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PROBABILISTIC APPROACH TO RELIABILITY-BASED OPTIMUM STRUCTURAL DESIGN

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In the paper foundations are given for the design of structures and structural optimization, based on the theory of probability. In this approach, loads, dimensions and material properties are considered as random values. The safety of the structure should be assured and the constraints should be satisfied with acceptable probabilities. A programming method is proposed with random constraints to reduce the probabilistic optimization problem to a deterministic one. The theoretical considerations are applied to an example in which optimal dimensions of a frame box beam are determined.

1. Design and optimization based on the probability theory

The first attempts to take into account in an actual design the probabilistic character of the quantities involved date back to the 1930, when the papers by Strzelecki [39] and Wierzbicki [42] initiated this new and important field in the theory of structural design. For a historical review of main approaches and solutions the reader is referred to the papers by Eimer [7] and Murzewski [27]. Below only a short outlook of these works is presented.

In structural optimization, propabilistic considerations have appeared in papers by Prot [31, 32], Rzhanicyn [35, 36] and Wierzbicki [43].

M. Prot proposed to optimize the probability of the safety of a structure by considering its overall cost, including the costs of tests and investigations realised to obtain information necessary for the design and also the insurance costs.

In Rzhanicyn's concept the cost covered only the cost of the structure itself and the cost of failure multiplied by its probability.

Papers by Freudenthal [9, 10, 11] and Johnson [16] stimulated a large development of the investigations in probabilistic optimization. As a result, valuable works in this field were published by Hilton and Feigen [13], Kalaba [17], Turkstra [41] and Haugen [12]. Among Polish authors a book by Murzewski [27] should be mentioned, where two main categories of optimal probabilistic problems are formulated.

All the achievements published in the above mentioned papers do not mean that almost all variables were not considered as random before 1930.

The admissible stresses or loads prescribed by design codes and regulations were always based, to a greater or smaller extent, on probabilistic estimates derived from observation of real structures and their loadings or from laboratory tests.

However, no consistent probabilistic approach was used, and the codified parameters were largely due to experience and intuition rather than to the concepts of mean values and dispersions of loads and material strengths.

This state of the methods of structural design and optimization resulted above all from the lack of data which would allow a complete statistical analysis. Even at the present time the observational or experimental data are insufficient; this impedes the development of application of probabilistic methods. Also, as it has been pointed out by LIND [20], the traditional views and habits prevailing in building practice have hindered the introduction of probability into design. Unlike other processes, in which mathematical statistics and the probability theory can be used throughout, structural analysis must lead to deterministic conclusions defining precise dimensions of structural elements and their configurations.

This situation in building practice has changed to some extent for the last twenty years since in most countries codes of practice and design recommendations are now based on probabilistic assumptions. The use of limit states design methods in all their variety is generally accepted both by code—makers and even by professional civil engineers. The present development of experimental methods of testing and observation of structures and their loadings, new facilities for data collection and data processing, and also the development of suitable mathematical methods, make it possible to replace intuitive handling of the fact that the inputs in a design problem are nondeterministic by a rational application of probabilistic methods. The quantitative development of the building industry and the increasing sizes of individual special-purpose engineering structures stimulate further the search for safe and economical solutions on these lines. It is therefore more justified than before to formulate the optimization problems and methods using probabilistic notions.

2. PROBABILISTIC APPROACH TO SAFETY AND RELIABILITY OF STRUCTURES

The safety of a structure can be estimated and represented by the probability of its failure, i.e. the occurrence of an ultimate limit state manifesting itself as, say, the formation of yield hinges, tupture, overturning, etc. In addition to ultimate limit states, we also distinguish limit states of serviceability, involving phenomena such as excessive cracking, displacements or vibrations, which do not cause a structural failure but make it impossible to use the structure according to its intended function; states in which a given building is unserviceable for reasons independent of its structure are not included in this category.

The concepts of safety and of serviceability so understood make up jointly what we call the reliability of the structure. Let us note that these terms are not always used in the sense given above. Depending on context, various authors define them in a somewhat different manner.

The probability of the occurrence of a limit state in a given structure can be calculated from the probability distribution of the loading and strength of the structure and all the other relevant parameters. The probability distributions of the random variables characterizing the loading can be determined by observation and measurement, and those of material and structural strength by laboratory or field tests.

In the present study we are concerned with the ultimate limit states; extending it to the limit states of serviceability does not present any serious conceptual or formal difficulties.

Failure of structure may concern the structure as a whole or may be attributed to the failure of its individual members. Depending on the kind of the structure, the relationship between the failure of a single member and that of the whole structure may be different. In statically determinate structures, for example, a failure of the weakest member causes a failure of the whole structure (weakest-link structures), while hyperstatic structures fail only if several members reach their capacity simultaneously (fail-safe structures).

The probability of various failure modes and the influence of the statistical correlation between them on the overall probability of failure P_f were examined theoretically by STEVENSON [38]. The relationship between the failure probability and the optimum weight of a structure was also investigated by LIND [21]. MOSES and KINSER [26] showed that the effect of the correlation of the optimal design depends largely on the load to strength variation coefficient ratio and only to a smaller extent on the chosen allowable failure probability P_f.

