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UNSTEADY FREE CONVECTION BOUNDARY LAYER FLOW AT A THREE-DIMENSIONAL STAGNATION POINT

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The unsteady free convection flow at the stagnation point of a three-dimensional body under boundary layer approximations has been studied when the wall temperature varies arbitrarily with time. The semi-similar equations governing the flow have been solved numerically using an implicit finite-difference scheme. The heat transfer is found to be strongly dependent on time whereas the skin friction is weakly dependent on it. The increase in the Prandtl number results in the increase in heat transfer but skin friction is reduced. The mass transfer affects both the heat transfer and skin friction significantly, but for large times its effect on the heat transfer is very weak.

NOMENCLATURE

c	parameter characterizing the nature of the stagnation point flow,
C_f, $\bar{C}_f$	skin friction coefficients at the wall in the x and y directions, respectively,
f, s	dimensionless stream functions,
f_w	dimensionless mass transfer parameter,
F, S	dimensionless velocity components in the x and y directions, respectively,
g	acceleration due to gravity,
G	dimensionless temperature,
$(G_\eta)_w$	heat transfer parameter at the wall,
L	characteristic length,
Nu	Nusselt number,
Pr	Prandtl number,
t, t^*	dimensional and dimensionless times, respectively,
T	temperature,
T_{w0}	wall temperature at $t=0$ (or $t^*=0$),
T_∞	free stream temperature (constant),
u, v, w	velocity components in the x, y and z directions, respectively,
x, y, z	principal, transverse and normal directions, respectively,
β	bulk coefficient of thermal expansion,
ε	constant,
η	dimensionless independent variable,
ν	kinematic viscosity,
ϱ	density,
φ	function of t^*.

SUBSCRIPTS

i	denotes initial conditions
w	denotes conditions on the surface
t, t^*, x, y, z, η	denote derivatives with respect to t, t^*, x, y, z and η, respectively.

1. Introduction

The phenomenon of free convection arises in a fluid when the temperature changes cause density variations leading to buoyancy forces. This process of heat transfer is encountered in the natural world such as in atmospheric and oceanic circulations as well as in technology such as in power transformers, nuclear reactors etc. The heat transfer mechanism, involving flows in contact with the walls which undergo a thermal transient change, is important for its applications in various industries. There is, at present, a great deal of information about the steady, laminar free convection boundary layer over bodies of variuos shapes. Extensive studies of the pertinent literature are given in excellent review papers by EDE [1], GEBHART [2] and MORGAN [3]. The various aspects of the steady laminar free convection boundary layer on three-dimensional bodies have been studied by several authors [4–9]. However, unsteady free convection over three-dimensional bodies has attracted comparatively little attention in the literature.

The aim of the present anylysis is to study the unsteady free convection at the stagnation point of a three-dimensional body under (laminar) boundary layer approximations when the wall temperature varies arbitrarily with time. The semi-similar partial differential equations governing the flow have been solved numerically using an implicit finite-difference scheme [10]. The results have been compared with those already available [4, 7].

2. Governing equations

We consider the unsteady laminar free convection flow of an incompressible fluid with mass transfer at the stagnation point of a three-dimensional body when the wall temperature varies arbitrarily with time. It is assumed that the free stream temperature is constant and the viscous dissipation is negligible at the stagnation point. All fluid properties are assumed to be constant except for the density changes which give the buoyancy terms in the momentum equations. We choose a locally orthogonal set of coordinates $Oxyz$ at the body surface in such a way that the origin coincides with the lowest stagnation point, x and y denote the (orthogonal) coordinates in the wall surface and z denotes the coordinate which is perpendicular to the wall. The appropriate boundary layer equations governing the foregoing flow can be expressed as [4, 7]

$$
\begin{aligned}
&u_x+v_y+w_z=0\,,\\
&u_t+uu_x+vu_y+wu_z-g\beta x\,(T-T_\infty)/L=\nu u_{zz}\,,\\
&v_t+uv_x+vv_y+wv_z-g\beta cy\,(T-T_\infty)/L=\nu v_{zz}\,,\\
&T_t+uT_x+vT_y+wT_z=\mathrm{Pr}^{-1}\,\nu T_{zz}\,.
\end{aligned}
\tag{2.1}
$$

