ROZPRAWY INŻYNIERSKIE • *ENGINEERING TRANSACTIONS* • 33, 3, 235–253, 1985
Polska Akademia Nauk • Instytut Podstawowych Problemów Techniki

ON CERTAIN MODELS OF NON-LAMINAR FLOWS WITH TURBULENT DIFFUSION (*)

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A new model of turbulent flow is proposed in the paper; the model takes into account the situation when the mass transport proceeds along two mutually intersecting curvilinear cones. Laminar flow represents the particular case of the model when the cones are reduced to curves.

In the present paper we consider some models describing the fluid motion with intensive turbulent diffusion.

Visualisation of the flow with intensive turbulent diffusion shows that the mass transport takes place along curvilinear cones mutually penetrating each other. The natural question arises whether it is possible to build a fenomenological model of fluid motion in which the mass transport takes place not along curves, as in the laminar case, but along curvilinear cones mutually penetrating each other. Our aim is to propose some models of this kind. These models are of interest for the kinetics of chemical reactions [7], because they may take into account the self-mixing effects which are of great importance for the chemical reactions.

We assume that the fluid motion with intensive turbulent self-mixing in the time interval $< t^0, t >$, (t^0 — the beginning of the observation), results instantaneously at the time t and each point x, so that the mass transport takes place at the instantaneous velocities $v(t, x, \alpha)$, $\alpha \in A$, (A — a region in R^3). We assume throughout the paper that v is continuously differentiable. Roughly speaking, it means that mass m located at time t in a very small neighbourhood of x is decomposed into components $m(\alpha)$, $\alpha \in A \subset R^3$, and each of these mass components moves at the instantaneous velocity $v(t, x, \alpha)$. Hence, at each point x we have an infinite bundle of instantaneous velocities, and through each point $p \in R^3$ at time t^0 passes

(*) Paper partialy given at XVI Symposium on Advanced Problems and Methods in Fluid Mechanics, Spała, 4–10 September, 1983.

an infinite bundle of instantaneous trajectories

$$\mathcal{T}_\alpha(t^0, p): x = x(t, t^0, p, \alpha), \quad p = x(t^0, t^0, p, \alpha), \quad \frac{dx}{dt} = v(t, x, \alpha).$$

The mass located at time t and point $x \in \mathcal{T}_\alpha(t, x)$, moving with the instantaneous velocity $v(t, x, \alpha)$ (Fig. 1) may reach, at the time $t', t' > t$, (due to

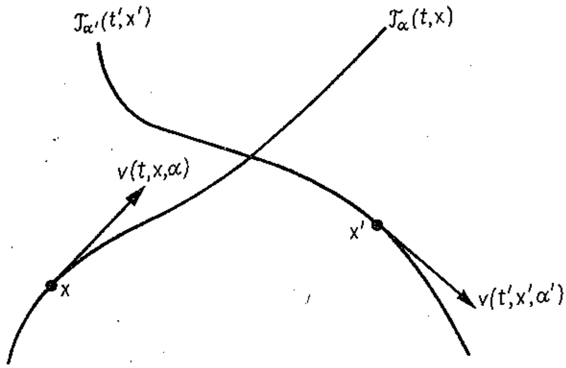

FIG. 1.

intensive self-mixing) the point $x' \in \mathcal{T}_{\alpha'}(t', x')$, and move with the instantaneous velocity $v(t', x', \alpha')$, $\alpha \neq \alpha'$. Hence, the instantaneous motion along the instantaneous trajectories does not give the complete description of the flow.

Besides the α — velocities $v(t, x, \alpha)$ we introduce here other physical α-quantities, that is the quantities depending on α, what enables us to formulate the basic conservation lows. This leads to a number of closed systems of *PDE*'s describing the corresponding models.

It is assumed throughout the paper that all functions considered are continuously differentiable.

Integrating the α-quantities over the suitable subsets of $R^3 \times A$ we obtain the mean values measured in the flow with intensive turbulent diffusion such as the mean pressure $\hat{p}(t, x)$, mean density $\hat{\varrho}(t, x)$ or the mean velocity $\hat{v}(t, x)$. If for $(t, x) \in G \subset R^4$ and $\alpha \in A$ we have: $v(t, x, \alpha) = v^*(t, x)$ and $\mu(t, x, \alpha) = 0$, where the α-quantity μ describes the mass transport caused by the turbulent diffusion, then the mean quantities $\hat{p}(t, x)$, $\hat{\varrho}(t, x)$, $\hat{v}(t, x)$ satisfy the laminar Navier–Stokes equations. Thus in our model we may consider the transition from turbulent to the laminar flow as well as the inverse transition.

There is a formal analogy between the proposed model and the classical theory of mixtures [1–5]. However, there exist also important differences:

1. In constrast to the finite number of components in the mixture theory, the set A describing in the present theory different trajectories is infinite.

2. Different components in the mixture theory are chemicaly or physicaly distinguishable, while in the present model of self-mixing the difference is only instantaneous and lies primarily in kinetics.

1. Kinetics

In the laminar model the x-densities are employed, that is the real physical quantities are obtained by integration of the corresponding expressions over some regions or surface in R^3. For example, there exists no mass at the time t at point x moving with the laminar velocity $v(t, x)$. However, for a small region $\omega_x \subset R^3$, $x \in \omega_x$, the mass contained at t in ω_x,

$$m(\omega_x, t) = \int_{\omega_x} \varrho(t, x)\, dx,$$

moves with the approximate velocity $v(t, x)$. Moreover, we have

$$v(t, x) = \lim_{|\omega_x| \to 0} \frac{\operatorname{imp}(\omega_x, t)}{m(\omega_x, t)},$$

where $|\omega_x|$ is the volume of ω_x, and $\operatorname{imp}(\omega_x, t)$ denotes the momentum of the mass portion $m(\omega_x, t)$,

$$\operatorname{imp}(\omega_x, t) = \int_{\omega_x} \varrho(t, x)\, v(t, x)\, dx.$$

In our model we use the x, α-densities, $x \in R^3$, $\alpha \in A$, which are constructed by means of different α-quantities. It means that the actually observed physical quantities are obtained by integration of the corresponding expressions over subsets of the Cartesian product $R^3 \times A$.