SHIRAISHI and FURUTA [37] presented the safety analysis for the design of rigid frames using the minimum-weight criterium. Considering various failure modes, upper and lower bounds for a solution were defined. The optimal solutions were obtained by the proposed iterative method.

Let us consider the simplest case of probabilistic safety analysis, in which the structure consists of a single bar of strength R, loaded by a tensile force P, where R and P are random variables of known distributions (MOSES [23, 24].) The probability P_f of failure for such a model, i.e. the probability that $P>R$, may be computed from

$$P_f=\int_0^{\infty} [F_R(t)]\, f_P(t)\, dt=1-\int_0^{\infty} [F_P(t)]\, f_R(t)\, dt, \tag{2.1}$$

where $F(t)$ denotes the probability distribution and $f(t)$ the density or frequency distribution. It was assumed, for simplicity, that both the load and the strength are normally distributed; in this case P_f can be computed as it is shown in example given in [23] and also discussed in [1]. This example is useful as an illustration of a manner of reasoning and a way to compute the failure probability. For a real structure, all structural members and their possible failure modes under various loading conditions during the entire lifetime of the overall structure must be taken into consideration.

In this approach a coefficient $n=\bar{R}/\bar{P}$ is introduced, which corresponds to what in a deterministic approach is termed the safety factor; here $\bar{R}$ and $\bar{P}$ are mean values of R and P, respectively. The conventional safety factors in most specifications have been developed in an evolutionary manner according to experience based on the existing structures. In recent years a great deal of work has been done on deriving safety factors from probabilistic safety analysis; their definitions vary depending on the parameters they relate to and aim at computational facility. A set of partial safety factors accomodated to semi-probabilistic limit state design was proposed in 1970, by the European Committee of Concrete and the International Federation of Prestressing in the form of International Recommendations [44]. A new, improved version of the Recommendations was published in 1978, [45]. For lack of sufficient statistical data, only conservative estimates were given for most of the coefficients, but underlying their derivation was the probabilistic concept of reliability; hence the term "semiprobabilistic method". This approach permits the results of new investigations and observations to be introduced gradually as they are acquired and elaborated statistically.

Further developments of this approach in standards and recommendations are aimed at more complete exploitation of the probabilistie concept and of the statistical data. At present these developments named "level two" and "level three" methods are not yet entirely operational for effective structural design and are used for calibration of the above described "semi-probabilistic method", which is also called "level one" method — the numerals reflect the degree of consideration of the probabilistic concept. In this convention the deterministic method is the "level zero" method.

3. Reliability-based structural optimization

The classical formulation of an optimum design problem reads (Moses [23], Brandt, *et al.* [1]):

Minimize $F(x_i)$ subject to $g_j(x_i) \geqslant 0$; $i=1, 2, ..., n$, $j=1, 2, ..., m$, x_i denote design variables, $F(x_i)$ is an objective function and $g_j(x_i)$ are constraints.

The design variables x_i represent the geometrical and the mechanical properties of the structure that must be determined. The objective function $F(x_i)$ measures the volume, weight or cost of the structure, or some other quantity chosen as a criterion of optimization, e.g. elastic strain energy. Constraints $g_j(x_i)$ follow from strength and strain limit values, but may also include fabrication or functional requirements. Those constraints which limit the stresses or strains to some permissible values involve the conventional safety factors which, in the best of situations, are determined from probabilistic and statistical analysis. Constraints imposed by the conditions of execution or use of the structure are regarded as deterministic.

In the reliability-based approach the numerous constraints on stresses and strains are replaced by a single condition on the failure probability P_f as a function of the design variables:

Minimize $F(x_i)$ subject to $P_f(x_i) \leqslant P_f$ permissible.

Technological or other side constraints can also be imposed of the form $g_j(x_i) \geqslant 0$.

The limit value of the failure probability should be determined with regard to all possible failure modes, the value of the structure and the cost of its failure and its consequences. The probability P_f is derived from the statistical distributions of the random variables representing the loading and strength of the structure. The optimization problem consists in determining the values of the design variables such that the objective function F, e.g. volume or cost, will be minimum and probability of failure will not exceed the allowable value.

For multi-member structures the failure probability P_f can be approximated by the sum of the probabilities of failure of the individual members; these can be determined in a manner similar to that mentioned above in the fundamental one-member one-load case. The approximation is good if all the probabilities involved are small. The optimal structure has its members proportioned so that the overall objective function F is minimum and the failure probabilities of the individual members add up to an overall probability of failure not exceeding a prescribed permissible value. Problems formulated in this fashion were considered by HILTON and FEIGEN [13], SWITZKY [40], KHACHATURIAN and HAIDER [19], MOSES and KINSER [26] and PARIMI and COHN [30] among others.

Now we consider, following DAVIDSON, FELTON and HART [5] and MOSES [25], a formulation which includes constraints on individual failure modes as well.

