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ON INTERNALLY STATICALLY AND KINEMATICALLY DETERMINATE PROBLEMS IN THE LINEAR THEORY OF SHELLS

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Modified conditions of static and kinematic determinateness for the interior domain of an arbitrary elastic shell are found on the basis of estimates of the error of shell theory solutions in an energetic norm. The conditions are illustrated by an example and confronted with their earlier counterparts formulated in terms of the residual error in shell theory equations.

1. Introduction

Approximate particular solutions in shell theory can often be found by reducing the problem to a sequence of statically and kinematically determinate problems. Then the equilibrium and compatibility equations are no longer simultaneous and may be solved successively. The approximate solution thus obtained is known to be correct under two conditions [1]: (i) the stress distribution over the thickness is nearly uniform, or, conversely, it is close to that of inextensional bending and (ii) the residual error produced in the equilibrium equations is small compared to the load. In the context of (i) the question arises whether (and when) problems other than the two extreme ones of the membrane and inextensional bending theories can be reduced to a sequence of statically and kinematically determinate problems. A second question, related to (ii), is that concerning the actual (not residual) error of the approximate solution (two-dimensional) compared with the exact solution, for real bodies the latter being three-dimensional.

In this paper both questions are answered by the hypersphere method suggested by Prager and Synge [2] and then used by Koiter [3] and Danielson [4] for establishing bounds on the error of shell theory solutions. We shall make special use of Danielson's results who has shown that, given an exact shell theory solution in two dimensions, one may construct an approximate solution in three dimensions, which has an error of relative order ε when compared to the exact three-dimensional solution, ε depending on the shell geometry and the shell theory solution. In constrast to this we start from not an exact but an approximate solution in two dimensions which is then used to construct a three-dimensional solution having a relative error δ, with δ generalizing the parameter ε. Furthermore, by requiring δ to be small relative to unity, a set of modified conditions of static and kinematic determinateness is obtained, which proves to be less restrictive than the classical conditions (i) and (ii). It is shown, in particular, that some non-membrane and noninextensional bending problems can also be reduced to a sequence of statically and kinematically determinate problems.

In opposition to [3, 4] our analysis is based on the introduction of two wavelengths describing variations of the membrane and bending strains. This makes the conditions of static and kinematic determinateness less restrictive then they would be with one single wavelength.

The general considerations are illustrated by an example concerning a circular cylinder under normal load.

2. Classical conditions of static and kinematic determinateness

A successive integration of the membrane equilibrium equations and the membrane strain-displacement relations is often used to determine particular solutions in shell theory [1]. For our purposes it is convenient to summarize this method without explicit reference to the displacements.

The full linear equilibrium and compatibility equations of shell theory can be written in the following vector form:

$$\mathbf{E}(\mathbf{n}, \mathbf{m}) = \mathbf{q}, \qquad \mathbf{G}(\mathbf{n}, \mathbf{m}) = \mathbf{0}, \tag{2.1}$$

where $\mathbf{E}$ and $\mathbf{G}$ are vector-valued linear differential operators, $\mathbf{n}$ and $\mathbf{m}$ are membrane forces and moments, $\mathbf{q}$ is a given load. Equations (2.1) may be integrated successively as follows:

$$\begin{array}{lll} \text{step 1:} & \text{integrate } \mathbf{G}(\mathbf{0}, \mathbf{m}) = \mathbf{0} & \text{for} \quad \mathbf{m}, \\ \text{step 2:} & \text{given } \mathbf{m} \text{ integrate } \mathbf{E}(\mathbf{n}, \mathbf{m}) = \mathbf{q} & \text{for} \quad \mathbf{n}, \\ \text{step 3:} & \text{given } \mathbf{n} \text{ integrate } \mathbf{G}(\mathbf{n}, \hat{\mathbf{m}}) = \mathbf{0} & \text{for} \quad \hat{\mathbf{m}}. \end{array} \tag{2.2}$$