The appropriate initial and boundary conditions are

$$\begin{aligned}
&u(x,y,z,0)=u_i(x,y,z), \qquad v(x,y,z,0)=v_i(x,y,z),\\
&w(x,y,z,0)=w_i(x,y,z), \qquad T(x,y,z,0)=T_i(x,y,z),\\
&u(x,y,0,t)=v(x,y,0,t)=0, \qquad w(x,y,0,t)=(w)_w,\\
&T(x,y,0,t)=T_\infty+(T_{w0}-T_\infty)\,\varphi(t^*), \qquad T(x,y,\infty,t)=t_\infty,\\
&u(x,y,\infty,t)=v(x,y,\infty,t)=0.
\end{aligned} \tag{2.2}$$

The set of partial differential equations (2.1) containing four independent variables x, y, z, t can be reduced to semi-similar equations involving two independent variables η, t^* by using the following transformations:

$$\begin{aligned}
&\eta=\mathrm{Gr}^{1/4}L^{-1}z, \qquad \mathrm{Gr}=\beta g\,(T_{w0}-T_\infty)\,L^3/\nu^2,\\
&t^*=\nu L^{-2}\,\mathrm{Gr}^{1/2}\,t, \qquad u=\nu L^{-2}\,x\mathrm{Gr}^{1/2}\,F(\eta,t^*),\\
&v=\nu c L^{-2}\,y\,\mathrm{Gr}^{1/2}\,S(\eta,t^*), \qquad w=-\nu L^{-1}\,\mathrm{Gr}^{1/4}\,(\mathrm{f}+\mathrm{cs}),\\
&(T-T_\infty)/(T_{w0}-T_\infty)=G(\eta,t^*), \qquad (T_w-T_\infty)=\varphi(t^*)\,(T_{w0}-T_\infty).
\end{aligned} \tag{2.3}$$

Consequently, Eq. $(2.1)_1$ is identically satisfied and Eqs. $(2.1)_2$ to $(2.1)_4$ can be expressed as

$$\begin{aligned}
&F_{\eta\eta}+(f+cs)\,F_\eta-F^2-F_{t^*}+G=0,\\
&S_{\eta\eta}+(f+cs)\,S_\eta-cS^2-S_{t^*}+G=0,\\
&\mathrm{Pr}^{-1}\,G_{\eta\eta}+(f+cs)\,G_\eta-G_{t^*}=0.
\end{aligned} \tag{2.4}$$

The relevant boundary conditions are

$$\begin{aligned}
&F=S=0, \qquad G=\varphi(t^*) \quad \text{for} \quad \eta=0,\\
&F=S=G=0 \quad \text{as} \quad \eta\to\infty, \quad \text{for} \quad t^*\geqslant 0,
\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
&f=\int_0^\eta F\,d\eta+f_w, \qquad s=\int_0^\eta S\,d\eta, \qquad s_w=0,\\
&f_w=-(w)_w\,L/(\nu\,\mathrm{Gr}^{1/4}).
\end{aligned} \tag{2.6}$$

If the normal velocity at the wall $(w)_w$ is a constant, then f_w will be a constant. The parameter $f_w \gtrless 0$, according to whether there is suction or injection. Here $\varphi(t^*)$ is an arbitrary function of t^* indicating the unsteadiness in the wall temperature. It has been assumed here that the flow is initially steady and changes to unsteady for $t^*>0$. Hence the initial conditions for F, S and G at $t^*=0$ are given by the steady flow equations obtained by putting

$$\varphi(t^*)=1, \qquad F_{t^*}=S_{t^*}=G_{t^*}=0 \tag{2.7}$$

in Eq. (2.4). These steady state equations are essentially the same as those of POOTS [4], except for some minor differences due to slightly different transformations. However, they are identical for $c=0$ (two-dimensional case) and $c=1$ (axi-symmetric case). Also, they are identical with those of BANKS [7] if we replace cs by $\bar{s}$.