The minimum-weight optimization problem for a structure with random parameters may be stated as follows:

Minimize the objective function

$$F(\mathbf{x}) \tag{3.1}$$

subject to the constraints

$$P_f = P_0(\mathbf{x}) \leqslant p_0, \tag{3.2}$$

$$P_i(\mathbf{x}) = P[g_i(\mathbf{x}) > G_i(\mathbf{x})] \leqslant p_i, \quad i = 1, ..., M, \tag{3.3}$$

where $\mathbf{x}^T = (x_1, x_2, ..., x_n, ..., x_N)$ is a vector of N random variables, the first n of which are the design variables; $P_0(\mathbf{x})$ denotes the overall probability of failure and p_0 is the allowable limit for this quantity; $g_i(\mathbf{x})$ is the i-th response quantity of the structure (e.g. displacement or stress) and $G_i(\mathbf{x})$ its allowable limit. Note that in many cases the allowable response G_i may be independent of $\mathbf{x}$. The expression $P[...] \leqslant p_i$ in Eq. (3.2) means that the probability $P_i(\mathbf{x})$ of the i-th response being greater than the allowable limit must not exceed a specified failure probability p_i. The probabilities p_0 and p_i will generally have to be very small for the constraints (3.2) to be active and sensitive to the particular distributions of the individual variables.

Assuming all the random variables to be normally distributed, we may relate p_i to response quantities by

$$p_i = 1 - \Phi(e_i), \tag{3.4}$$

where $\Phi(e_i)$ is the cumulative normal distribution function for which

$$e_i = \frac{|\bar{G}_i(\mathbf{x}) - \bar{g}_i(\mathbf{x})|}{[\sigma^2_{G_i}(\mathbf{x}) + \sigma^2_{g_i}(\mathbf{x})]^{1/2}} \tag{3.5}$$

is the coupling equation between response and allowable response. For a given p_i, e_i may be obtained from Φ-tables and the original constraint (3.3) replaced by the equivalent condition

$$\bar{g}_i(\mathbf{x})+e_i[\sigma^2_{G_i}(\mathbf{x})+\sigma^2_{g_i}(\mathbf{x})]^{1/2}-\bar{G}_i(\mathbf{x})\leqslant 0, \tag{3.6}$$

where e_i is required to be positive.

An important factor in the reliability analysis of a structure is the relationship between the overall probability of failure and the probabilities of the individual failure modes. For multiply-loaded hyperstatic structures which are assumed to fail when any constraint is violated ("weakest-link" model), failure modes are usually neither completely statistically dependent nor completely statistically independent and exact correlations are difficult to determine. However, these two extreme cases provide a lower and an upper bounds on the overall reliability:

$$\max P_i(\mathbf{x})\leqslant P_0(\mathbf{x})\leqslant \sum_{i=1}^{M} P_i(\mathbf{x}). \tag{3.7}$$

If the design involves relatively few active failure modes, which is often the case, the upper bound can reasonably be used for actual evaluation of P_0:

$$P_0(\mathbf{x})=\sum_{i=1}^{M} P_i(\mathbf{x}). \tag{3.8}$$

In this relation $P_i(\mathbf{x})$ is obtained from a modified form of Eq. (3.4)

$$P_i(\mathbf{x})=1-\Phi(\bar{e}_i), \tag{3.9}$$

where $\bar{e}_i$ is the value of e_i satisfying the relation (3.6) as an equality.

The above considerations allow the original optimization problem to be restated as:

Minimize

$$F(\mathbf{x}) \tag{3.10}$$

subject to

$$\sum_{i=1}^{M} P_i(\mathbf{x})-p_0\leqslant 0, \tag{3.11}$$

$$\bar{g}_i(\mathbf{x})+e_i[\sigma^2_{G_i}(\mathbf{x})+\sigma^2_{g_i}(\mathbf{x})]^{1/2}-G_i(\mathbf{x})\leqslant 0,$$

$$i=1, \ldots, M.$$

It was shown above that the solution of reliability — based structural optimization with random parameters has been transformed to the deterministic inequality-constrained minimization problem which can be solved using well-known nonlinear programming methods.

If the function $F(x_i)$ represents the cost of the srtucture, it can — in the simplest case — be taken as proportional to the volume or weight. For more realistic design, however, it is necessary to include in it both economy and safety components. One alternative formulation (F. MOSER [23]) defines the total cost C as the sum

$$C = C_i + P_f\, C_f, \tag{3.12}$$

where C_i is the initial cost of erecting the structure and C_f is the cost associated with structural failure; the failure cost consists of the cost of reconstruction assumed to be equal to the initial cost and another term C' expressing the consequences of failure (damage):

$$C_f = C_i + C'. \tag{3.13}$$

Another approach to reliability optimization is to set a permissible value of $F(x_j)$, e.g. the material volume of the structure, and to seek the distribution of material over different parts or members for a minimum of the failure probability P_f. If the optimum P_f is too large, then either the assigned volume or the feasibility of the kind of the structure adopted must be re-evaluated (HILTON and FEIGEN [13], MOSES and KINSER [26], MOSES [23], ROSENBLUETH and MENDOZA [34], MAU and SEXMITH [22]).

