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# ON THE CONVECTIVE MOTION OF CONDUCTING FLUID IN MAGNETIC FIELD

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The problem of convection in a conducting, viscous and incompressible liquid subject to a magnetic field is considered. The uniqueness theorem is proved and the structure of the fluctuation spectrum is determined. It is proved that the action of the magnetic field increases the stability of motion of the non-uniformly heated, conducting, viscous and incompressible liquid. This conclusion complies with the experimental and numerical results concerning the behaviour of a horizontal layer of liquid.

## 1. Formulation of the problem; fundamental equations

In the Boussinesq approximation, the linear set of equations describing the convective motion of a nonuniformly heated, conducting, viscous and incompressible liquid contained in a cavity and subject to magnetic field, has the following form [1]

$$\frac{\partial v}{\partial t} = -\nabla(p+MaH)+\Delta v+RTk_3+M(a\nabla)H, \tag{1.1}$$

$$P\frac{\partial T}{\partial t} = \Delta T+(vk_3), \tag{1.2}$$

$$P_m\frac{\partial H}{\partial t} = \Delta H+M(a\nabla)v, \tag{1.3}$$

$$\operatorname{div} v = 0, \tag{1.4}$$

$$\operatorname{div} H = 0 \quad \text{in} \quad \Omega. \tag{1.5}$$

Here $v, p, T$ denote the respective dimensionless fluctuations of velocity, pressure and temperature of the liquid, $H$ is the dimensionless magnetic field intensities, $R = (g\beta AL^4)/(\nu\chi)$ — Rayleigh number, $P = \nu/\chi$ — Prandtl number, $M = \frac{h_0 L}{c}\left(\frac{\sigma}{\varrho\nu}\right)^{1/2}$ — Hartmann number, $P_m = (4\pi\sigma\nu)/c^2$ — magnetic Prandtl number, $L$ — characteristic dimension of the cavity, $g$ — acceleration of gravity, $\nu, \chi, \beta$ — coefficients of kinematic viscosity, temperature conduction and thermal expansion, respectively, $-Ak_3$ — the equilibrium temperature gradient, $k_3$ — a vertical unit vector directed upwards, $H_0 = h_0\, a$ — field intensity in the state of mechanical equilibrium, $a$ — a unit vector parallel to the external field, $c$ — light speed, $\varrho, \nu$ — density and electric conductivity of the liquid, $\Omega$ — region filled by the liquid.

The velocity, temperature and field fluctuactions are assumed to vanish at the boundary of the cavity, what yields the following boundary conditions at the boundary $S$:

$$v = 0, \quad T = 0, \quad H = 0 \quad \text{on} \quad S. \tag{1.6}$$

The problem (1.1)–(1.6) is considered under the initial conditions

$$v|_{t=0} = v(0), \quad T|_{t=0} = T(0), \quad H|_{t=0} = H(0). \tag{1.7}$$

## 2. Operator formulation. Existence theorem

Let us denote by $L_2(\Omega)$ the Hilbert space of the vector functions square summable in $\Omega$, and by $L_{2,0}(\Omega)$ its subspace consisting of solenoidal vectors with vanishing normal components at $S$. Let us introduce the space $W^1_{2,0}(\Omega)$ which is obtained as the complement of the set of infinitely differentiable, finite in $\Omega$ solenoidal vectors in the metric space corresponding to the scalar product

$$(u, v) = \int_\Omega \nabla u\, \nabla v\, d\Omega + \int_S uv dS.$$

Denote by $H_2(\Omega)$ the Hilbert space consisting of all scalar functions which are square-summable in $\Omega$, and by $H^1_2(\Omega)$ — the S. L. Sobolev space with the norm

$$\|T\|^2_{H^1_{2,0}(\Omega)} = \int_\Omega |\text{grad } T|^2\, d\Omega + \int_S |T|^2\, dS.$$

Let $H^1_{2,0}(\Omega)$ be the subspace of $H^1_2(\Omega)$ consisting of functions vanishing at $S$. Let $\Pi$ be the orthogonal projector from $L_2(\Omega)$ onto $L_{2,0}(\Omega)$. In papers [2, 3] it was shown that the operator $\Pi\Delta$ in $W_{2,0}(\Omega)$ may be extended to a self-adjoint, positive definite operator $A_1$. It is known [4] that operator $\Delta$ may be extended, according to Friedrichs, to the self-adjoint, positive-definite operator $A_2$.