The skin friction and heat transfer coefficients can be expressed in the form

$$
\begin{aligned}
C_f &= \mu\, (\partial u/\partial z)_w/[\rho \nu^2\, x/L^3] = \mathrm{Gr}^{3/4}\, (F_\eta)_w\,, \\
\bar{C}_f &= \mu\, (\partial v/\partial z)_w/[\rho \nu^2\, y/L^3] = \mathrm{Gr}^{3/4}\, c\, (S_\eta)_w\,, \qquad (2.8)\\
\mathrm{Nu} &= L\, (\partial T/\partial z)_w/(T_{w0} - T_\infty) = \mathrm{Gr}^{1/4}\, (G_\eta)_w\,.
\end{aligned}
$$

It may be remarked that even though the stagnation point boundary layer solutions are valid in a small region in the neighbourhood of the stagnation point of a three-dimensional body, they represent several physical flows of engineering significance. The stagnation point solutions serve as a starting solution for the solution over the entire body. The governing equations for the stagnation point flow can be reduced to a set of partial differential equations involving two independent variables η, t^*. On the other hand, the solution over the entire three-dimensional body would involve four independent variables, thus rendering the problem more complex.

3. RESULTS AND DISCUSSION

The set of partial differential equations (2.4) under the initial conditions obtained by using the relations (2.7) in Eqs. (2.4) and the boundary conditions (2.5) has been solved numerically using an implicit finite difference scheme. Here the method is not presented as it is described in great detail in [10]. The step size $\Delta\eta$ and Δt^* and the edge of the boundary layer represented by η_∞ have been optimized and the results presented here are independent of the step size and the edge of the boundary layer up to at least third decimal place. Here the computations have been carried out with $\Delta\eta=0.01$, $\Delta t^*=0.05$ and $4\leqslant\eta_\infty\leqslant 8$. Since most bodies of practical interest lie between the cylinder ($c=0$) and the sphere ($c=1$), the computation has been carried out for $0\leqslant c\leqslant 1$. Also, the variation of wall temperature with time has been taken to be of the form $\varphi\,(t^*)=1-\varepsilon t^*$.

It may be noted that the present method is capable of handling the parabolic type of partial differential equations involving any number of independent variables which govern forced, free and mixed convection flows over any arbitrary body.

In order to test the accuracy of the present method, we compared our skin friction and heat transfer results for the steady-state case ($t^*=0$) with those of Poots [4] and Banks [7] and found them to be in excellent agreement (see Fig. 1).

The effects of time t^*, the Prandtl number and Prandtl mass transfer f_w on the skin friction and heat transfer parameters $(F_\eta)_w$, $(S_\eta)_w$, $-(G_\eta)_w$ for $\varphi\,(t^*)=1+\varepsilon t^*$ ($\varepsilon>0$) are shown in Figs. 2–4 and Table 1. The skin friction parameters $(F_\eta)_w$, $(S_\eta)_w$ decrease as time t^* increases because wall temperature decreases with t^*. Consequently, the fluid energy decreases with t^* (boundary layer thickness grows with t^*) and this causes reduction in the skin friction (Figs. 2–3). However, the heat transfer parameter $-(G_\eta)_w$ significantly increases with time because the difference between

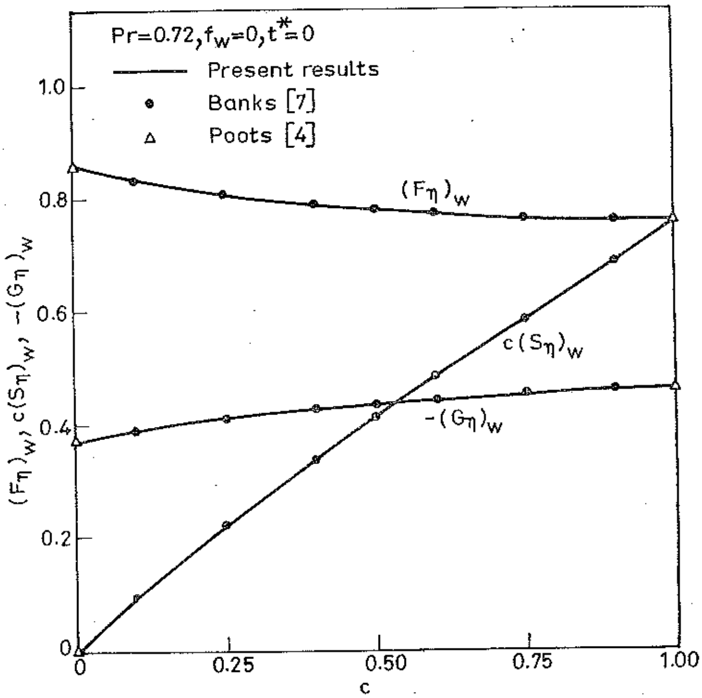

FIG. 1. Comparison of skin friction and heat transfer results.