Both approaches to reliability optimization — design for minimum total cost subject to a failure probability constraint and design for minimum failure probability with a volume constraint — are discussed and illustrated with examples in recent papers by FRANGOPOL and RONDAL [8] and by BURY [2].

An instructive example of a minimum-volume reliability-based design of an isostatic truss was given by KHACHATURIAN [18]. The problem is solved for cases where both the concentrated load and strength of each bar of the truss follow the Lognormal or the Gamma probability distributions, with the overall probability of failure prescribed as 10^{-3}, 10^{-4} or 10^{-5} and for several different values of the coefficients of variation of load (γ_x) and strength (γ_{yk}). The results show that the validity of the optimal solution depends on the degree of knowledge of the load and strength probability distributions and on the choice of an appropriate level of safety.

Let us note, after Khachaturian, that the design considered in the example ignores certain important factors, e.g. deformability or serviceability requirements, that the type of structure and its geometry are fixed and that the structure is idealized to a pin-jointed truss.

STEVENSON [38] considered reliability-based optimum design of hyperstatic structures. His analysis can be applied to frames, trusses and grids, and even to plates treated by the yield lines method, that is, to all cases where the ultimate limit state function can be represented as a linear combination of the random variables of load and strength.

An example of relation between the optimum material cost, the overall probability of failure and the coefficients of variation of material strength and load intensity was presented by MOSES [23] for a single story frame. Figure 1 shows that cost increases considerably when imposed probability of failure decreases and this is an obvious effect of the additional structural safety. Cost increases also with the coefficients of variations, which reflects the influence of the material quality and of the characteristic of the load. The influence of assumed distribution normal or log normal is negligible in this example.

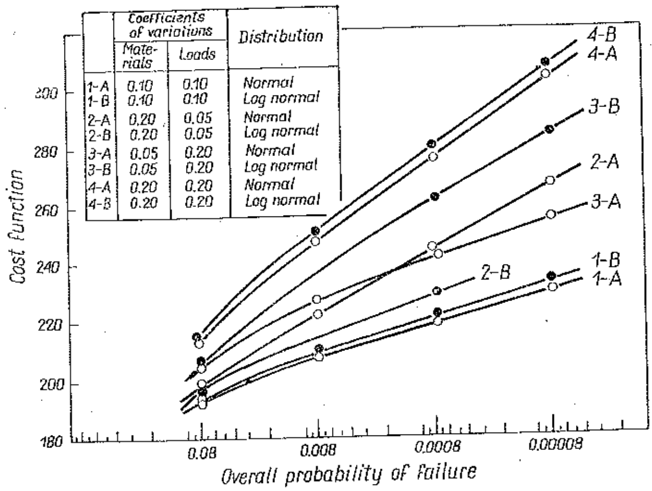

FIG. 1. Cost function plotted against overall probability of failure [23].

By analysing optimal solutions obtained under various assumptions it is possible to observe and formulate certain relationships between different parameters of an optimum structure. Such relationships can be useful in simplifying specific optimization problems. As an example, let us mention a result obtained by SWITZKY [40] — developing the ideas of HILTON and FEIGEN [13]; he showed that in an optimum structure under a single load the following relationship holds:

$$\frac{\text{weight of member 1}}{\text{total weight}} = \frac{P_f \text{ of member 1}}{P_f \text{ overall allowable}}.$$

BURY [2] has considered minimum weight design with reliability constraints under sequential random loads and has derived the relations between reliability and design variables.

CARMICHAEL [3] has derived the iterative equations for structural systems modelling based on the Bellman principle of optimality and control theory. DAVIDSON, FELTON and HART [6], NIGAM [29] and NARAYANAN and NIGAM [28] have considered structural optimization under dynamic loads using nonlinear and stochastic programming.

4. A GENERAL NONLINEAR STOCHASTIC PROGRAMMING PROBLEM

The problem can be stated as follows [14, 15]. Find a vector $\mathbf{x}=(x_1, x_2, ..., x_n)$ which minimizes the objective function $F(\mathbf{y})$ subject to the constraints

$$P\,[g_j(\mathbf{y})\geqslant 0]\geqslant p_j, \quad j=1, 2, ..., m, \tag{4.1}$$

where $\mathbf{y}$ is the vector of N random variables $y_1, y_2, ..., y_N$, which include the design variables $x_1, ..., x_n$ and all other parameters involved in the problem and considered to be random variables. The case where $\mathbf{x}$ is deterministic is a special case of the present formulation.

In what follows we shall assume that all the random variables are independent and normally distributed.

The stochastic problem stated above can be converted into an equivalent deterministic problem by a chance-constrained programming technique (see e.g. CHARNES and COOPER [4] or RAO [33].