For example, actually no mass is moving at time t and point x with the instantaneous velocity $v(t, x, \alpha)$. If, however, $\omega_x \subset R^3$, $x \in \omega_x$, $a_\alpha \subset A$, $\alpha \in a$, where ω_x, a_α are small regions, then

$$m(a_\alpha, \omega_x, t) = \int_{a_\alpha} d\alpha \int_{\omega_x} \varrho(t, x, \alpha)\, dx,$$

where $\varrho(t, x, \alpha)$ is the α-density, represents the mass contained at time t in ω_x which is moving approximately with the instantaneous velocity $v(t, x, \alpha)$.

Moreover, we have

$$v(t, x, \alpha) = \lim_{|\omega_x|, |a_\alpha| \to 0} \frac{\operatorname{imp}(a_\alpha, \omega_x, t)}{m(a_\alpha, \omega_x, t)},$$

where $\operatorname{imp}(a_\alpha, \omega_x, t)$ denotes the momentum of the mass portion $m(a_\alpha, \omega_x, t)$,

$$\operatorname{imp}(a_\alpha, \omega_x, t) = \int_{a_\alpha} d\alpha \int_{\omega_x} \varrho(t, x, \alpha)\, v(t, x, \alpha)\, dx.$$

Thus we have obtained the momentum imp (a_α, ω_x, t) by integrating the x, α-densities ϱv over $\omega_x \times a_\alpha \subset R^3 \times A$.

If $\alpha, \beta \in A$, $\alpha \neq \beta$, and if $\alpha_x \in a_\alpha$, $\beta \in a_\beta$, a_α, $a_\beta \subset A$ are small subregions, then at the same time t the small region ω_x is filled up by two different mass portions $m(a_\alpha, \omega_x, t)$, $m(a_\beta, \omega_x, t)$ moving approximately with different instantaneous velocities $v(t, x, \alpha) \neq v(t, x, \beta)$.

Under the assumption of $v \in C^1$ for each $\alpha \in A$, there exists a corresponding α-trajectory $\mathcal{T}_\alpha(t^0, p,)$: $x = x(t, t^0, p, \alpha)$, i.e. the mapping

$$Y_{\alpha, t^0}(t): \omega \to \omega_{\alpha, t^0}(t),$$

where $\omega \subset R^3$ is an arbitrary region $Y_{\alpha,t^0}(t)\, p = x$, $x = x(t, t^0, p, \alpha)$ is a diffeomorfism if $v \neq 0$. The inverse mapping $Y^{-1}_{\alpha, t^0}(t)\, x = p$ is denoted by $p = p(t, t^0, x, \alpha)$

For regions $a \subset A$, $\omega \subset R^3$ we introduce the following notations (see Fig. 2)

$$\Omega_{a,t^0}(t) = \bigcup_{\alpha \in a} \omega_{\alpha, t^0}(t), \qquad \Omega_{t^0}(t) = \Omega_{A, t^0}(t).$$

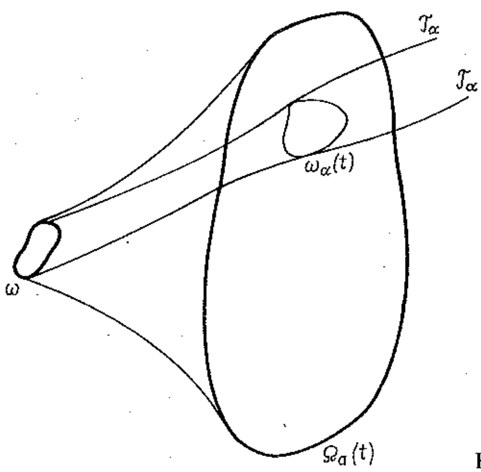

FIG. 2.

In what follows the index t^0 will be omitted if this does not lead to misunderstanding. The closure of $\Omega_a(t)$ is decomposed into the following two disjoint sets

$$\bar{\Omega}_a(t) = B\Omega_a(t) \bigcup I\Omega_a(t).$$

$I\Omega_a(t)$ will be called the inner set. $x \in I\Omega_a(t)$ if and only if all trajectories $\mathcal{T}_\alpha(t, x)$, $\alpha \in a$, passing x at time t, intersect the region ω at time $t^0 < t$. This means that the trajectories $\mathcal{T}_\alpha(t, x)$ may be parametrically represented in the following way:

$$x = x(\vartheta, t^0, p, \alpha), \quad t^0 \leqslant \vartheta \leqslant t, \quad p \in \omega$$

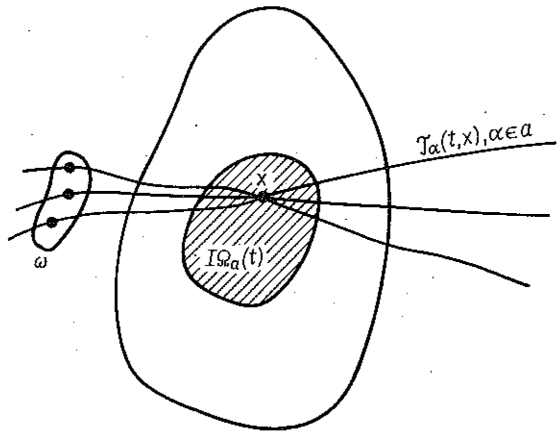

FIG. 3.

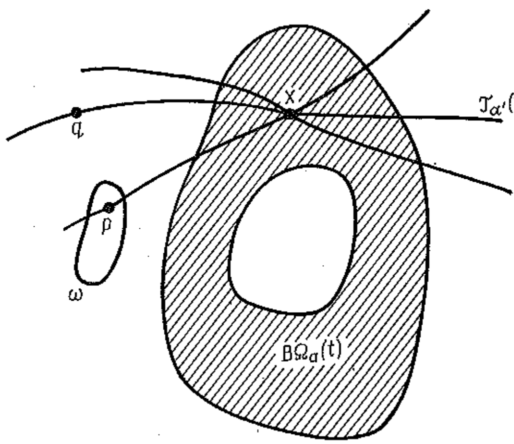

FIG. 4.

(see Fig. 3). $B\Omega_a(t)$ will be called the boundary set. $x \in B\Omega_a(t)$ if and only if there exists at least one point $\alpha' \in a$ such that the trajectory $\mathcal{T}_\alpha(t, x)$ passing x at time t has no common point with ω at time $t^0 < t$. Thus, $\mathcal{T}_\alpha(t, x)$ may be represented in the parametric form only $x = x(\vartheta, t^0, q, \alpha)$, $t^0 \leqslant \vartheta \leqslant t$, $q \notin \omega$ (see Fig. 4).