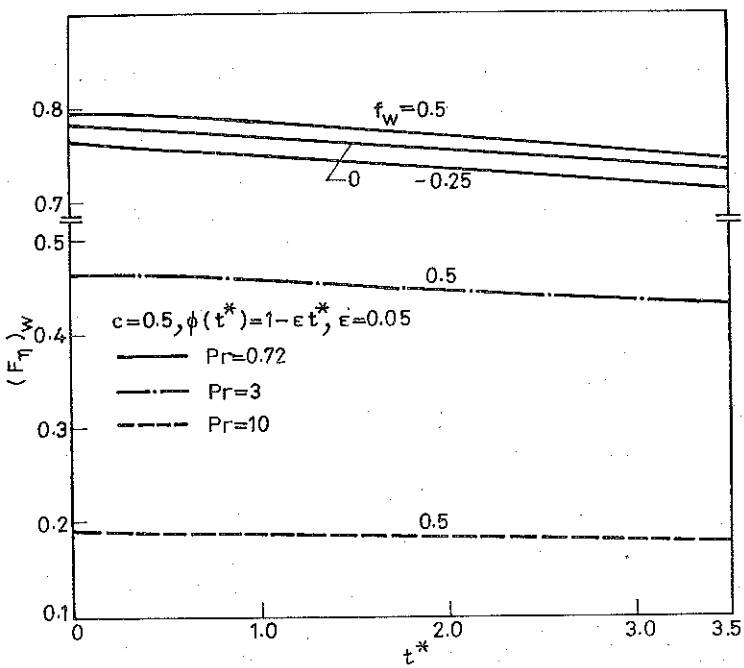

FIG. 2. Effect of the Prandtl number and mass transfer on the skin friction parameter in the *x*-direction.

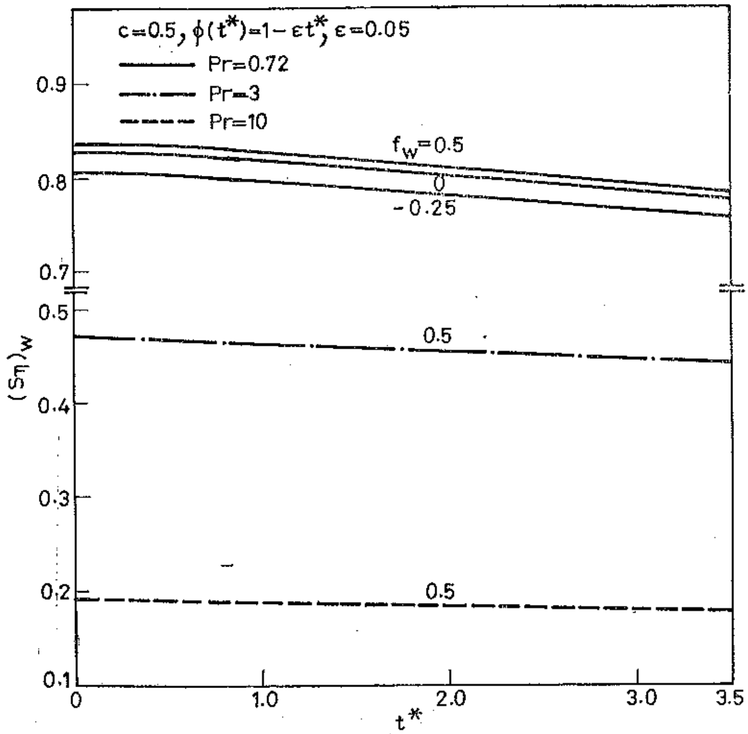

FIG. 3. Effect of the Prandtl number and mass transfer on the skin friction parameter in the y-direction.

the wall temperature and the free stream temperature increases with time, which causes more transfer of heat (Fig. 4).