We begin by resolving the objective function into a Taylor series about the expected values of y_i:

$$F(\mathbf{y})=F(\bar{\mathbf{y}})+\sum_{i=1}^{N}\left(\frac{\partial F}{\partial y_i}\bigg|_{\bar{\mathbf{y}}}\right)(y_i-\bar{y}_i)+\text{higher order derivative terms.} \tag{4.2}$$

If the standard deviations of y_i, σ_{y_i} are small, $F(\mathbf{y})$ can be approximated by the first two terms of Eq. (4.2), i.e.

$$F(\mathbf{y})\approx F(\bar{\mathbf{y}})-\sum_{i=1}^{N}\left(\frac{\partial F}{\partial y_i}\bigg|_{\bar{\mathbf{y}}}\right)\bar{y}_i+\sum_{i=1}^{N}\left(\frac{\partial F}{\partial y_i}\bigg|_{\bar{\mathbf{y}}}\right)y_i\equiv\psi(y). \tag{4.3}$$

Being a linear function of normally distributed variables y_i, $\psi(\mathbf{y})$ also follows normal distribution. The mean value and the variance of ψ are given by

$$\bar{\psi}=\psi(\bar{\mathbf{y}}) \tag{4.4}$$

and

$$\sigma_\psi^2=\sum_{i=1}^{N}\left(\frac{\partial F}{\partial y_i}\bigg|_{\bar{\mathbf{y}}}\right)^2\sigma_{y_i}^2 \tag{4.5}$$

because all y_i are independent.

A new deterministic objective function can be defined as

$$\mathcal{F}(\mathbf{y})=k_1\bar{\psi}+k_2\sigma_\psi, \tag{4.6}$$

where k_1 and k_2 are nonnegative weights indicating the relative importance for minimization of the mean and the standard deviation. Setting $k_2=0$ would mean that the expected value of F is to be minimized with no regard to the standard deviation, while the choice $k_1=0$ would imply that we are only interested in minimizing the dispersion of F about an arbitrary mean value. The case $k_1=k_2=1$ attaches equal importance to both characteristics of F.

An alternative possibility is to chose the mean value ψ as the objective function and to introduce the additional constraint $\sigma_\psi\leqslant k_3\psi$, where k_3 is a constant.

The inequalities constraints (4.1) can be written as

$$\int_0^\infty f_{g_j}(g_j)\,dg_j\geqslant p_j, \quad j=1,2,...,m, \tag{4.7}$$

where $f_{g_j}(g_j)$ is the probability density function of the random variable g_j, whose range is assumed to be $-\infty$ to ∞; to within the first order terms of its Taylor series expansion about the mean $\bar{\mathbf{y}}$, the constraint function $g_j(\bar{\mathbf{y}})$ can be represented as

$$g_j(\mathbf{y}) \approx g_j(\bar{\mathbf{y}}) + \sum_{i=1}^{N} \left(\frac{\partial g_j}{\partial y_i} \bigg|_{\bar{\mathbf{y}}} \right) (y_i - \bar{y}_i). \tag{4.8}$$

Hence the mean value $\bar{g}_j$ and the standard deviation σ_{g_j} are calculated as

$$\bar{g}_j = g_j(\bar{\mathbf{y}}), \tag{4.9}$$

$$\sigma_{g_j} = \left[\sum_{i=1}^{N} \left(\frac{\partial g_j}{\partial y_i} \bigg|_{\bar{\mathbf{y}}} \right)^2 \sigma_{y_i}^2 \right]^{1/2}. \tag{4.10}$$

The inequalities constraints (4.7) by standardization can be written in the following form:

$$P\left[-\frac{g_j - \bar{g}_j}{\sigma_{g_j}} \geqslant \frac{-\bar{g}_j}{\sigma_{g_j}} \right] \geqslant p_j, \quad j = 1, 2, \ldots, m, \tag{4.11}$$

where $(g_j - \bar{g}_j)/\sigma_{g_j}$ is a standard normal variable with zero mean and unit variance. If e_j denotes the value of the standard normal variable at which the standard normal distribution function

$$\Phi(e_j) = p_j, \tag{4.12}$$

then the inequality constraints (4.11) can be stated as

$$\Phi\left(-\frac{\bar{g}_j}{\sigma_{g_j}} \right) \geqslant \Phi(e_j), \quad j = 1, 2, \ldots, m. \tag{4.13}$$

These inequalities will be satisfied only if

$$-\frac{\bar{g}_j}{\sigma_{g_j}} \geqslant e_j$$

or

$$\bar{g}_j + e_j \sigma_{g_j} \leqslant 0, \quad j = 1, 2, \ldots, m. \tag{4.14}$$

By substituting Eq. (4.10) in the inequalities (4.11), we obtain

$$\bar{g}_j + e_j \left[\sum_{i=1}^{N} \left(\frac{\partial g_j}{\partial y_i} \bigg|_{\bar{\mathbf{y}}} \right)^2 \sigma_{y_i}^2 \right]^{1/2} \leqslant 0, \quad j = 1, 2, \ldots, m. \tag{4.15}$$

Thus the original stochastic optimization problem has been transformed into the deterministic problem of minimizing the objective function (4.6) subject to constraints (4.15).

As an illustration of the above stochastic programming method let us consider the following example.

EXAMPLE. Design the cross-section of a steel frame beam shown in Fig. 2 so that it may carry the load uniformly distributed p and has a minimum volume. All the dimensions of the frame beam as well as the load and the permissible stresses are assumed to be random variables with normal distributions.