The final approximate solution is $(\mathbf{n}, \mathbf{m}+\hat{\mathbf{m}})$ and it is known to be correct under two conditions [1]:

$$\begin{gathered} h\mathbf{n} \gg \mathbf{m}+\hat{\mathbf{m}} \quad \text{or} \quad h\mathbf{n} \ll \mathbf{m}+\hat{\mathbf{m}}, \\ \mathbf{E}(\mathbf{0}, \hat{\mathbf{m}}) \ll \mathbf{q}, \end{gathered} \tag{2.3}$$

where the vector and tensor inequalities are meant to hold for the absolute maximum values of the components, and h denotes the thickness of the shell. The condition $(2.3)_1$ requires the deformation to be either nearly membrane or that of nearly inextensional bending, whereas the relation $(2.3)_2$ demands the error produced in the equilibrium equations by the moments $\hat{\mathbf{m}}$ to be small compared to the load. It is to be noted that the problems in steps 1 and 3 are kinematically determinate, as they consist in integrating three compatibility equations for three unknown moments with given membrane forces. In step 2 we have a statically determinate problem where three equilibrium equations must be solved for three membrane forces with given moments. Thus the problem of determining particular solutions to generally coupled equilibrium and compatibility equations reduces to a sequence of statically and kinematically determinate problems, the conditions (2.3) being the classical conditions of static and kinematic determinateness.

3. Modified conditions of static and kinematic determinateness

Although widely used, the classical conditions of static and kinematic determinateness are not fully satisfactory. The restriction $(2.3)_1$ on the type of deformation is intuitive but has not been proved to be necessary. Likewise, the condition $(2.3)_2$ ensures only that the residual error is small, the actual error being unknown. To answer these questions we resort to results due to Danielson [4], adapted for our purposes, first, by introducing two wavelengths of the deformation pattern and, second, by replacing the exact shell theory solution of [4] by the approximate solution obtained in Sect. 2.

Equation (3.3) of [4] immediately implies the following fundamental inequality:

$$\|\tilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma}\|/\|\tilde{\boldsymbol{\sigma}}\| \leqslant \delta, \tag{3.1}$$

where

$$\delta = \|\tilde{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}\|/\|\tilde{\boldsymbol{\sigma}}\|. \tag{3.2}$$

This means that an exact, practically unknown, stress field $\boldsymbol{\sigma}$ in three dimensions may be approximated by two known stress fields: a statically admissible stress $\tilde{\boldsymbol{\sigma}}$ and a kinematically admissible stress $\hat{\boldsymbol{\sigma}}$. The relative error is δ and it is seen to depend only on $\tilde{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\sigma}}$. The energetic norm used is, when squared, equal to the complementary energy corresponding to a given stress. The above relations are valid under certain "regular" boundary conditions [3, 4] which require the prescribed stress to be equal to $\tilde{\boldsymbol{\sigma}}$ and the prescribed displacements to produce, through the stress-displacement relations, the stress field $\hat{\boldsymbol{\sigma}}$. Dealing only with the interior problem of shell theory, we may freely assume the boundary conditions to be "regular".

Our main task is to construct from the approximate shell theory solution given in Sect. 2 the three-dimensional fields $\tilde{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\sigma}}$. These fields can be easily obtained from those of [4]:

$$\tilde{\sigma}_{\alpha\beta} = \hat{\sigma}_{\alpha\beta} = \frac{n_{\alpha\beta}}{h} - \frac{12zm_{\alpha\beta}}{h^3} + O\left(\frac{n}{R}, \frac{hn}{L_N^2}, \frac{m}{hR}, \frac{m}{L_M^2}\right), \tag{3.3}$$

(3.3) [cont.]
$$\tilde{\sigma}_{\alpha 3} = \hat{\sigma}_{\alpha 3} = -\frac{z}{h} n^{\beta}_{\alpha|\beta} + \frac{3}{2h}\left(\frac{4z^2}{h^2}-1\right) m^{\beta}_{\alpha|\beta} + O\left(\frac{n}{R}, \frac{hn}{L_N^2}, \frac{m}{hR}, \frac{m}{L_M^2}\right),$$
$$\tilde{\sigma}_{33} = \hat{\sigma}_{33} = O\left(\frac{n}{R}, \frac{hn}{L_N^2}, \frac{m}{hR}, \frac{m}{L_M^2}\right),$$

where $n_{\alpha\beta}$ and $m_{\alpha\beta}$ are membrane forces and moments whose absolute maximum values are denoted by n and m, h is the thickness of the shell, z denotes the distance from the midsurface, R is the least principal radius of curvature, a vertical stroke marks surface covariant differentiation. L_N and L_M are the characteristic wavelengths of $n_{\alpha\beta}$ and $m_{\alpha\beta}$ and are defined to be the largest numbers such that