Equations (1.1) and (1.3) are now projected onto the space $L_{2,0}(\Omega)$. From the equations (1.4) and (1.5) and the boundary conditions (1.6) it follows that

$$\frac{dv}{dt} = -A_1\, v + RB_{12}\, T + MB_{13}\, H, \tag{2.1}$$

$$P_m \frac{dH}{dt} = -A_1 H + MB_{13} v. \tag{2.2}$$

The equations (1.2) with the boundary condition (1.6) assumes in the space $H_2(\Omega)$ the form

$$P \frac{dT}{dt} = -A_2 T + B_{21} v. \tag{2.3}$$

Here

$$B_{12} T \equiv \Pi (Tk_3), \qquad B_{21} v \equiv (vk_3), \qquad B_{13} u \equiv \Pi (a\nabla) u.$$

The set of Eqs. (2.1)–(2.3) may be considered as a single ordinary differential equation in the Hilbert space

$$L_{2,0}(\Omega) \times H_2(\Omega) \times L_{2,0}(\Omega)$$

and, namely, it is reduced to the problem

$$\frac{dx}{dt} = -\mathscr{A}x - \mathscr{B}x, \qquad x|_{t=0} = x(0), \tag{2.4}$$

where

$$x = \begin{Vmatrix} v \\ T \\ H \end{Vmatrix}, \qquad \mathscr{A} = \begin{Vmatrix} A_1 & 0 & 0 \\ 0 & P^{-1} A_2 & 0 \\ 0 & 0 & P_m^{-1} A_1 \end{Vmatrix}, \tag{2.5}$$

$$\mathscr{B} = \begin{Vmatrix} 0 & -RB_{12} & -MB_{13} \\ -P^{-1} B_{21} & 0 & 0 \\ -P_m^{-1} B_{13} & 0 & 0 \end{Vmatrix}.$$

THEOREM 1. *Equation (2.4) is an abstract parabolic equation; the corresponding Cauchy problem is uniformly correct, and the corresponding semigroup is analytic in the sector containing the positive semiaxis.*

Proof. It was mentioned before that the operators $A_1$, $A_2$ are self-adjoint, positive definite in $L_{2,0}(\Omega)$ and $H_2(\Omega)$, respectively; $P$ and $P_m$ are positive parameters. Consequently, operator $\mathscr{A}$ is also self-adjoint and positive definite in $L_{2,0}(\Omega) \times H_2(\Omega) \times L_{2,0}(\Omega)$.

From Eq. (2.5) it follows that

$$\|\mathscr{B}x\|^2 \leqslant R^2 \|B_{12} T\|^2 + M^2 \|B_{13} H\|^2 + P^{-2} \|B_{21} v\|^2 + P_m^{-2} M^2 \|B_{13} v\|^2. \tag{2.6}$$

Operators $B_{12}, B_{21}$ are bounded, and operators $A_1, A_2$ — unbounded. The following estimates are true

$$\begin{aligned} \|B_{12}\, T\|^2 &\leqslant \alpha_1 \|T\|^2 \leqslant \alpha_2 \|A_2\, T\|\, \|T\|, \\ \|B_{21}\, v\|^2 &\leqslant \alpha_3 \|v\|^2 \leqslant \alpha_4 \|A_1\, v\|\, \|v\|. \end{aligned} \tag{2.7}$$

Moreover, due to the condition (1.6) we obtain

$$\|B_{13}\, H\|^2 = \int_\Omega (a\nabla H)(a\nabla H)\, d\Omega \leqslant |a|^2 \int_\Omega |\nabla H|^2\, d\Omega = |a|^2 \int_\Omega |\Delta H H|\, d\Omega$$

whence

$$\|B_{13}\, H\|^2 \leqslant \alpha_5 \|A_1\, H\|_{L_{2,0}(\Omega)}. \tag{2.8}$$

Similarly,

$$\|B_{13}\, v\|^2 \leqslant \alpha_6 \|A_1\, v\|_{L_{2,0}(\Omega)} \|v\|_{L_{2,0}(\Omega)}. \tag{2.9}$$

Using Eqs. (2.7)–(2.9), we obtain from Eq. (2.6)

$$\begin{gathered} \|\mathscr{B}x\|^2 \leqslant \alpha_7 \{\|A_1\, v\|\, \|v\| + \|A_2\, T\|\, \|T\| + \|A_1\, H\|\, \|H\|\} \leqslant \alpha_7 \|\mathscr{A}x\|\, \|x\|, \\ \|\mathscr{B}x\| \leqslant \alpha_7^{1/2} \|\mathscr{A}x\|^{1/2} \|x\|^{1/2}. \end{gathered}$$

Here

$$\alpha_7 = \max\{R^2 \alpha_2, M^2 \alpha_5, P^{-2} \alpha_4 + P_m^{-2} M^2 \alpha_6\}.$$

The latter inequality proves that operator $\mathscr{B}$ is subject to $\mathscr{A}$, so that, in view of the results of [5] (theorem 7.2, p. 183), the theorem is proved.