The sets $I\Omega_a(t)$, $B\Omega_a(t)$ may also be defind in the following, equivalent way

$$I\Omega_a(t) = \bigcap_{\alpha\in a} \omega_\alpha(t), \qquad B\Omega_a(t) = \bigcup_{\alpha\in a} \partial\omega_\alpha(t),$$

where $\partial\omega_\alpha(t)$ denotes the boundary of the region $\omega_\alpha(t)$. The proof of this fact may be found in [8].

In our model we distinguish two types of mass portions and the corresponding physical quantities. The first is the mass portion $m(a, \omega, t)$ which is defined by the following two conditions:

1. The mass portion $m(a, \omega, t)$ is contained at time t in ω.

2. The mass portion $m(a, \omega, t)$ being at time t at the point $x \in \omega$, moves with the instantaneous velocities $v(t, x, \alpha)$, $\alpha \in a$.

In the special case $m(\omega, t) = m(A, \omega, t)$ denotes the whole mass contained in ω at time t. We have

$$m(\omega, t) \geqslant m(a, \omega, t), \qquad a \subset A.$$

If we now introduce x, α-densities of different physical quantities, for instance: $\varrho(t, x, \alpha)$ — the x, α-mass density, $\varrho(t, x, \alpha)\, v(t, x, \alpha)$ — the x, α-momentum density, then the corresponding physical quantities for the mass portion $m(a, \omega, t)$ are introduced in the following way. The mass:

$$m(a, \omega, t) = \int_a d\alpha \int_\omega \varrho(t, x, \alpha)\, dx; \tag{1.1}$$

momentum of the mass $m(a, \omega, t)$:

$$\operatorname{imp}(a, \omega, t) = \int_a d\alpha \int_\omega \varrho(t, x, \alpha)\, v(t, x, \alpha)\, dx, \tag{1.2}$$

and so on.

From (1.1) it follows, that the measured mean density $\hat{\varrho}(t, x)$ is in our model given by the following integral

$$\hat{\varrho}(t, x) = \int_A \varrho(t, x, \alpha)\, d\alpha. \tag{1.3}$$

Moreover, using Eqs. (1.1) and (1.2), the measured mean velocity $\hat{v}(t, x)$ may be expressed by the following formula

$$\hat{v}(t, x) = \lim_{r \to 0} \frac{\operatorname{imp}\big(K_r(x), t\big)}{m\big(K_r(x), t\big)}, \tag{1.4}$$

where $K_r(x) = \{y : |y - x| < r\}$ and

$$\operatorname{imp}\big(K_r(x), t\big) = \int_A d\alpha \int_\omega \varrho(t, x, \alpha)\, v(t, x, \alpha)\, dx$$

is the momentum of the whole mass contained at time t in the ball $K_r(x)$.

Let us now formulate the general principle used throughout the paper in our model for expressing the physical quantities by the α-quantities and the corresponding, x, α-densities in the case of the mass portion $m(a, \omega, t)$.

A. In order to express the physical quantity connected with the mass $m(a, \omega, t)$ and the x, α-density $D(t, x, \alpha)$, we take one of the following integrals

$$\int_a d\alpha \int_\omega D(t, x, \alpha)\, dx \qquad \text{or} \qquad \int_a d\alpha \int_{\partial\omega} D(t, x, \alpha)\, dx.$$

B. If the quantity considered has a laminar analogy then, performing for each $\alpha \in a$ the corresponding laminar considerations for the flow along the α-trajectories, we write the proper x-density $L(t, x, \alpha)$ for ω or $\partial\omega$ and take one of the following integrals:

$$\int_a d\alpha \int_\omega L(t, x, \alpha)\, dx \quad \text{or} \quad \int_a d\alpha \int_{\partial\omega} L(t, x, \alpha)\, dx.$$

Introduce now the second type of mass portions, the moving mass portions $m^v(a, \omega, t)$. This mass portion is defined by the following two conditions.

1. The mass portion $m^v(a, \omega, t)$ is contained at time t in $\Omega_a(t)$.

2. The mass of the portion $m^v(a, \omega, t)$ being at time t at the point $x \in \Omega_a(t)$ moves only along the instantaneous trajectories $\mathcal{T}_\alpha$, $\alpha \in a$ which satisfy the condition

$$\mathcal{T}_\alpha(t, x) = \mathcal{T}_\alpha(t^0, p), \qquad p \in \omega.$$

The mass of the moving portion $m^v(a, \omega, t)$ moves only along such instantaneous trajectories $\mathcal{T}_\alpha$, $\alpha \in a$, which at time t^0 pass through the points $p \in \omega$.

The mass $m^v(a, \omega, t)$ does not represent the whole mass contained at time t in $\Omega_a(t)$. If $\varrho(t, x, \alpha) > 0$, then we have, in general,

$$m\big(\Omega_a(t), t\big) > m\big(a, \Omega_a(t), t\big) > m^v(a, \omega, t)$$

and for $a = A$:

$$m\big(\Omega(t), t\big) = m\big(A, \Omega(t), t\big) > m^v(A, \omega, t).$$

Velocities $v(t, x, \alpha)$ are only instantaneous and besides the instantaneous mass transport along the trajectories $\mathcal{T}_\alpha$ we have to consider the mass transport caused by the turbulent diffusion. Therefore it may happen that for $t' < t''$ the portion $m^v(a, \omega, t'')$ contains only a part or does not contain at all the mass of the portion $m^v(a, \omega, t')$.

The physical quantities connected with the moving mass portion $m^v(a, \omega, t)$ are expressed in our model by means of the x, α-densities in the following way. Mass:

$$m^v(a, \omega, t) = \int_a d\alpha \int_{\omega_\alpha(t)} \varrho(t, x, \alpha)\, dx. \tag{1.5}$$

Momentum of the mass $m^v(a, \omega, t)$:

$$\operatorname{imp}^v(a, \omega, t) = \int_a d\alpha \int_{\omega_\alpha(t)} \varrho(t, x, \alpha)\, v(t, x, \alpha)\, dx, \tag{1.6}$$

and so on. This is the natural way of expressing these quantities, because the equality

$$\omega = \omega_\alpha(t^0) = \Omega_{a, t^0}(t^0), \qquad \alpha \in a$$

implies that the following important conditions are fulfilled

$$m(a, \omega, t^0) = m^v(a, \omega, t^0), \qquad \operatorname{imp}(a, \omega, t^0) = \operatorname{imp}^v(a, \omega, t^0).$$

Let us now formulate the general principle used throughout the paper for expressing the physical quantities by the α-quantities and the corresponding x, α-densities in the case of the mass portion $m^v(a, \omega, t)$.