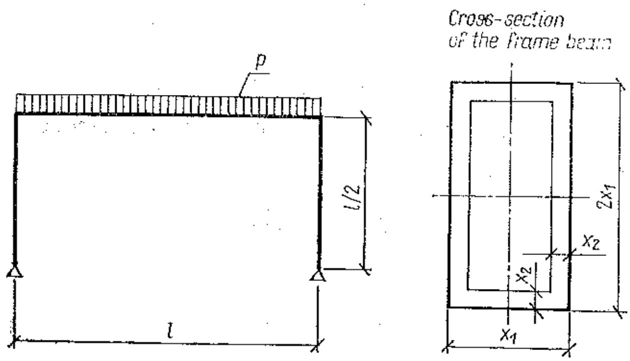

FIG. 2. Dimensions and load of the frame.

The objective function, i.e. the cross-sectional area, is given by

$$A=2x_2\,(3x_1-2x_2),$$

where x_1 is the width of the beam and x_2 denotes the flange and web thicknesses. The depth of the beam is equal to $2x_1$. The design variables are the mean thickness $\bar{x}_1$ and $\bar{x}_2$. The standard deviations of x_1 and x_2 are assumed to be related to the mean values by

$$\sigma_{x_1}=\alpha_1\,\bar{x}_1 \quad \text{and} \quad \sigma_{x_2}=\alpha_2\,\bar{x}_2,$$

where α_1 and α_2 are constants.

A minimum of the function A is sought subject to the following constraints: stress constraint at the cross-section edges

$$g_1=\frac{Mx_1}{J}+\frac{H}{A}-f\leqslant 0$$

i.e.

$$\frac{pl^2\,x_1}{16J}+\frac{pl}{8A}-f\leqslant 0,$$

where

$$J=\frac{2}{3}\,x_1^4-\frac{2}{3}\,(x_1-2x_2)\,(x_1-x_2)^3$$

and f is the permissible stress;

side constraints, i.e. limitations on the minimum and maximum thicknesses of the web and the flanges

$$g_2=-x_1+x_{1\min}\leqslant 0,$$

$$g_3=x_1-x_{1\max}\leqslant 0,$$

$$g_4=-x_2+x_{2\min}\leqslant 0,$$

$$g_5=x_2-x_{2\max}\leqslant 0.$$

Each constraint is required to hold with a probability not less than a prescribed probability p_i.

The random variable vector **y** has the following components, each assumed to be normally distributed with the mean and standard deviation as indicated:

uniformly distributed load

$$y_1=(\bar{p},\ \sigma_p),$$

permissible stress

$$y_2=(\bar{f},\ \sigma_f),$$

beam length

$$y_3=(\bar{l},\ \sigma_l),$$

beam width

$$y_4=(\bar{x}_1,\ \sigma_{x_1}),$$

flange and web thickness

$$y_5=(\bar{x}_2,\ \sigma_{x_2}).$$

To construct a deterministic objective function we find the derivatives at $\mathbf{y}=\bar{\mathbf{y}}$ of the function

$$\bar{\psi}=A(\bar{\mathbf{y}})=2\bar{x}_2(3\bar{x}_1-2\bar{x}_2)$$

with respect to the variables y_i:

$$\frac{\partial A}{\partial y_1}=\frac{\partial A}{\partial y_2}=\frac{\partial A}{\partial y_3}=0,$$

$$\frac{\partial A}{\partial y_4}=6\bar{x}_2,\qquad \frac{\partial A}{\partial y_5}=6\bar{x}_1-8\bar{x}_2.$$

Thus the new objective function takes the form

$$A=k_1\bar{\psi}+k_2\sigma_\psi=k_1[2\bar{x}_2(3\bar{x}_1-2\bar{x}_2)]+$$
$$+k_2[36\bar{x}_2^2(\alpha_1\bar{x}_1)^2+(6\bar{x}_1-8\bar{x}_2)^2(\alpha_2\bar{x}_2)^2].$$

To obtain the deterministic constraints for the function A we compute the partial derivatives of the constraint functions, viz. $\left.\frac{\partial g_i}{\partial y_i}\right|_{\bar{\mathbf{y}}}$ and use the formula (4.15), i.e.

$$\bar{g}_j+e_j\left[\sum_{i=1}^{5}\left(\left.\frac{\partial g_j}{\partial y_i}\right|_{\bar{\mathbf{y}}}\right)^2\sigma_{y_i}^2\right]^{1/2}\leqslant 0,\qquad j=1,2,\ldots,5.$$

The resulting constraints are:

For $j=1$:

$$\frac{\bar{p}l^2\bar{x}_1}{16\bar{J}}+\frac{\bar{p}l}{8\bar{A}}-\bar{f}+e_1\left\{\left(\frac{l^2_{\bar{x}_1}}{16\bar{J}}+\frac{l}{8\bar{A}}\right)^2\sigma_p^2+\sigma_f^2+\left(\frac{\bar{p}l\bar{x}_1}{8\bar{J}}+\frac{\bar{p}}{8\bar{A}}\right)^2\sigma_1^2+\right.$$
$$+\left[\frac{\bar{p}l^2}{8\bar{J}^2}\left[-\bar{x}_1^4+\frac{1}{3}(\bar{x}_1-\bar{x}_2)^2(3\bar{x}_1^2-4\bar{x}_1\bar{x}_2-2\bar{x}_2^2)\right]-\frac{3}{4}\frac{\bar{p}l\bar{x}_2}{\bar{A}^2}\right]^2\alpha_1^2\bar{x}_1^2+$$
$$\left.+\left[-\frac{\bar{p}l^2\bar{x}_1}{24\bar{J}^2}(\bar{x}_1-\bar{x}_2)^2(5\bar{x}_1-8\bar{x}_2)-\frac{\bar{p}l}{4\bar{A}^2}(3\bar{x}_1-4\bar{x}_2)\right]^2\alpha_2^2\bar{x}_2^2\right\}^{1/2}\leqslant 0.$$

For $j=2$: $g_2=-\bar{x}_1+x_{1\min}-e_2\,\alpha_1\,\bar{x}_1\leqslant 0$,
For $j=3$: $g_3=\bar{x}_1-x_{1\max}-e_3\,\alpha_1\,\bar{x}_1\leqslant 0$,
For $j=4$: $g_4=-\bar{x}_2+x_{2\min}-e_4\,\alpha_2\,\bar{x}_2\leqslant 0$,
For $j=5$: $g_5=\bar{x}_2-x_{2\max}-e_5\,\alpha_2\,\bar{x}_2\leqslant 0$.

The deterministic problem so formulated has been solved with the following numerical data:

$$\bar{p}=800\text{ kN/m},\quad \sigma_p=40\text{ kN/m},\quad \bar{f}=200\text{ MPa},\quad \sigma_f=10\text{ MPa},\quad \bar{l}=5\text{ m},$$
$$\sigma_1=0.25\text{ m},\quad \alpha_1=\alpha_2=0.05,\quad\text{and}\quad p_j=0.95\quad (j=1,\dots,5),$$

with the corresponding value (from normal distribution tables) $e_j=1.645$.

The presence of only two design variables has allowed a graphical solution. The feasible region and objective contours are shown in Fig. 3, when $k_1=1$ and $k_2=0$. In other cases, when $k_1=0$, $k_2=1$ and $k_1=1$, $k_2=1$, the feasible region is the same and objective contours are similar.

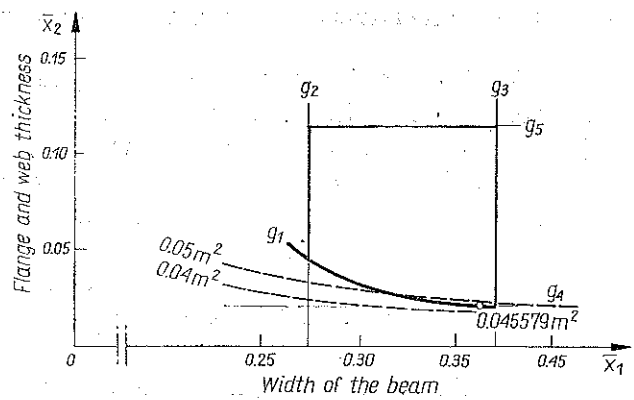

FIG. 3. Graphical solution of the optimization problem.

In all cases, the minimum of the objective function is realized at the intersection of the g_1 and g_4. It corresponds to an allowable minimum thickness of webs and flanges equal to $\bar{x}_2=21.8$ mm and to the width of the box cross-section determined by the strength condition and equal to $\bar{x}_1=363$ mm. The values of the objective function A in the three cases considered are 0.045579 m^2, 0.000008 m^2 and 0.045587 m^2.

If all the quantities involved in the problem are treated as deterministic, the optimum solution is found at the intersection of the same two constraints, with $x_1=334$ mm, $x_2=20$ mm and the minimum cross-sectional area equal to 0.038480 m^2.

The total cross sectional area is shown in Fig. 4 as a function of the probability of failure. This area increases when the overall failure probability decreases.

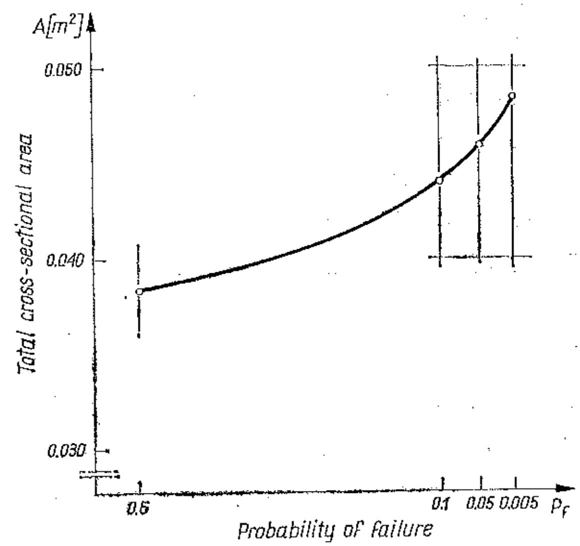

FIG. 4. Cross-sectional area plotted against probability of failure P_f.