## 3. NORMAL FLUCTUATIONS

Let us now consider the normal fluctuations problem and find the particular solutions which are exponentially time-dependent,

$$(u, T, H) = \exp(-\lambda t)(u_1, T_1, H_1)$$

Here $u_1, T_1, H_1$ are functions of the coordinates only. From Eqs. (2.1)–(2.3) we obtain for these functions the relations

$$\lambda v_1 = A_1\, v_1 - RB_{12}\, T_1 - MB_{13}\, H_1, \tag{3.1}$$

$$\lambda T_1 = P^{-1} A_2\, T_1 - P^{-1} B_{21}\, v_1, \tag{3.2}$$

$$\lambda H_1 = P_m^{-1} A_1\, H_1 - P_m^{-1} MB_{13}\, v. \tag{3.3}$$

Let us consider the set of Eqs. (3.1)–(3.3) as a single equation in the space $L_{2,0}(\Omega)\times H_2(\Omega)\times L_{2,0}(\Omega)$ and namely

$$\lambda x_1 = \mathscr{A}x_1 + \mathscr{B}x_1. \tag{3.4}$$

THEOREM 2. *The whole spectrum of the problem* (3.4) *consists of normal eigenvalues. Independently of their number* $\varepsilon, \varepsilon > 0$ *all of them lie (except, probably, a finite number) within the angles*

$$-\varepsilon < \arg\lambda < \varepsilon, \qquad \pi-\varepsilon < \arg\lambda < \pi+\varepsilon.$$

*The system of eigenvectors of the problem* (3.4) *is complete in the space* $W_{2,0}^1(\Omega)\times H_{2,0}^1(\Omega)\times W_{2,0}^1(\Omega)$.

Proof. It was shown in papers [2–4] that the operators $A_1, A_2$ are self-adjoint, positive-definite and possess compact inverse operators, what means that their spectra are discrete.

Consequently, operator $\mathscr{A}$ is self-adjoint and positive definite with a discrete spectrum and possesses the compact inverse operator which acts from $L_{2,0}(\Omega)\times H_2(\Omega)\times L_{2,0}(\Omega)$ to $W_{2,0}^1(\Omega)\times H_{2,0}^1(\Omega)\times W_{2,0}^1(\Omega)$.

Owing to the results published in [6], operator $\mathscr{A}^{-1}$ is of finite order

$$s_n(\mathscr{A}^{-1}) \leqslant cn^{-1/2}.$$

Operator $\mathscr{B}\mathscr{A}^{-1}$ is bounded, what follows from the chain of continuous transformations

$$L_{2,0}(\Omega)\times H_2(\Omega)\times L_{2,0}(\Omega) \xrightarrow{\mathscr{A}^{-1}} W_{2,0}^2(\Omega)\times H_{2,0}^2(\Omega)\times W_{2,0}^2(\Omega) \to \\ \xrightarrow{\mathscr{B}} W_{2,0}^1(\Omega)\times H_{2,0}^1(\Omega)\times W_{2,0}^1(\Omega).$$

As a result, operator $\mathscr{A}^{-1}\mathscr{B}\mathscr{A}^{-1}$ is compact in $W_{2,0}^1(\Omega)\times H_{2,0}^1(\Omega)\times W_{2,0}^1(\Omega)$ and is of a finite order:

$$s_n(\mathscr{A}^{-1}\mathscr{B}\mathscr{A}^{-1}) \leqslant \|\mathscr{B}\mathscr{A}^{-1}\|\, s_n(\mathscr{A}^{-1}) \leqslant c_1 n^{-1/2}.$$

Thus all the conditions of Theorem 10.1 of the book [7] are satisfied for the operators $\mathscr{A}+\mathscr{B}$, what proves our theorem.