A^v. In order to express the physical quantity connected with the mass $m^v(a, \omega, t)$ and the x, α-density $D(t, x, \alpha)$, we take one of the following integrals

$$\int_a d\alpha \int_{\omega_\alpha(t)} D(t, x, \alpha)\, dx \quad \text{or} \quad \int_a d\alpha \int_{\partial\omega_\alpha(t)} D(t, x, \alpha)\, dx.$$

B^v. If the quantity considered has a laminar analogy then, performing for each $\alpha \in a$ the corresponding laminar considerations on the flow along the α-trajectories, the proper x-density $L(t, x, \alpha)$ will be written for $\omega_\alpha(t)$ or $\partial\omega_\alpha(t)$, and one of the following integrals will be taken:

$$\int_a d\alpha \int_{\omega_\alpha(t)} L(t, x, \alpha)\, dx \quad \text{or} \quad \int_a d\alpha \int_{\partial\omega_\alpha(t)} L(t, x, \alpha)\, dx.$$

Considering now a stationary flow with intensive turbulent diffusion, and let us perform at point $p \in R^3$ the visualization of the mass transport by introducing during the time interval $< t^0, t >$ a smoke trace at point p. Assuming that the flow may be described by means of our model in the stationary case, curvilinear cone $K(p)$ occupied at time t by the smoke should satisfy the following condition

$$K(p) \subset S(p) = \bigcup_{\alpha \in A} \mathscr{L}_\alpha(p),$$

where $\mathscr{L}_\alpha(p): x = x(t, \alpha)$, $p = x(t^0, \alpha)$, $dx/dt = v(x, \alpha)$, the α-stream line passing through the point p. Indeed, in the stationary case the whole mass located at the time t at point $x \in K(p)$ is transported with the instantaneous velocities $v(x, \alpha)$, $\alpha \in A$. Therefore the mass of the smoke may move only with these velocities. If we assume that

$$\inf |v(x, \alpha)| > \gamma > 0,$$

then there should exist a number $r > 0$ such that for the ball $B_r(p) = \{x: |x-p| < r\}$ we have

$$K(p) \cap B(p) = S(p) \cap B(p).$$

In this manner our model may be verified by visualization of the mass transport.

The instantaneous motion with velocity $v(x, \alpha)$ is only the instantaneous result of the motion with turbulent diffusion and does not describe the flow fully. Owing to the intensive turbulent diffusion, the set $\Omega_{a, t^0}(t)$ may not contain any mass which at time t^0 was in ω. Due to this we cannot determine in our model to what extent the set $K(p)$ coincides with $S(p)$.

2. The mass conservation law

In order to formulate the mass conservation law, let us introduce a new, real-valued x, α-density $\mu(t, x, \alpha)$ which describes the amount of mass added to the mass portions $m(a, \omega, t)$ and $m^v(a, \omega, t)$, due to the turbulent diffusion in a time unit. The physical meaning of this α-quantity without laminar analogue is given by the following formulation of the mass conservation law for the portion $m(a, \omega, t)$:

$$\frac{d}{dt}\int_a d\alpha \int_\omega \varrho(t, x, \alpha)\, dx = -\int_a d\alpha \int_{\partial\omega} \varrho \langle v, n\rangle\, dx + \int_a d\alpha \int_\omega \mu(t, x, \alpha)\, dx \tag{2.1}$$

and for the portion $m^v(a, \omega, t)$:

$$\frac{d}{dt}\int_a d\alpha \int_{\omega_\alpha(t)} \varrho(t, x, \alpha)\, dx = \int_a d\alpha \int_{\omega_\alpha(t)} \mu(t, x, \alpha)\, dx, \tag{2.2}$$

where the principles A, B and A^V, B^V from Sect. 1 are used. In this way we distinguish in the model two components of the mass transport.

1. The instantaneous with velocity $v(t, x, \alpha)$.
2. The turbulent diffusion described by the x, α-density $\mu(t, x, \alpha)$.

The integral

$$\int_A d\alpha \int_\omega \mu(t, x, \alpha)\, dx \tag{2.3}$$

describes the mass added to the mass $m(\omega, t)$ in unit time due to the turbulent diffusion. The portion $m(\omega, t)$ represents the whole mass located at time t in ω, hence Eq. (2.3) gives the amount of the whole mass transported in unit time through the closed surface $\partial\omega$ due to the turbulent diffusion. If the flow takes place inside a vessel V with solid walls ∂V, then the following condition will be satisfied

$$\int_A d\alpha \cdot \int_V \mu(t, x, \alpha)\, dx = 0.$$

If we now perform the differentiation d/dt in Eq. (2.1), change the integral over $\partial\omega$ into the integral over ω, and perform the localisation (a and ω are arbitrary), the following differential form of the mass conservation law for $m(a, \omega, t)$ is obtained:

$$\partial_t \varrho + \operatorname{div}(\varrho v) = \mu, \qquad \alpha \in A. \tag{2.4}$$

Using for the differentiation d/dt in (2.2) the formula

$$\frac{d}{dt}\int_a d\alpha \int_{\omega_\alpha(t)} f\, dx = \int_a d\alpha \int_{\omega_\alpha(t)} [\partial_t f + \operatorname{div}(fv)]\, dx,$$

we obtain, after localisation of (2.2), the equations (2.4) as the differential form of the mass conservation law for the moving portion $m^v(a, \omega, t)$.

Introduce now one more real-valued x, α-density, $M(t, x, \alpha)$, its physical meaning being given by the following formula

$$\int_a d\alpha \int_{\omega_{\alpha,t^0}(t)} M(t, x, \alpha)\, dx = \int_{t^0}^{t} \Big[\int_a d\alpha \int_{\omega_{\alpha,t^0}(\vartheta)} \mu(\vartheta, x, \alpha)\Big]\, d\vartheta. \tag{2.5}$$

In this way $M(t, x, \alpha)$ is the x, α-density of the mass amount which was added in the time interval $< t^0, t >$, (due to the turbulent diffusion) to the instantaneous motion along the trajectory $\mathcal{T}_\alpha(t^0, p(t, t^0, x, \alpha))$, where

$$p(t, t^0, x, \alpha) = Y_{\alpha,t^0}^{-1}(t)\, X.$$

One may say that M describes the history of mixing.