Figures 2–4 also show the effect of the Prandtl number Pr on the skin friction and heat transfer parameters $(F_\eta)_w$ and $(S_\eta)_w$, $-(G_\eta)_w$. The skin friction parameters $(F_\eta)_w$ and $(S_\eta)_w$ decrease as Pr increases, but the heat transfer parameter $-(G_\eta)_w$ increases. The reason for this behaviour is that the velocity boundary layer thickness increases with Pr (higher Pr implies more viscous fluid); this causes deceleration in the fluid and thereby $(F_\eta)_w$ and $(S_\eta)_w$ are reduced. On the other hand, the thermal boundary layer thickness decreases as Pr increases. Consequently, heat transfer $-(G_\eta)_w$ increases with Pr. Also, we find that the heat transfer parameter $-(G_\eta)_w \gtreqless 0$ as $t^* \lesseqgtr t_1^*$ (for example, $t_1^* \approx 1.3$ for Pr=10); this implies that there is a heat transfer from the wall to the fluid when $t^* < t_1^*$ and from the fluid to the wall when $t^* > t_1^*$ and there is no heat transfer when $t^* = t_1^*$ (Fig. 4). This behaviour is due to the reduction of wall temperature with time, the wall temperature at $t^*=0$ being maintained at a higher temperature than the free stream temperature (i.e. $T_{w0} > T_\infty$). When the wall temperature is taken as increasing with time $\varphi(t^*) = 1 + \varepsilon t^*$ $(\varepsilon > 0)$ with initial wall temperature higher than the free stream temperature (i.e. $T_{w0} > T_\infty$), there is a transfer of heat from the wall to the fluid for all t^* (not shown in the figure). Also, both skin friction parameters increase with time because fluid energy increases with time and this makes the boundary layer thickness decrease.

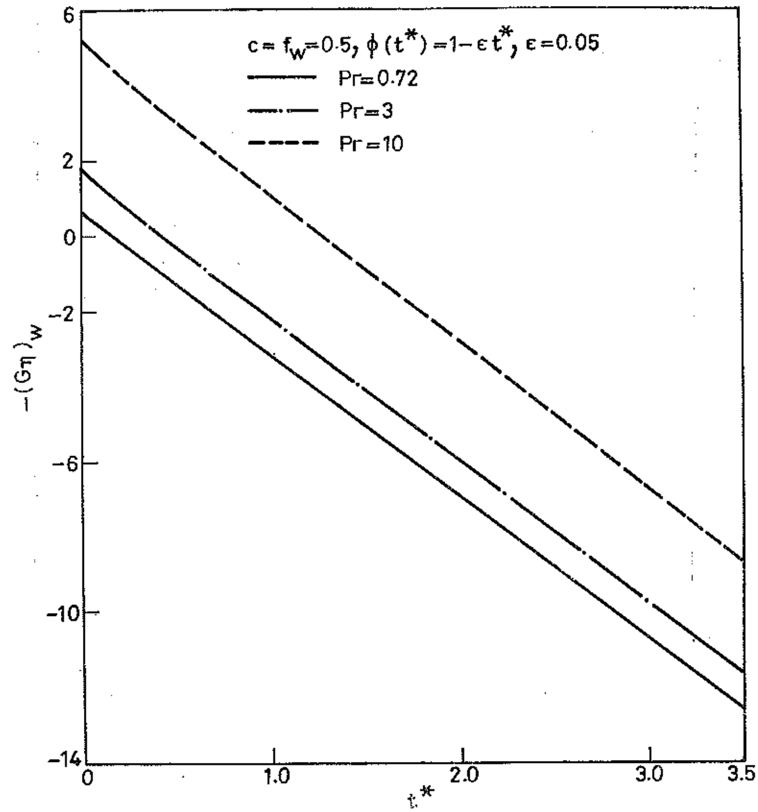

FIG. 4. Effect of the Prandtl number on the heat transfer.

