moment intensity has been introduced:

$$m_e^2 = m_\rho^2 - m_\rho m_\theta + m_\theta^2. \tag{3.6}$$

Equation (3.5) is identically satisfied; therefore in further formulation of a boundary problem only one constitutive equation (3.4) will be taken. A simplified form of Eq. (3.4) will be given dividing the members of Eq. (3.4) by Eq. (3.5):

$$\bar{\varphi}_{,\rho} = \frac{1}{\rho}\bar{\varphi}\,\frac{2m_\rho - m_\theta}{2m_\theta - m_\rho}. \tag{3.7}$$

The integral of Eq. (2.15) for a certain load of the plate $\bar{q}(\rho)$ is the function of the transverse force $q_\rho(\rho)$. Therefore the governing set of equations in the numerical solution of the boundary problem is reduced to the equilibrium equation (2.14), physical relation (3.7) and geometric expression (2.20). Introducing the parametrization (3.2), this set of equations takes the form

$$\psi_{,\rho} = \left\langle \frac{1}{n\rho}\cos\left(\psi + \frac{\pi}{6}\right) \frac{(3-n)\cos\left(\psi - \frac{\pi}{6}\right) + (2n-3)\cos\left(\psi + \frac{\pi}{6}\right)}{2\cos\left(\psi + \frac{\pi}{6}\right) - \cos\left(\psi - \frac{\pi}{6}\right)} - \right.$$
$$\left. - \frac{1}{\rho}\cos\left(\psi + \frac{\pi}{6}\right) + \frac{q\rho}{\left(\frac{4}{3}\right)^{\frac{n-1}{2n}} \frac{2n}{1+2n}} \left[\frac{\bar{\varphi}}{\rho(\sqrt{3}\cos\psi - 3\sin\psi)}\right]^{-\frac{1}{n}} \right\rangle \Big/$$
$$\Big/ \sin\left(\psi - \frac{\pi}{6}\right) - \frac{1}{n}\cos\left(\psi - \frac{\pi}{6}\right) \frac{\sqrt{3}\sin\psi + 3\cos\psi}{\sqrt{3}\cos\psi - 3\sin\psi}, \tag{3.8}$$

$$(3.9)\qquad \bar{\varphi}_{,\rho}=\frac{1}{\rho}\bar{\varphi}\frac{2\cos\left(\psi-\frac{\pi}{6}\right)-\cos\left(\psi+\frac{\pi}{6}\right)}{2\cos\left(\psi+\frac{\pi}{6}\right)-\cos\left(\psi-\frac{\pi}{6}\right)},$$

$$(3.10)\qquad \bar{w}_{,\rho}=-\bar{\varphi}.$$

The solution of the boundary problem is reduced to the integration of three ordinary differential equations (time is a parametr) with respect to the unknown functions $\psi(\rho)$, $\bar{\varphi}(\rho)$ and $\bar{w}(\rho)$. The known form of the parametrizing function $\psi-(\rho)$ makes it possible to calculate the time of the first cracks by means of Eqs. (3.1) and (2.25).

4. Algorithm of calculation

The algorithm of the calculation will be shown in the example of the solution of an annular simply supported plate along the external edge, Fig. 1. The statical boundary conditions will be formulated for a plate loaded uniformly with a moment M_0 at the internal edge of the plate $r=r_a$ and load q applied to the upper surface.

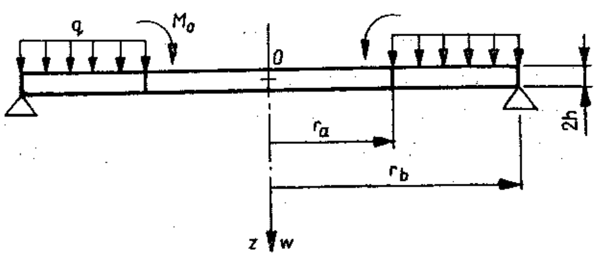

FIG. 1.