Differentiating (2.5) with respect to t and performing the localisation, as in the case of Eq. (2.2), we obtain the following differential equations

$$\partial_t M + \operatorname{div}(Mv) = \mu, \qquad \alpha \in A. \tag{2.6}$$

In the laminar case, if the velocity field $v(t, x)$ is given, the mass conservation law leads to the closed system: one equation and one unknown function $\varrho(t, x)$. In our case the mass conservation law leads to two families of equations (2.4), (2.6), and assuming that $\dot{v}(t, x, \alpha)$ is known, to three families of unknown functions ϱ, μ, M, $\alpha \in A$. In order to close this system we introduce the constitutive equation of mixing

$$\partial_t M = \mathfrak{M}(\mu, M, \varrho, v, \ldots).$$

It may be suggested here that a fluid with given chemical properties may, in different situations, follow different self-mixing states described by different mixing constitutive equations. The simplest possibility of the mixing constitutive equation is

$$\partial_t M = \mu. \tag{2.7}$$

For the physical explanation of this condition let us introduce a new α-quantity $\mathfrak{M}(t, p, \alpha)$ satisfying the following condition

$$\int_a d\alpha \int_\omega \mathfrak{M}(t, p, \alpha)\, dp = \int_{t^0}^{t} \Big[\int_a d\alpha \int_{\omega_\alpha(\vartheta)} \mu(t, x, \alpha)\, dx\Big]\, d\vartheta.$$

This means that the quantity $\mathfrak{M}$ describes the amount of mass, which is added, as a result of the turbulent diffusion, to the instantaneous motion along the trajectory $\mathcal{T}_\alpha(t^0, p)$ in the time interval $\langle t^0, t\rangle$. In this way $\mathfrak{M}$ describes the future of mixing.

Let us denote by $J(t, p, \alpha)$ the Jacobian of the mapping $Y_{\alpha,t^0}\, p = x$, $x = x(t, t^0, p, \alpha)$; then, in the stationary case the constitutive equation (2.7) may be justified by the following assumptions

$$\left|\frac{\partial_t J}{J^2}\right| \ll 1, \quad \left|\frac{\partial_p J}{J^2}\right| \ll 1, \quad \sigma = 1, 2, 3$$

and

$$\langle \operatorname{grad}_p \mathfrak{M}(t, p, \alpha), v(p, \alpha)\rangle = 0.$$

The last condition means that the direction of maximal change of $\mathfrak{M}$ (t9, p, α) is perpendicular to $v(p, \alpha)$. The proof and the general instationary case is considered in more detail in [8].

3. The momentum conservation law

For the moving mass portion $m^v(a, \omega, t)$ the momentum conservation law may be formulated in the following way

$$\frac{d}{dt}\int_a d\alpha \int_{\omega_\alpha(t)} \varrho v \, dx = F(t) + F_B(t) + F_I(t), \tag{3.1}$$

where

$$F(t) = \int_a d\alpha \int_{\omega_\alpha(t)} \varrho(t, x, \alpha) f(t, x) \, dx$$

denotes the external mass forces with intensity $f(t, x)$. F_B and F_I are the boundary and internal forces which appear as a result of interaction of the mass portion $m^v(a, \omega, t)$ and the rest of the flow.

The boundary forces are acting in the boundary set $B\Omega_a(t)$. We assume the existence of the function $\tau(t, x, \alpha, n)$.

$$\tau: R^4 \times A \times S_2 \to R^3, \quad S_2 = \{x : |x| = 1\}$$

such that

$$F_B(t) = \int_a d\alpha \int_{\partial\omega_\alpha(t)} \tau(t, x, \alpha, n) \, dx,$$

where in the integral over $\partial\omega_\alpha(t)$, n is the outer normal vector to $\partial\omega_\alpha(t)$. τ is the α-stress vector; for more details see [8].

The internal forces are due to the momentum transfer between different masses located at the same time and the same point, and moving with different velocities $v(t, x, \alpha)$, $v(t, x, \beta)$, $\alpha \neq \beta$. We assume that

$$F_I(t) = \int_a d\alpha \int_{\omega_\alpha(t)} i(t, x, \alpha) \, dx,$$

where $i: R^4 \times A \to R^3$ is a new unknown α-quantity. We suggest the following expression for $i(t, x, \alpha)$:

$$i(t, x, \alpha) = \int_A \varphi(\alpha, \beta) \frac{D[\varrho(t, x, \beta) v(t, x, \beta) - \varrho(t, x, \alpha) v(t, x, \alpha)]}{Dt} dx,$$

where

$$\varphi: A \times A \to R, \quad \varphi(\alpha, \beta) = \varphi(\beta, \alpha), \quad \varphi(\alpha, \alpha) = 0$$

and

$$\frac{Df(t, x, \beta)}{Dt} = \partial_t f(t, x, \beta) + \sum_{\sigma=1}^{3} v_\sigma(t, x, \beta) \partial_{x_\sigma} f(t, x, \beta),$$

$$\frac{Df(t, x, \alpha)}{Dt} = \partial_t f(t, x, \alpha) + \sum_{\sigma=1}^{3} v_\sigma(t, x, \alpha) \partial_{x_\sigma} f(t, x, \alpha)$$

are the β, α material derivatives. In different models we may choose different functions $\varphi(\alpha, \beta)$. For more details see [8].

For the α-stress $\tau(t, x, \alpha, n) = (\tau_1, \tau_2, \tau_3)$, the following equality holds

$$\tau_i(t, x, \alpha, n) = \sum_{j=1}^{3} T_{ij}(t, x, \alpha) n_j, \tag{3.2}$$

where $n = (n_1, n_2, n_3)$, $|n| = 1$. Moreover, $T_{ij} = T_{ji}$ if and only if the angular momentum is conserved. For the proof see [8]. Tensor T will be called the α-stress tensor of the boundary forces.