THEOREM 3. *If the liquid is heated from above, that is if the Rayleigh number R is negative, the spectrum of the problem will lie in the right-hand halfplane.*

Proof. From the set of Eqs. (3.1)–(3.3) it follows that

$$\begin{aligned}
&(\lambda^*-\lambda)\,\|T_1\| = P^{-1}\,[(B_{21}v_1, T_1)-(B_{21}v_1, T)^*],\\
&(\lambda^*-\lambda)\,\|v_1\|^2 = R\,[(B_{12}T_1, v_1)-(B_{12}T_1, v_1)^*]+\\
&\qquad\qquad +M\,[(B_{13}H_1, v_1)-(B_{13}H_1, v_1)^*],\\
&(\lambda^*-\lambda)\,\|H_1\|^2 = MP_m^{-1}\,[(B_{13}v_1, H_1)-(B_{13}v_1, H_1)^*],
\end{aligned} \tag{3.5}$$

$$(3.6)\quad \begin{aligned} &(\lambda^*+\lambda)\,\|v_1\|^2 = 2\,\|A_1^{1/2}v_1\|^2 - R\,[(B_{12}\,T_1, v_1)^* + (B_{12}\,T_1, v_1)] - \\ &\qquad\qquad - M\,[(B_{13}\,H_1, v_1)^* + (B_{13}\,H_1, v_1)], \\ &(\lambda^*+\lambda)\,\|T_1\|^2 = 2P^{-1}\,\|A_2^{1/2}\,T_2\|^2 - P^{-1}\,[(B_{21}\,v_1, T_1)^* + (B_{21}\,v_1, T_1)], \\ &(\lambda^*+\lambda)\,\|H_1\|^2 = 2P_m^{-1}\,\|A_1^{1/2}\,H_1\|^2 - MP_m^{-1}\,[(B_{13}\,v_1, H_1)^* + \\ &\qquad\qquad + (B_{13}\,v_1, H_1)]. \end{aligned}$$

Let us observe that

$$(3.7)\quad (B_{12}\,T_1, v_1) = \int_\Omega (k_3\,T_1)\,v_1^*\,d\Omega = \int_\Omega (k_3\,v_1)^*\,T_1\,d\Omega = \\ = \int_\Omega ((k_3\,v), T_1^*)^*\,d\Omega = (B_{21}\,v_1, T)^*.$$

The boundary conditions (1.6) yield the result

$$(3.8)\quad (B_{13}\,H_1, v_1) = \int_\Omega (a\nabla)\,H_1\,v_1^*\,d\Omega = -\int_\Omega ((a\nabla)\,v_1\,H_1^*)^*\,d\Omega = \\ = -(B_{13}\,v_1, H_1)^*.$$

Let the Rayleigh number $R$ be negative; then from Eqs. (3.5)–(3.8) it follows that

$$(3.9)\quad \operatorname{Jm}\lambda = \frac{-|R|\,\{\operatorname{Jm}(B_{21}\,v_1, T_1) - \operatorname{Jm}(B_{12}\,T_1, v_1)\} + 2M\operatorname{Jm}(B_{13}\,H_1, v_1)}{\|v_1\|^2 + P\,|R|\,\|T_1\|^2 + P_m\,\|H_1\|^2},$$

$$(3.10)\quad \operatorname{Re}\lambda = \frac{\|A_1^{1/2}\,v_1\|^2 + \|A_2\,T_1\|^2\,|R| + \|A_1^{1/2}\,H_1\|^2}{\|v_1\|^2 + P\,|R|\,\|T_1\|^2 + P_m\,\|H_1\|^2}.$$

Relations (3.9), (3.10) indicate that the spectrum of the problem lies in the right-hand halfplane.

THEOREM 4. *If the fluid is heated from below, that is if the Rayleigh number $R$ is positive, the spectrum of the problem will lie in the right- and left-hand halfplanes. If the Rayleigh number satisfies the condition*

$$R < \gamma_1\,\gamma_2 + c,$$

*where*

$$\gamma_1 = \lambda_{\min}(A_1), \qquad \gamma_2 = \lambda_{\min}(A_2), \qquad c = \frac{\|A_1^{1/2}\,H\|^2}{4\gamma_1\,\gamma_2\,\|v_1\|\,\|T_1\|\,R^{1/2}},$$

*then the spectrum will lie in the right-hand halfplane only.*

Proof. Let $R$ be positive; then it follows from Eqs. (3.5)–(3.8) that

$$(3.11)\quad \operatorname{Jm}\lambda = \frac{-2M\operatorname{Jm}(B_{13}\,H_1, v_1)}{\|v_1\|^2 + RP\,\|T_1\|^2 + P_m\,\|H_1\|^2},$$

$$(3.12)\quad \operatorname{Re}\lambda = \frac{\|A_1^{1/2}\,v_1\|^2 + \|A_2\,T_1\|^2\,R + \|A_1^{1/2}\,H_1\| - 2RRe(B_{12}\,T_1, v_1)}{\|v_1\|^2 + PR\,\|T_1\|^2 + P_m\,\|H_1\|^2}.$$

Relations (3.11), (3.12) show that with $R > 0$ the spectrum lies in both the halfplanes.