(3.4)
$$|n_{\alpha\beta|\varkappa}| \leq n/L_N, \quad |n_{\alpha\beta|\varkappa\lambda}| \leq n/L_N^2,$$
$$|m_{\alpha\beta|\varkappa}| \leq m/L_M, \quad |m_{\alpha\beta|\varkappa\lambda}| \leq m/L_M^2.$$

The estimates in the relations (3.3) and (3.4) only make sense for the midsurface coordinates having the dimension of length, which is also assumed to hold in what follows.

In [4] only one single deformation wavelength L was defined: in view of the relations (3.4) it is verified to be the minimum of L_N and L_M. The distributions (3.3) are actually those of [4] with the error terms estimated here more precisely by means of the two wavelengths.

The main feature of $\tilde{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\sigma}}$ is that these stresses are statically and kinematically admissible in three dimensions. In [4] this fact was proved for $\tilde{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\sigma}}$ constructed from the exact shell theory solution ($n_{\alpha\beta}$, $m_{\alpha\beta}$) this solution being obviously both statically and kinematically admissible in two dimensions. In actual fact, however, any statically admissible two-dimensional solution, when introduced into the relations (3.3), produces a statically admissible stress field in three dimensions. Similarly, a kinematically admissible two-dimensional solution provides through the relations (3.3) a solution which is kinematically admissible in three dimensions. This applies, in particular, to the approximate solution from Sect. 2: according to the relation $(2.2)_2$, $(\mathbf{n}, \mathbf{m})$ is a statically admissible solution, and, in view of the relations $(2.2)_1$ and $(2.2)_3$, $(\mathbf{n}, \mathbf{m}+\hat{\mathbf{m}})$ is a kinematically admissible solution. Consequently, $\tilde{\boldsymbol{\sigma}}(\mathbf{n}, \mathbf{m})$ and $\hat{\boldsymbol{\sigma}}(\mathbf{n}, \mathbf{m}+\hat{\mathbf{m}})$ are statically and kinematically admissible stress fields in three dimensions. They may easily be calculated from the relations (3.3) to give

(3.5)
$$\tilde{\boldsymbol{\sigma}}(\mathbf{n}, \mathbf{m}) - \hat{\boldsymbol{\sigma}}(\mathbf{n}, \mathbf{m}+\hat{\mathbf{m}}) = O\left(\frac{n}{R}, \frac{hn}{L_N^2}, \frac{m}{hR}, \frac{m}{L_M^2}, \frac{\hat{m}}{h^2}\right),$$
$$\tilde{\boldsymbol{\sigma}}(\mathbf{n}, \mathbf{m}) = O\left(\frac{n}{h}, \frac{m}{h^2}\right),$$

n, m and $\hat{m}$ now being the absolute maximum values of $\mathbf{n}$, $\mathbf{m}$ and $\hat{\mathbf{m}}$, whereas L_N and L_M are wavelengths of $\mathbf{n}$ and $\mathbf{m}$. These estimates are meant to be valid for all the components of the stress tensors involved. They are verified to be also valid for the energetic norms of the stresses, provided the terms on the right-hand side of Eq. (3.5) are multiplied by a certain constant to compensate for the difference in units between stress and its energetic norm. When the norms of Eq. (3.5) are introduced into Eq. (3.2), the constant cancels and we find

$$\delta = \max\left(\frac{n}{R}, \frac{hn}{L_N^2}, \frac{m}{hR}, \frac{m}{L_M^2}, \frac{\hat{m}}{k^2}\right) \Big/ \max\left(\frac{n}{h}, \frac{m}{h^2}\right). \tag{3.6}$$