For a given load, the integral of Eq. (2.15) with the boundary condition

$$(4.1)\qquad \bar{q}(\rho_a)=0$$

is the function of the transverse force

$$(4.2)\qquad q_\rho=\frac{\bar{q}}{2\rho}(\rho^2-\rho_a^2).$$

The two further boundary conditions for bending moments will be formulated on the internal edge

$$(4.3)\qquad m_\rho(\rho_a)=-\frac{M_0\,t_0^{\frac{1}{n}}}{\sigma_c h^2}=-\bar{M}_0$$

and the external edge of the plate

$$(4.4)\qquad m_\rho(\rho_b)=0.$$

According to the parametrization (3.2), the first condition (4.3) yields the relation

$$(4.5)\qquad \bar{\varphi}(\rho_a)=\frac{1}{4}\left(1+\frac{1}{2n}\right)^n\left(\frac{3}{4}\right)^{\frac{n-1}{2}}\rho_a\left\{\frac{-\bar{M}_0}{\cos\left[\psi(\rho_a)-\frac{\pi}{6}\right]}\right\}^n\times$$
$$\times\left[\sqrt{3}\cos\psi(\rho_a)-3\sin\psi(\rho_a)\right].$$

The second condition (4.4) for the positive circumferential bending moment on the external edge of the plate $(m_\theta(\rho_b)>0)$ yields

$$(4.6)\qquad \psi(\rho_b)=\frac{2}{3}\pi.$$

The boundary conditions (4.5) and (4.6) formulate a two-point problem for Eqs. (3.8)–(3.9). This consists in such a choice of the magnitudes $\psi(\rho_a)$ in the expression (4.5) so that after integration of Eqs. (3.8)–(3.9) the function $\psi(\rho)$ satisfies simultaneously the boundary condition (4.6). For the purpose of choosing the values $\psi(\rho_a)$, the linear interpolation method was used in the paper. It proved to be of rapid convergence, even satisfying the condition (4.6) with very high accuracy. The function $\bar{\varphi}(\rho)$ obtained in the numerical way allows to calculate the function of deflection $\bar{w}$ as the integral of Eq. (3.10) with the boundary condition

$$(4.7)\qquad \bar{w}(\rho_b)=0.$$

In agreement with the adopted Kachanov law (2.25), the first cracks appear in the place where the maximum tensile stress occurs. In the solution of an annular plate this will be the circumferential stress s_θ. The coordinate ρ_0 of the point where the first cracks will appear (on the upper $\zeta=-1$ or lower surface $\zeta=1$ of the plate) agrees with the coordinate of the maximum circumferential moment

$$(4.8)\qquad m_\theta^{\max}=m_\theta(\rho_0).$$

The stresses which are constant during the development of damages till the first cracks appear enable transformation of Kachanov's law to a form allowing for the determination of time τ^* without need of numerical integration. Regarding the relations (3.3) this time can be calculated according to the expression

$$(4.9)\qquad \tau^*=\frac{1}{\bar{A}(n_0+1)\left(1+\frac{1}{2n}\right)^{n_0}\max\limits_{\rho_0}\left[m_\theta(\rho_0)\right]^{n_0}}.$$

In the numerical analysis of brittle crrep rupture of metal plates at elevated temperatures shown below, the constants describing the properties of the plate will be taken as follows:

$$(4.10)\qquad n=5,\quad n_0=3.5,\quad \bar{A}=0.11.$$

These data correspond to creep and damage behaviour, typical of metals used in high temperature applications.

In examples of the solutions, for the schema of load shown in Fig. 1, the influence of $\bar{q}$ (for $\bar{M}_0$=const) and $\bar{M}_0$ (for $\bar{q}$=const) on the shape of the functions m_ρ, m_θ and $\bar{w}$ is investigated separately. Particularly, the influence of the load $\bar{q}$ on the form of the radial bending moment, for the fixed value $\bar{M}_0=-0.35\cdot10^{-2}$, is shown in Fig. 2. Corresponding functions of the circumferential bending moment are presented in Fig. 3. The coordinates ρ connected with the maximum values of the functions

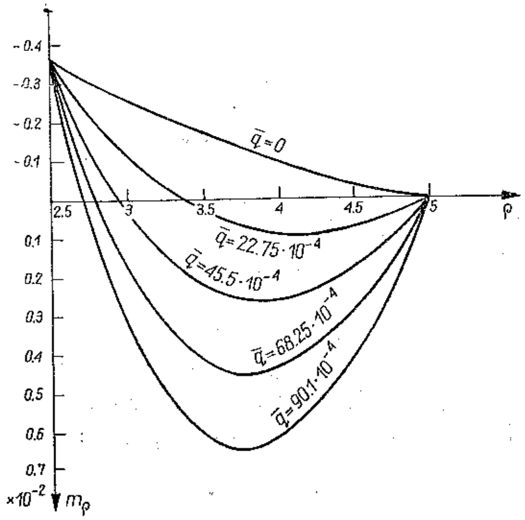

FIG. 2.