Operators $A_1, A_2$ are self-adjoint and positive definite and, hence, the estimates hold true

$$\|A_1^{1/2} v_1\|^2 \geqslant \gamma_1 \|v_1\|^2, \tag{3.13}$$

$$\|A_2^{1/2} T_1\|^2 \geqslant \gamma_2 \|T_1\|^2, \tag{3.14}$$

where

$$\gamma_1 = \lambda_{\min}(A_1), \gamma_2 = \lambda_{\min}(A_2).$$

It is easily seen that

$$|\mathrm{Re}\,(B_{12} T_1, v_1)| \leqslant |(B_{12} T_1, v_1)| \leqslant \|T_1\| \, \|v_1\|. \tag{3.15}$$

Equations (3.13)–(3.15) make it possible to estimate the numerator of the expression (3.12)

$$\|A_1^{1/2} v_1\|^2 + \|A_2^{1/2} T_1\|^2 R + \|A_1^{1/2} H_1\|^2 - 2R\,\mathrm{Re}\,(B_{12} T_1, v_1) \geqslant$$
$$\geqslant \gamma_1 \|v_1\|^2 + \gamma_2 R \, \|T_1\|^2 + \|A_1^{1/2} H_1\|^2 - \frac{2\sqrt{R}}{\gamma_1^{1/2} \gamma_2^{1/2}} \|T_1\| \, \gamma_2^{1/2} R^{1/2} \|v_1\| \, \gamma_1^{1/2}.$$

It follows that

$$R < \gamma_1 \gamma_2 \left(1 + \frac{\|A_1^{1/2} H_1\|^2}{2\gamma_1^{1/2} \gamma_2^{1/2} \|v_1\| \, \|T_1\| \sqrt{R}}\right), \tag{3.16}$$

then $\mathrm{Re}\,\lambda \geqslant 0$, what means that the spectrum lies in the left-hand halfspace, and the theorem is proved.

If the magnetic field is absent, conditions (3.16) will be reduced to $R < \gamma_1 \gamma_2$, what secures the stability of motion, as shown in [8].

The results of Theorem 3.4 indicate that heating from above leads to vibrational fluctuations in the liquid and all the vibrations are damped, contrary to the case of absence of the magnetic field; heating from below produces vibrational fluctuations due to the magnetic field. Undamped vibrations are produced in such a case due to the action of the operators $B_{12}$ and $B_{21}$, independently of the magnetic field. Action of the magnetic field increases the Rayleigh number, what means that the motion of the fluid becomes more stable. These results coincide with the experimental and numerical data given in [1].

## REFERENCES

1. G. Z. GERSHIN, E. M. MUKHOVICKIĬ, *Convective stability of incompressible fluid* [in Russian], Nauka, p. 329, 1972.

2. S. G. KREĬN, *On the functional properties of operators in the vector analysis and hydrodynamics* [in Russian] Dokl. AN SSR, **93**, 6, 1953.
3. S. G. KREĬN, *Differential equations in Banach spaces and their applications to hydromechanics* [in Russian], Uspekhi Mat. Nauk, **12**, 1, 1957.
4. S. G. MIKHLIN, *Problem of minimum of a quadratic functional*, [in Russian], Gostekhizdat, 1952.
5. S. G. KREĬN, *Linear differential equations in a Banach space* [in Russian], Nauka 1967.
6. V. I. PARASKA, *On the asymptotics of eigenvalues and singular points of linear operators increasing the smoothness* [in Russian], Mat. Sb., **68**, 4, 1965.
7. I. C. GOKHBERG, M. G. KREĬN, *Introduction to the theory of linear operators in a Hilbert space* [in Russian], Nauka, 1965.
8. A. G. ZARUBIN, NGO ZUĬ KAN, *Convection in the liquid filling a cavity of a solid body in motion* [in Russian], Prikl. Mat. i Mekh., **44**, 6, 1980.

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