Since δ is the relative error of the approximate solution from Sect. 2, it must be small compared to unity. This with Eq. (3.6) gives the modified conditions of internal static and kinematic determinateness

$$\frac{h}{R} \ll 1, \quad \frac{h^2}{L_N^2} \ll 1, \quad \frac{m}{hn}\frac{h^2}{L_M^2} \ll 1, \quad \frac{\hat{m}}{hn} \ll 1 \quad \text{for} \quad hn \geqslant m, \tag{3.7}$$

$$\frac{h}{R} \ll 1, \quad \frac{h^2}{L_M^2} \ll 1, \quad \frac{hn}{m}\frac{h^2}{L_N^2} \ll 1, \quad \frac{\hat{m}}{m} \ll 1 \quad \text{for} \quad hn < m. \tag{3.8}$$

Compared with the classical conditions, the modified conditions of static and kinematic determinateness are seen to be far less restrictive. At a glance the latter are greater in number than the former. However, the first three relations in the sets (3.7) and (3.8) are actually the requirements [5] imposed on any solution in shell theory if it is to be a valid approximation to the exact elasticity theory solution. Observe that neither the set (3.7) nor the set (3.8) demands the deformation to be a membrane or that of inextensional bending, in contrast to the classical condition $(2.3)_1$, unless L_M is as short as h in the case of the set (3.7) or L_N is as short as h in the case of the set (3.8). Also, of the two wavelengths only one is required to be large compared to the thickness; this would not be so for one single wavelength $L = \min(L_N, L_M)$, as introduced in [3, 4], since then the inequalities $(3.7)_{2,3}$ and $(3.8)_{2,3}$ would reduce to

$$h^2/L^2 \ll 1. \tag{3.9}$$

Finally, it should be stressed that the modified conditions of static and kinematic determinateness are concerned with the error of the solution, whereas the classical restriction $(2.3)_2$ is imposed on the error in the equations.

4. Example

As an illustration consider an infinitely long circular cylinder of a constant thickness subjected to a rotationally-symmetric normal load $q(x)$; the Cartesian coordinates x and y are assumed to be arc lengths along the generator and circumference, respectively.

The general formulae in absolute notation, Sect. 2, may now be specified in their physical components. The nonzero membrane forces and moments are

$$\mathbf{n} = \{n_x, n_y\}, \quad \mathbf{m} = \{m_x, m_y\}. \tag{4.1}$$

They are interrelated by the equilibrium equations

$$\{\mathbf{E}(\mathbf{n}, \mathbf{m}) = \mathbf{q}\} = \{dn_x/dx = 0, n_y/R - d^2 m_x/dx^2 = q\} \tag{4.2}$$

and compatibility conditions

$$\{\mathbf{G}(\mathbf{n}, \mathbf{m}) = \mathbf{0}\} = \{dm_y/dx = 0, d^2 n_y/dx^2 + 12m_x/(h^2 R) = 0\}, \tag{4.3}$$

from which the strains have been eliminated using the standard first-order constitutive equations in which the Poisson's ratio was assumed to be zero, for the sake of simplicity.

When applied to Eqs. (4.2) and (4.3), the approximate integration procedure (2.2) gives

$$\begin{aligned} &\text{step 1:} \quad \mathbf{m} = \{m_x = 0, m_y = 0\}, \\ &\text{step 2:} \quad \mathbf{n} = \{n_x = 0, n_y = qR\}, \\ &\text{step 3:} \quad \hat{\mathbf{m}} = \{\hat{m}_x = -(h^2 R^2/12)\, d^2 q/dx^2, \hat{m}_y = 0\}, \end{aligned} \tag{4.4}$$

where taking $\mathbf{m} = \mathbf{0}$ in the first step is the natural assumption adopted when searching for a nearly membrane solution [1].