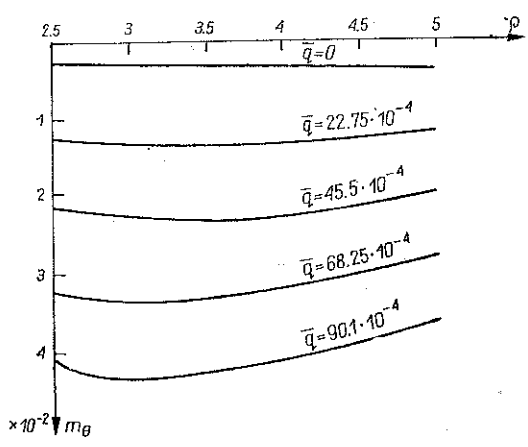

FIG. 3.

m_θ show localization of the points where the cracks will appear. The dependence of the relative damage time (referred to the damage time of the plate loaded only with a bending moment $\bar{M}_0$, $\bar{q}=0$) on the load $\bar{q}$ is shown in Fig. 4. Considering appreciable differences in values of the damage time on the load $\bar{q}$, a logarithmic scale has been taken for the τ-axis. For the same reasons a logarithmic scale has been accepted

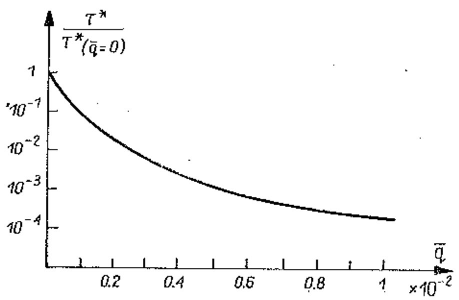

FIG. 4.

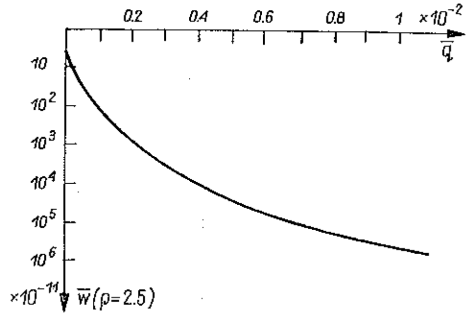

FIG. 5.

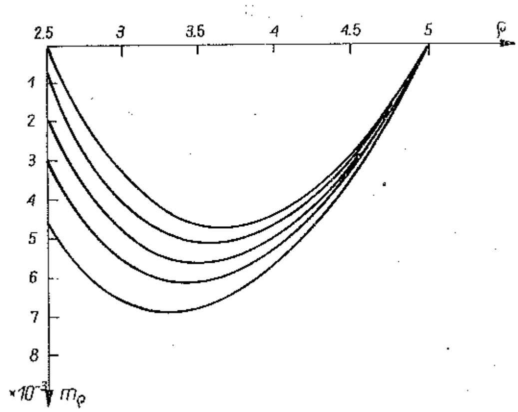

FIG. 6.

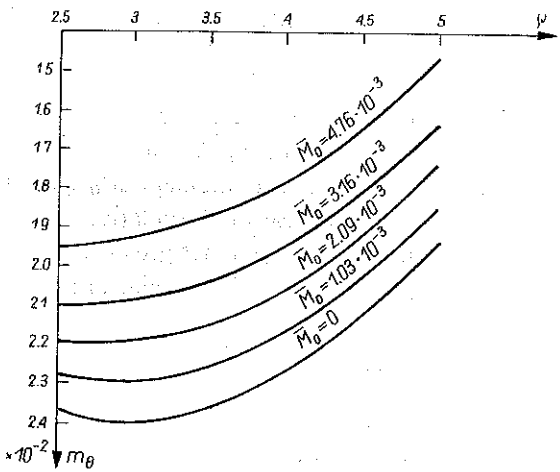

FIG. 7.

in order to show the function of deflection of the internal edge of the plate $\bar{w}$ ($\rho=2.5$) against the load $\bar{q}$, Fig. 5.