Figure 2–4 and Table 1 also display the effect of mass transfer (f_w) on the skin friction and heat transfer parameters $(F_\eta)_w$, $(S_\eta)_w$, $-(G_\eta)_w$. As expected, the suction ($f_w>0$) increases both the skin friction and heat transfer and the effect of injection ($f_w<0$) is just the reverse. This is due to the fact that suction reduces both velocity and thermal boundary layer thicknesses which enhance both the velocity and temperature gradients. We also find that the heat transfer is weakly dependent on mass transfer for large t^* (Table 1).

Table 1. Effect of mass transfer parameter f_W on heat transfer parameter $-(G_\eta)_W$ for $c=0.5$, Pr=0.72 and $\varphi(t^*)=1-\varepsilon t^*$ ($\varepsilon=0.05$).

f_W/t^*	0	1	2	3	4
−0.25	0.3440	−3.4503	−7.2207	−10.9882	−14.7543
0	0.4323	−3.3642	−7.1369	−10.9067	−14.6749
0.5	0.6426	−3.1591	−6.9370	−10.7118	−14.4852

The effect of c, the parameter characterizing the nature of the stagnation point, on the skin friction and heat transfer is presented in Fig. 5 and Table 2. It is observed that in contrast to the forced flow [11], both the skin friction and heat transfer

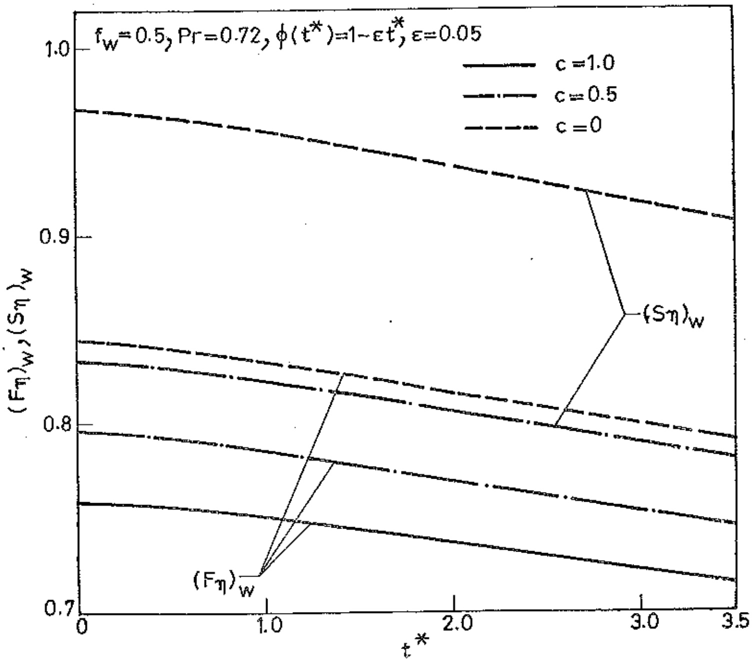

FIG. 5. Effect of the parameter c on the skin friction parameters.

parameters decrease for $t^* \geqq 0$ as c increases from 0 to 1, except the heat transfer parameter $-(G_\eta)_w$ for $t^*=0$ (steady case) which increases with c. A similar effect has been observed by POOTS [4] and BANKS [7] for the steady state case.

Table 2. Effect of c on the heat-transfer parameter $-(G_\eta)_W$ for $\mathrm{Pr}=0.72$, $f_W=0.5$ and $\varphi(t^*)=1-\varepsilon t^*$ ($\varepsilon=0.05$).

c/t^*	0	1	2	3	4
1.0	0.6754	−3.1294	−6.9088	−10.6840	−14.4523
0.5	0.6426	−3.1591	−6.9370	−10.7118	−14.4852
0	0.6002	−3.2012	−6.9779	−10.7514	−14.5234

The velocity profiles in the x-direction (F) and the temperature profiles (G) have been shown in Fig. 6. Since the velocity profiles in the y-direction (S) is similar to those of F profiles, they are not presented here. The maximum value of F (or S) decreases as Pr or t^* increases because it (Pr or t^*) increases the boundary layer thickness. This results in deceleration of the fluid and the velocity gradients are reduced. On the other hand, the temperature profiles (G) become steeper as Pr or t^* increases because of the reduction in the thermal boundary layer which in turn increases the temperature gradient. Since in the figure we have presented results only for the case when the wall temperature decreases with time t^*, the effect produced by it on the temperature profiles (G) is similar to that of Pr.