Now we are able to calculate the differential form of the momentum conservation law (3.1). Using (3.2) we may change in (3.1) the integrals over $\partial\omega_\alpha(t)$ into the integrals over $\omega_\alpha(t)$. Performing the differentiation with respect to t in Eq. (3.1) we find

$$\int_a d\alpha \int_{\omega_\alpha(t)} \{[\partial_t(\varrho v_j) + \operatorname{div}(\varrho v_j v)] - \varrho f_j - (\operatorname{div} T)_j - i_j\} dx, \quad j = 1, 2, 3;$$

after localisation and application of (2.4) we obtain the following differential system

$$\varrho\Big(\partial_t v + \sum_{\sigma=1}^{3} v_\sigma \partial_{x_\sigma} v - f\Big) = \operatorname{div} T + i - \mu v, \alpha \in A. \tag{3.3}$$

In order to obtain closed systems we must introduce a constitutive equation connecting the α-stress tensor with the kinematical and thermodynamical α-quantities and their derivatives. We shall use an analogy to the Navier–Stokes laminar constitutive equation, which may be written in the following form

$$T = \mathbf{1}(-p + \lambda \operatorname{div} v) + 2\gamma D(v),$$

where $\mathbf{1}$ the unit matrix, $\lambda=\text{const}$, $\gamma=\text{const}$ and

$$D(v)=\left(\frac{\partial_{x_i} v_j+\partial_{x_j} v_i}{2}\right), \quad i,j=1,2,3.$$

In our model we assume the following form of the constitutive equation

$$T=\mathbf{1}\,[-p+\psi(\mu)+\lambda_1 \operatorname{div} v+\lambda_2 \operatorname{div}(\operatorname{grad}_x \mu)]+ \\ +2\gamma_1 D(v)+2\gamma_2 D(\operatorname{grad}_x \mu), \tag{3.4}$$

where λ_1, λ_2, γ_1, γ_2 are constants, and $p(t,x,\alpha)$ is a new α-quantity, the pressure. For the α-pressure and other thermodynamical α-parameters such as the α-density $\varrho(t,x,\alpha)$, α-internal energy $U(t,x,\alpha)$ etc., we shall use the state equations of the equilibrium thermodynamics. The term $\psi(\mu)$, $\psi: R\to R$, in (3.4) enables us to take into account the situation in which the surface forces acting in the perpendicular direction to $\partial\omega_\alpha(t)$, depend not only on the thermodynamical parameter $p(t,x,\alpha)$ but also depend in an essential way on the turbulent diffusion described by the α-quantity μ. The term $2\gamma_1 D(v)$ introduces viscosity γ_1 similarly to the Navier–Stokes model. The vector $\operatorname{grad}_x \mu$ gives the direction of the maximum change of the turbulent diffusion. Hence $\operatorname{grad}_x \mu$ may be considered as the direction of an increased mass stream caused by the turbulent diffusion. That is why we have introduced additional viscosities λ_2, and γ_2, connected with this mass stream.

4. The energy conservation law

The energy conservation law for the mass portion $m^V(a,\omega,t)$ may be written in the following way:

$$\frac{d}{dt}\int\limits_a d\alpha \int\limits_{\omega_\alpha(t)} \varrho\left(U+\frac{|v|^2}{2}\right)dx=P(t)+P_B(t)+P_I(t)+Q(t), \tag{4.1}$$

where $U(t,x,\alpha)$ is the α-internal energy, $P(t)$, $P_B(t)$, $P_I(t)$ — intensities of the forces $F(t)$, $F_B(t)$, $F_I(t)$ acting on the mass portion $m^V(a,\omega,t)$, $Q(t)$ — the non-mechanical energy flux.

Applying the principles A^V, B^V formulated in Sect. 1, we may write

$$P(t)=\int\limits_a d\alpha \int\limits_{\omega_\alpha(t)} \langle \varrho f, v\rangle\, dx,$$

$$P_B(t)=\int\limits_a d\alpha \int\limits_{\partial\omega_\alpha(t)} \langle Tn, v\rangle\, dx=\int\limits_a d\alpha \int\limits_{\omega_\alpha(t)} \Big[\langle v, \operatorname{div} T\rangle+\sum_{i,j=1}^{3} T_{ij}\,\partial_{x_i} v_j\Big]\, dx,$$

$$P_I(t)=\int\limits_a d\alpha \int\limits_{\omega_\alpha(t)} \langle i, v\rangle\, dx.$$

The flux of non-mechanical energy is decomposed into the boundary and internal parts:

$$Q(t) = Q_B(t) + Q_I(t)$$

and introduce the corresponding two new α-quantities. The α-boundary flux of energy is $q(t, x, \alpha)$, $q: R^4 \times A \to R^3$ with the following physical meaning:

$$Q_B(t) = \int_\alpha d\alpha \int_{\partial\omega_\alpha(t)} \langle n, q\rangle\, dx = -\int_a d\alpha \int_{\omega_\alpha(t)} \operatorname{div} q\, dx;$$

The second new α-quantity is the x, α-mean of internal flux of energy $Q(t, x, \alpha)$, $Q: R^4 \times A \to R$ with the following physical meaning:

$$Q_I(t) = -\int_a d\alpha \int_{\omega_\alpha(t)} Q(t, x, \alpha)\, dx.$$

After differentiating with respect to t and localisation of (4.1) we obtain the following differential form of (4.1):

$$\left(U + \frac{1}{2}|v|^2\right)[\partial_t \varrho + \operatorname{div}(\varrho v)] + \partial_t U + \sum_{\sigma=1}^{3} v_\sigma \partial_{x_\sigma} U +$$
$$+ \langle v, \varrho(\partial_t v + \sum_{\sigma=1}^{3} v_\sigma \partial_{x_\sigma} v)\rangle = \langle v, \varrho f\rangle + \langle v, \operatorname{div} T\rangle +$$
$$+ \sum_{i,j=1}^{3} T_{ij}\partial_{x_i} v_j + \langle v, i\rangle - \operatorname{div} q - Q.$$

Applying the mass conservation law (2.4) and the equation of motion (3.3), we finally obtain the following set of energy equations:

$$\partial_t U + \sum_{\sigma=1}^{3} v_\sigma \partial_{x_\sigma} U = \sum_{i,j=1}^{3} T_{ij}\partial_{x_i} v_j - \mu U + \mu\frac{|v|^2}{2} - \operatorname{div} q - Q, \qquad \alpha \in A. \tag{4.2}$$

For example, in the case when $U(t, x, \alpha) = C\Theta(t, x, \alpha)$, where $C = \text{const}$ and $\Theta(t, x, \alpha)$ is the α-temperature, the α-quantities $q(t, x, \alpha)$ and $Q(t, x, \alpha)$ may be expressed in the following way:

$$q(t, x, \alpha) = D(\alpha)\,\Theta(t, x, \alpha),$$
$$Q(t, x, \alpha) = \int_A \vartheta(\alpha, \beta)[\Theta(t, x, \beta) - \Theta(t, x, \alpha)]\, dx,$$

where

$$\vartheta(\alpha, \alpha) = 0, \qquad \vartheta(\alpha, \beta) = \vartheta(\beta, \alpha).$$

In a special case we may put $D = \text{const}$, $\vartheta = \text{const}$.