We specify the load to have the simple form

$$q(x) = a + b \sin(x/l), \tag{4.5}$$

a and b being certain constant pressures and l being a certain length. For such a load we have

$$q = O[\max(a, b)], \; d^n q/dx^n = O(b/l^n). \tag{4.6}$$

From the relations (4.4)–(4.6) and (3.4) we can calculate the amplitudes and wavelengths of internal forces and moments to find

$$\begin{aligned} &n = O[\max(a, b) R], \quad m = 0, \quad \hat{m} = O(bh^2 R^2/l^2), \\ &L_N = O\left[\max\left(\sqrt{a/b}\, l, l\right)\right], \quad L_M = O(l). \end{aligned} \tag{4.7}$$

This implies that $hn > m$; consequently, the modified conditions of static and kinematic determinateness are the relations (3.7) and reduce, together with the relations (4.7), to

$$h/R \ll 1, \quad \hat{m}/(hn) = \min(1, b/a)\, hR/l^2 \ll 1, \tag{4.8}$$

where the remaining two conditions, resulting from the inequalities $(3.7)_{2,3}$, are readily verified to be fulfilled automatically when the relations (4.8) are true.

An interesting fact is that the conditions (4.8) permit the bending strain wavelength to be as short as the thickness. Indeed, for $L_M = h$ Eq. $(4.7)_5$ gives $l = h$, and the conditions (4.8) hold provided that

$$h/R \ll 1, \; b/a \ll h/R, \tag{4.9}$$

which are true for a thin shell subjected to a nearly constant load. Thus we have constructed a solution which satisfies the modified conditions of static and kinematic determinateness in terms of two deformation wavelengths and, at the same time, violates the conditions restricted to one wavelength, see the inequality (3.9). This proves the usefulness of introducing the two wavelengths.

The conditions (4.8), are, in fact, equivalent to the statement that the error of the solution (4.4) is

$$\delta = \max\,[h/R, \min\,(1, b/a)\, hR/l^2]. \tag{4.10}$$

Substituting the relations $(4.4)_{2,3}$ into Eqs. (4.2), the residual error α of the same solution is found to be of the relative order

$$\alpha = O\,[h^2 R^2 (d^4 q/dx^4)/q]. \tag{4.11}$$

In view of the relation (4.6), this gives

$$\alpha = O\,[\min\,(1, b/a)\, h^2 R^2/l^4]. \tag{4.12}$$

Comparing the relation (4.10) with the other (4.12), it is immediately verified that the following relations are true simultaneously:

$$\sqrt{b/a} \ll 1, \quad hR/l^2 \gg 1, \quad \alpha = O\,(1), \quad \delta \ll 1. \tag{4.13}$$

This means that for loads consisting of two parts: a large constant part and a rapidly varying of small amplitude, the residual error may happen to be large, whereas the actual error is small. In such a case, the classical condition $(2.3)_2$ which restricts the residual error is seen to be unsatisfactory.

5. Concluding remarks

The modified conditions of static and kinematic determinateness obtained in this report require the energetic-norm error of the solution to a statically and kinematically determinate problem to be small compared to unity. When satisfied for one, say, classical shell theory, this demand is also satisfied for all the variants of shell theory which differ from classical theory by small terms of relative order δ in their constitutive equations. In particular, the conditions (3.7) and (3.8) are the same for both classical and the Sanders–Koiter shell theories which are of most practical interest.

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Streszczenie

O WEWNĘTRZNIE STATYCZNIE I KINEMATYCZNIE WYZNACZALNYCH ZAGADNIENIACH LINIOWEJ TEORII POWŁOK

Otrzymano zmodifikowane warunki statycznej i kinematycznej wyznaczalności wewnętrznego zagadnienia liniowej teorii powłok sprężystych — na podstawie oszacowania błędu rozwiązań teorii powłok w normie energetycznej. Przedstawione warunki zilustrowano na przykładzie oraz porównano z warunkami znanymi dotychczas, opartymi na oszacowaniu błędu residualnego w równaniach teorii powłok.

Резюме

О ВНУТРЕННЕ СТАТИЧЕСКИ И КИНЕМАТИЧЕСКИ ОПРЕДЕЛЯЕМЫХ ПРОБЛЕМАХ ЛИНЕЙНОЙ ТЕОРИИ ОБОЛОЧЕК

Получены модифицированные условия статической и кинематической определяемости внутренней проблемы линейной теории упругих оболочек на основе оценки погрешности решений теории оболочек в энергетической норме. Представленные условия иллюстрированы на примере, а также сравнены с условиями известными до сих пор, опираюшимися на оценке резидуальной погрешности в уравнениях теории оболочек.

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