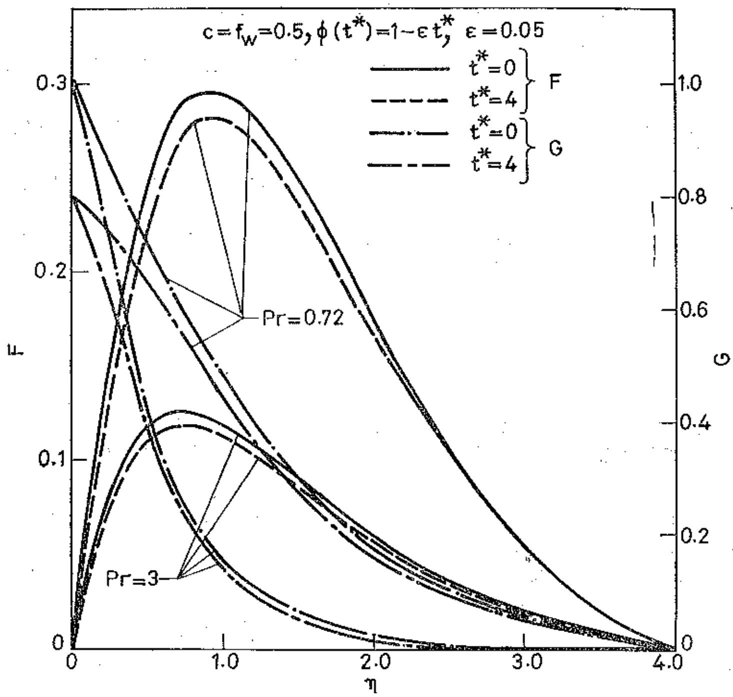

FIG. 6. Velocity and temperature profiles.

4. CONCLUSIONS

The heat transfer is found to be strongly dependent on time whereas the skin friction is weakly dependent on it. The heat transfer is enhanced by the increase in the Prandtl number, but the skin friction is reduced. The mass transfer affects both the heat transfer and skin friction, but its effect on the heat transfer is very weak for large times.

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Streszczenie

NIEUSTALONY SWOBODNY PRZEPŁYW KONWEKCYJNY WARSTWY PRZYŚCIENNEJ W PUNKCIE STAGNACJI TRÓJWYMIAROWEGO CIAŁA

Przeanalizowano nieustalony swobodny przepływ konwekcyjny w punkcie stagnacji trójwymiarowego ciała w pobliżu warstwy przyściennej przy założeniu dowolnej zmienności temperatury ścianki w czasie. Semi-podobne równania rządzące przepływem rozwiązano numerycznie za pomocą uwikłanego schematu różnic skończonych. Stwierdzono, że przepływ ciepła zależy silnie od czasu w przeciwieństwie do tarcia powierzchniowego. Wzrost liczby Prandtla prowadzi do zwiększenia przepływu ciepła lecz zmniejsza tarcie powierzchniowe. Przepływ masy wpływa istotnie zarówno na przewodnictwo jak i tarcie, choć dla dużych czasów wpływ ten na przewodnictwo ciepła jest nieznaczny.

Резюме

НЕУСТАНОВИВШЕЕСЯ СВОБОДНОЕ КОНВЕКЦИОННОЕ ТЕЧЕНИЕ ПОГРАНИЧНОГО СЛОЯ В ТРЕХМЕРНОЙ КРИТИЧЕСКОЙ ТОЧКЕ

Проанализировано неустановившееся свободное конвекционное течение в критической точке трехмерного тела, в приближении пограничного слоя, при предположении произвольной переменности температуры стенки во времени. Полумодельные уравнения, управляющие течением, решены численно при помощи неявной схемы конечных разностей. Констатировано, что перенос тепла сильно зависит от времени, в противовес к поверхностному трению. Рост числа Прандтля приводит к увеличению переноса тепла, но уменьшает поверхностное трение.-Перенос массы влияет существенно так на проводность, как и на трение, хотя для больших времен это влияние на теплопроводность незначительно.

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