The independent investigations of the influence of the load moment M_0 (for $\bar{q}=$ $=56.87 \cdot 10^{-3}$) on the form of the functions solving the problem of brittle, creep

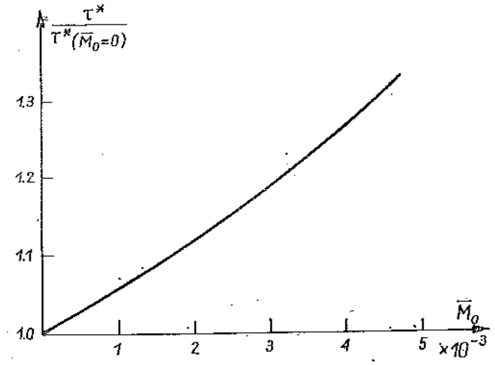

FIG. 8.

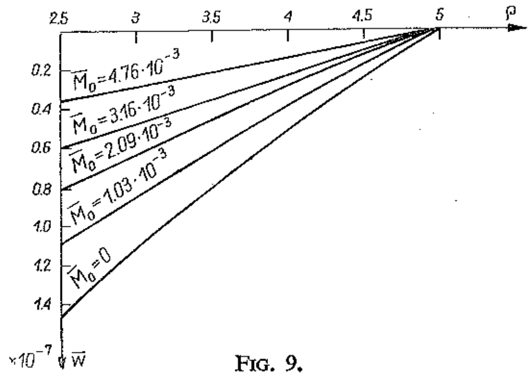

FIG. 9.

rupture of plate are presented in Figs. 6, 7, 8 and 9. The functions of the radial moment m_ξ for different loads

$$\bar{M}_0 = m_\rho\ (\rho = 2.5)$$

are shown in Fig. 6. The functions of circumferential moments corresponding to them are shown in Fig. 7. It is seen from Fig. 7 that the values of circumferential moments decrease with an increasing load moment $\bar{M}_0$ (for fixed load $\bar{q}$). This influence will be marked with an increase of the relative rupture time (related to the time of damage for $\bar{M}_0=0$) when $\bar{M}_0$ becomes greater. The functions of deflection of the midspan for the taken values of $\bar{M}_0$ (Fig. 6) are presented in Fig. 9.

5. Final remarks

The algorithm of calculation, based on the parametrization of bending moments presented in Chapter 4, is limited in application to the analysis of creep and damage processes of plates with nonzero boundary conditions for the angle of deflection $\bar{\varphi}$. According to the parametrization (3.2), the sought value $\bar{\varphi}$ leads to vanishing of the bending moments m_ρ and m_θ. Hence the algorithm of calculation is valid for annular simple-supported plates. In the case of clamped plates or circular plates without hole, a separate algorithm of calculation should be connected with the clamped place or middle point of the plate, respectively.

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Streszczenie

PARAMETRYZACJA MOMENTÓW ZGINAJĄCYCH W ANALIZIE PEŁZANIA SWOBODNIE PODPARTYCH PŁYT KOŁOWO-SYMETRYCZNYCH

Przedstawiono metodę rozwiązania opartą na parametryzacji funkcji momentów zginających dla pierścieniowych płyt swobodnie podpartych w warunkach pełzania ustalonego. W wyniku parametryczacji rozwiązanie zagadnienia brzegowego sprowadza się do rozwiązania układu trzech równań różniczkowych zwyczajnych pierwszego rzędu. Rozdzielony układ równań względem pochodnych umożliwia zastosowanie standardowych procedur całkowania Rungego–Kutty. Czas pierwszych pęknięć obliczono posługując się prawem zniszczenia Kaczanowa–Rabotnowa.

Резюме

ПАРАМЕТРИЗАЦИЯ ИЗГИБНЫХ МОМЕНТОВ В АНАЛИЗЕ ПОЛЗУЧЕСТИ СВОБОДНО ПОДПЕРТЫХ КРУГОГОСИММЕТРИЧНЫХ ПЛИТ

Представлен метод решения, опирающийся на параметризации функций изгибных моментов для свободно подпертых кольцевых плит в условиях установившейся ползучести. В результате параметризации решение краевой задачи сводится к решению системы трех обыкновенных дифференциальных уравнений первого порядка. Разделенная по отношению к производным система уравнений дает возможность применить стандартные процедуры интегрирования Рунге–Кутта. Время появления первых трещин рассчитано, послуживаясь законом разрушения Качанова–Работнова.

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