5. Examples of closed systems

Let us now present some of the simplest examples of closed systems. For brevity we will consider only inviscid flows, and assume $\lambda_1 = \lambda_2 = \gamma_1 = \gamma_2 = 0$ in the constitutive equation (3.4).

Hence (3.4) takes the form $T = 1\left(-p+\psi(\mu)\right)$.

Incompressible flow. We put $\varrho(t, x, \alpha) = \varrho(\alpha)$, hence the measured mean density

$$\hat{\varrho}(t, x) = \int_A \varrho(\alpha)\, d\alpha = \text{const}.$$

The function $\varrho(\alpha)$ is chosen arbitrarily, for example $\varrho(\alpha) = \text{const}$. Taking into account (2.4), (2.6), (2.7), (3.3) we obtain the following closed system without the energy equation,

$$\tag{5.1} \begin{aligned} \varrho \operatorname{div} v &= \mu, \quad && \alpha \in A, \\ \partial_t M + \operatorname{div}(Mv) &= \mu, && \alpha \in A, \\ \partial_t M &= \mu, && \alpha \in A, \end{aligned}$$

$$\varrho\left(\partial_t v + \sum_{\sigma=1}^{3} v_\sigma \partial_{x_\sigma} v\right) = -\operatorname{grad}_x p + \operatorname{grad}_x \psi(\mu) - \mu v +$$

$$+ \int_A \varphi(\alpha, \beta) \frac{D\left[\varrho(t, x, \beta)\, v(t, x, \beta) - \varrho(t, x, \alpha)\, v(t, x, \alpha)\right]}{Dt}\, d\beta, \qquad \alpha \in A.$$

Here we have obtained six families of equations with six families of unknown functions

$$\mu, M, v_1, v_2, v_3, \qquad \alpha \in A.$$

Compressible flow with very intensive turbulent diffusion. In this model we assume that the turbulent diffusion is so strong that the influence of pressure may be neglected. Hence the constitutive equation (3.4) takes the form

$$T = \mathbf{1}\psi(\mu).$$

In this way by taking into account (2.4), 2.6) (2.7) and (3.3), we obtain the following closed system

$$\tag{5.2} \begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho v) &= \mu, \quad && \alpha \in A, \\ \partial_t M + \operatorname{div}(Mv) &= \mu, && \alpha \in A, \\ \partial_t M &= \mu, && \alpha \in A, \end{aligned}$$

$$\varrho\left(\partial_t v+\sum_{\sigma=1}^{3} v_\sigma\,\partial_{x_\sigma} v\right)=\operatorname{grad}\psi(\mu)-\mu v+$$
$$+\int_A \varphi(\alpha,\beta)\frac{D\left[\varrho(t,x,\beta)\,v(t,x,\beta)-\varrho(t,x,\alpha)\,v(t,x,\alpha)\right]}{Dt}\,d\beta,\qquad \alpha\in A.$$

of unknown functions

$$\mu, M, v_1, v_2, v_3,\qquad \alpha\in A.$$

Compressible not heat conducting flow. Taking into account the equations (2.4), (2.6), (2.7), (3.3) and the energy equation (3.7) we obtain for $T=\mathbf{1}\left(-p+\psi(\mu)\right)$ the following closed system

$$\partial_t\varrho+\operatorname{div}(\varrho v)=\mu,\qquad \alpha\in A,$$
$$\partial_t M+\operatorname{div}(Mv)=\mu,\qquad \alpha\in A,$$
(5.3)
$$\partial_t M=\mu,\qquad \alpha\in A,$$
$$\varrho\left(\partial_t v+\sum_{\sigma=1}^{3} v_\sigma\,\partial_{x_\sigma} v\right)=\operatorname{grad}p+\operatorname{grad}\psi(\mu)-\mu v+$$
$$+\int_A \varphi(\alpha,\beta)\frac{D\left[\varrho(t,x,\beta)\,v(t,x,\beta)-\varrho(t,x,\alpha)\,v(t,x,\alpha)\right]}{Dt}\,d\beta,\qquad \alpha\in A,$$
$$\partial_t U+\sum_{\sigma=1}^{3} v_\sigma\,\partial_{x_\sigma} U=\left(-p+\psi(\mu)\right)-\mu U+\mu\frac{|v|^2}{2},\qquad \alpha\in A,$$
$$p=f(\varrho, U),\qquad \alpha\in A,$$

where $p=f(\varrho, U)$ is the equilibrium termodynamical state equation. We have here eight families of equations and eight families of unknown functions

$$\mu, M, p, \varrho, U, v_1, v_2, v_3,\qquad \alpha\in A.$$

Stationary models. In the stationary case, in the systems (5.1), (5.2) and (5.3) let us disregard the time dependence and introduce additional changes in the second and third equations. In the stationary case the equation $\partial_t M(t,x,\alpha)=\mu(x,\alpha)$ is equivalent to the equality

$$M(t,x,\alpha)=(t-t^0)\,\mu(x,\alpha).$$

Therefore, the two equations containing M reduce to one equation

$$\operatorname{div}(\mu v)=0.$$

Thus, in the stationary case the systems (5.1), (5.2), (5.3) contain one family of equations less and also one family of unknown functions M, $\alpha\in A$ less.

The systems (5.1), (5.2), (5.3) are integro-differential equations, but by means of discretisation of the integrals over A, we may reduce them to partial ones (see [8]).