5. Prospects of reliability-based structural optimization

The goals to be achieved by exercising a probabilistic approach to safety are easy to formulate, although their realization does not seem possible in the near future. The first objective is to base the code safety factors which relate to strengths, loads and other random variables determining the safety of a structure, on a firm probabilistic foundation. An important issue is also to eliminate the inconsistencies present in deterministic design. The general objective is to reach economic gains by designing structures cheaper than those based on deterministic premises but equally reliable or, alternatively, to increase the safety of structures without increasing the cost.

Reliability-based optimization leads to designs in which structure and member sizes are optimal with respect to an adopted criterion, say of minimum volume or cost, the overall failure probability being equal to a prescribed allowable value. The safety factors of the individual members in such a design are not equal and depend on the degree of statistical correlation between the failures of those members. Also, to different failure modes of a given structure there correspond different safety factors.

The efforts which are being made to incorporate a probabilistic approach to safety into structural optimization should stimulate studies of random variables encountered in structural engineering. As we have indicated before, the choice of probability distributions and their parameters for a design strongly influences the optimal solution; the lack of sufficient observational and experimental data makes it impossible to choose correctly, and hence to obtain a useful result. Empirical studies of load conditions are particularly desirable as the existing statistical data in this area are much poorer than those relating to the strength of material and structural elements. As pointed out by LIND [20], small failure probability limitations and very meagre statistical data on load and strength probability distributions do not allow probability statements of the same confidence levels that statisticians usually have in other fields.

There is also the difficulty to provide a rational basis for determining the allowable probability of failure for a given structure. Although complete elimination of the subjective elements and intuition is hardly conceivable, detailed statistical data are indispensable if serious errors are to be avoided. The data should include detailed information about the safety of the existing structures of the kind considered; information on safety in other areas, e.g. transport, would also be useful for comparative studies.

Main obstacles in the development of the probability approach to structural safety in general and particularly to the reliability-based optimization are formulated by FRANGOPOL and RONDAL [8] in the following form.

The first of them is the lack of complete information on the statistical data of loads, strength and other characteristic values. This question has been mentioned above in more details.

The second obstacle is the reaction of public opinion against any possible decrease of the structural safety which may be suspected in this formulation of the optimum design. People are prepared to accept much more risk in other activities like sport or transportation than while living at home. This attitude may be reflected to some extent in the behaviour of professional designers of building and civil engineering structures.

Another obstacle is based on economical aspects: if in the optimization of structural dimensions exceptional actions are taken into account, the cost of structures should increase considerably. There actions like forces due to earthquake or to impact of a vehicle are already considered in the limit state design, codified in several countries. There is, however, no adequate optimization criterium which should be based on the simultaneous consideration of both: the probability of the appearance of such exceptional actions and the consequences of this event.

At the present state of knowledge the value of the existing solutions of probabilistic structural design problems is primarily theoretical. They allow verification of the design procedures used and give an insight into the possibilities of their further evolution. Above all, however, they make it possible to re-evaluate deterministic design solutions and to assess potential advantages of further work in this direction, based on thorough statistical studies.

Thus, although structural design and optimization continue to rely on deterministic methods which have indeed led to many interesting and important solutions, it seems more rational even at the present state of knowledge to treat loads, strength, deformability and other relevant quantities as random variables. We can thus expect a further development of structural optimization methods based on a probabilistic approach to safety and an analogous evolution of load standards, material specifications and methods of observation and testing of structures.

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Streszczenie

OPTYMALIZACJA KONSTRUKCJI Z UWZGLĘDNIENIEM ICH NIEZAWODNOŚCI METODAMI PROGRAMOWANIA STOCHASTYCZNEGO

W pracy omówiono podstawy projektowania i optymalizacji konstrukcji na podstawie teorii prawdopodobieństwa. W takim ujęciu obciążenia wymiary i wielkości charakteryzujące materiał mogą być wielkościami losowymi. Bezpieczeństwo konstrukcji i warunki ograniczające muszą być spełnione z dostatecznie dużym prawdopodobieństwem. Przedstawiono metodę programowania z ograniczeniami losowymi, umożliwiającą sprowadzenie problemu stochastycznego do deterministycznego. Rozważania zilustrowano przykładem wyznaczenia optymalnych wymiarów przekroju rygla ramownicy.

Резюме

ОПТИМИЗАЦИЯ КОНСТРУКЦИЙ С УЧЕТОМ ИХ НАДЕЖНОСТИ МЕТОДАМИ СТОХАСТИЧЕСКОГО ПРОГРАММИРОВАНИЯ

В работе обсуждены основы проектирования и оптимизации конструкций на основе теории вероятности. В таком подходе нагрузки, размеры и величины характеризующие материал могут быть случайными величинами. Безопасность конструкции и ограничивающие условия должны быть удовлетворены с достаточно большой вероятностью. Представлен метод программирования со случайными ограничениями, который дает возможность сведения стохастической проблемы к детерминистической проблеме. Рассуждения иллюстрированы примером определения оптимальных размеров сечения ригеля рамы.

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