6. Transition to the laminar flow

Let us now show that in the model described one may consider the transition from the turbulent to laminar flow as well as the inverse transition. Consider the system

$$\partial_t \varrho + \operatorname{div}(\varrho v) = \mu, \qquad \alpha \in A, \tag{6.1}$$

$$\varrho\,(\partial_t v + \sum_{\sigma=1}^{3} v_\sigma \partial_{x_\sigma} v) = -\operatorname{grad} p + \operatorname{grad}\psi(\mu) + (\lambda_1+\gamma_1)\operatorname{div} v + \tag{6.2}$$

$$+(\lambda_2+\gamma_2)\operatorname{div}(\operatorname{grad}\mu) + \gamma_1 \Delta v + \gamma_2 \Delta(\operatorname{grad}\mu) - \mu v +$$

$$+\int_A \varphi(\alpha,\beta)\,\frac{D\,[\varrho(t,x,\beta)\,v(t,x,\beta) - \varrho(t,x,\alpha)\,v(t,x,\alpha)]}{Dt}\,d\beta, \qquad \alpha \in A,$$

$$\partial_t U + \sum_{\sigma=1}^{3} v_\sigma \partial_{x_\sigma} U = -p\operatorname{div} v + \psi(\mu)\operatorname{div} v + \lambda_1 (\operatorname{div})^2 + \tag{6.3}$$

$$+\lambda_2\,[\operatorname{div}(\operatorname{grad}\mu)]^2 + \gamma_1 \sum_{ij} (\partial_{x_i} v_j + \partial_{x_j} v_i)\,\partial_{x_i} v_j +$$

$$+2\gamma_2 \sum_{ij} (\partial_{x_i}\partial_{x_j}\mu)^2 - \mu U + \mu\frac{|v|^2}{2} - \operatorname{div} q - \int_A \vartheta(\alpha,\beta)\,[\Theta(t,x,\beta) -$$

$$-\Theta(t,x,\alpha)]\,d\beta, \qquad \alpha \in A,$$

where Eq. (6.1) is the mass conservation law, Eqs. (2.4), (6.2) and (6.3) are the equations of motion, (3.3) and the energy equation (4.2) in the case when the constitutive equation (3.4) is taken into account. The corresponding system for the laminar compressible Navier–Stokes model is of the following form

$$\begin{gathered} \partial_t \varrho + \operatorname{div}(\varrho v) = 0, \\ \varrho\,(\partial_t v + \sum_\sigma v_\sigma \partial_{x_\sigma} v) = -\operatorname{grad} p + (\lambda+\lambda)\operatorname{div} v + \gamma \Delta v, \\ \partial_t U + \sum_\sigma v_\sigma \partial_{x_\sigma} U = -p\operatorname{div} v + \lambda(\operatorname{div} v)^2 + \\ +\gamma \sum_{ij} (\partial_{x_i} v_j + \partial_{x_j} v_i)\,\partial_{x_i} v_j - \operatorname{div} q. \end{gathered} \tag{6.4}$$

Let us now consider in our model the mean measured values: $\hat\varrho(t,x)$, $\hat v(t,x)$ given by (1.3), (2.4) and, in accordance with the principle B in Sect. 1:

$$\hat p(t,x) = \int_A [p(t,x,\alpha) - \psi(\mu(t,x,\alpha))]\,d\alpha,$$

the mean measured pressure,

$$\hat{U}(t,x) = \int_A U(t,x,\alpha)\,d\alpha,$$

the mean measured internal energy,

$$\hat{q}(t,x) = \int_A q(t,x,\alpha)\,d\alpha$$

the mean measured flux of non-mechanical energy. Then, assuming that all quantities considered are twice continuously differentiable, we may simply prove the following theorem:

THEOREM. *If* $\psi(0) = 0$ *and if in some* $G \subset R^4$ *for* $(t,x,\alpha) \in G \times A$ *we have* $\mu(t,x,\alpha) = 0$, $v(t,x,\alpha) = \overset{*}{v}(t,x)$, *then* $\hat{v}(t,x) = \overset{*}{v}(t,x)$ *and the functions* $\hat{\varrho}$, $\hat{p}$, $\hat{U}$, $\hat{v}$, $\hat{q}$ *satisfy for* $(t,x) \in G$ *the Navier–Stokes system* (6.4) *with* $\lambda = \lambda_1 |A|$, $\gamma = \gamma_1 |A|$.

Proof. The equality $\hat{v}(t,x) = \overset{*}{v}(t,x)$ follows immediately from Eq. (1.4) and the mean value theorem for integrals. We put in the system (6.1), (6.2), (6.3) $v = \hat{v}$, and $\mu = 0$, and perform the integration over $\alpha \in A$. Due to the symmetry of $\varphi(\alpha,\beta)$ and $\vartheta(\alpha,\beta)$ we have

$$\int_A d\alpha \int_A \varphi(\alpha,\beta) \frac{D[\varrho(t,x,\beta)\,v(t,x,\beta) - \varrho(t,x,\alpha)\,v(t,x,\alpha)]}{Dt}\,d\beta = 0,$$

and

$$\int_A d\alpha \int_A \vartheta(\alpha,\beta)\,[\Theta(t,x,\beta) - \Theta(t,x,\alpha)]\,d\beta = 0.$$

Hence, integrating the system of Eqs. (6.1), (6.2), (6.3) over $\alpha \in A$, we obtain the system (6.4) with $p = \hat{p}$, $\varrho = \hat{\varrho}$, $v = \hat{v}$, $q = \hat{q}$ and

$$\lambda = \lambda_1 \int_A d\alpha, \quad \gamma = \gamma_1 \int_A d\alpha,$$

what ends the proof.

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STRESZCZENIE

O PEWNYCH MODELACH NIELAMINARNYCH PRZEPŁYWÓW Z TURBULENTNĄ DYFUZJĄ

W pracy przedstawiono nowy model turbulentnych przepływów, w którym transport masy odbywa się wzdłuż krzywoliniowych stożków przenikających się wzajemnie. Przepływy laminarne występują w modelu jako szczególny przypadek, w którym stożki redukują się do krzywych.

РЕЗЮМЕ

О НЕКОТОРЫХ МОДЕЛЯХ НЕЛАМИНАРНЫХ ТЕЧЕНИЙ С ТУРБУЛЕНТНОЙ ДИФФУЗИЕЙ

В работе представлена новая турбулентная модель течений, в которой перенос массы совершается вдоль криволинейных конусов проникающихся взаимно. Ламинарные течения выступают в модели как частный случай, в котором конусы редуцируются к кривым.

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Received November 10, 1983.