K O M U N I K A T Y

SOME REMARKS ON THE WEAK INTERACTION OF MEAN AND TURBULENT FIELDS OF TEMPERATURE AND WIND VELOCITY

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Some spectral characteristics of the kinetic energy spectrum $E(\varkappa)$ corresponding to the case when interaction between the vorticities of the main and turbulent motions is weak and $b = (d\bar{u}/dz)^2 - \alpha\beta\,(dT/dz)$ can be treated as a small nonzero parameter, are examined with the help of a perturbation method as an alternative approach.

1. Introduction. Formulation of the problem

In the analysis of turbulent flow, the main source of turbulent energy is assumed to be the energy of the basic motion. The intensity of turbulence and turbulent transfer of momentum or the transfer of any conservative additive in turbulent flow depend on the velocity gradients (vorticity) of the main motion and also on the nature of interaction of the vorticity of the basuc stream with the vorticity of the turbulent motion. According to Tchen (cf. Hinze, [1]), we may have to do with two cases. In the first case, the vorticity of the main motion is small compared with the vorticity of the shear flow turbulence. In the second case the vorticity of the main notion is comparable with the vorticity of the turbulent motion. The interaction between the two vorticity fields in the first case can only be very weak and, accordingly, there is no resonance. In the second case, however, where the vorticities of the two motions are comparable, there may be a strong interaction between them causing violent resonance. Such interaction (weak or strong) as occurring in the atmosphere is determined by the vertical gradients of the mean wind velocity and that of the air temperature.

In the present paper our aim is to study spectral characteristics of atmospheric turbulence in the case when interaction between the vorticities of the main and turbulent motion is weak with the help of a perturbation technique described in the next section. For this purpose we shall concentrate our attention on the spectral equation obtained from the equations of motion and continuity by averaging and expanding in Fourier integrals (cf. GISINA [2]; VINNICHENKO *et al.*, [3]). For the equalibrium region ($\varkappa \gg L_0^{-1}$) of the spectrum, under the assumption of local unformity of turbulence, this equation is read as

$$\varepsilon = 2\nu \int_0^{\varkappa} \varkappa^{1^2} E(\varkappa^1)\, d\varkappa^1 - \frac{d\bar{u}}{dz} \int_{\varkappa}^{\infty} \tau(\varkappa^1)\, d\varkappa^1 + F(\varkappa) + \beta \int_{\varkappa}^{\infty} H(\varkappa^1)\, d\varkappa^1, \tag{1.1}$$

where $E(\varkappa)$ is turbulence energy spectrum, $\varkappa$ wave number, L_0 a length comparable with the scale of flow as a whole, ε total energy dissipation, $F(\varkappa)$ represents the transfer of energy through the hierarchy of eddies, $H(\varkappa)$ spectrum of vertical heat flux, ν kinematic viscosity, $\beta\,(=g/T)$ buoyancy parameter and $\tau(\varkappa)$ spectrum of turbulent friction stress.

The region concerned is chosen as such that the turbulence is not too remote from isotropy and HEISENBERG'S [4] assumption may be applied for the transfer apectrum $F(\varkappa)$.

Thus

$$F(\varkappa) = K(\varkappa) \cdot 2 \int_0^{\varkappa} (\varkappa^1)^2 E(\varkappa^1)\, d\varkappa^1, \tag{1.2}$$

where

$$K(\varkappa) = a \int_{\varkappa}^{\infty} [E(\varkappa^1)]^{1/2} (\varkappa^1)^{-3/2}\, d\varkappa^1, \tag{1.3}$$

a being a numerical coefficient.

Also for the case of weak interaction between the basic and turbulent motion in the wave number region $\varkappa \gg L_0^{-1}$, we may take

$$\int_{\varkappa}^{\infty} \tau(\varkappa^1)\, d\varkappa^1 = -K(\varkappa) \frac{d\bar{u}}{dz} \tag{1.4}$$

and

$$\int_{\varkappa}^{\infty} H(\varkappa^1)\, d\varkappa^1 = -\alpha K(\varkappa) \frac{dT}{dz}, \tag{1.5}$$

where α is the ratio of the coefficient of turbulent mixing for heat and momentum. By virtue of the relations (1.2)–(1.5), Eq. (1.1) is transformed into

$$\varepsilon = 2\nu \int_0^{\varkappa} (\varkappa^1)^2 E(\varkappa^1)\, d\varkappa^1 + 2a \int_{\varkappa}^{\infty} [E(\varkappa^1)]^{1/2} (\varkappa^1)^{-3/2}\, d\varkappa^1 \int_0^{\varkappa} (\varkappa^1)^2 E(\varkappa^1)\, d\varkappa^1 + \\ + ab \int_{\varkappa}^{\infty} [E(\varkappa^1)]^{1/2} (\varkappa^1)^{-3/2}\, d\varkappa^1, \tag{1.6}$$

where

$$b=\left(\frac{d\bar{u}}{dz}\right)^2-\alpha\beta\frac{dT}{dz}. \tag{1.7}$$

In the next section we obtain solutions of Eq. (1.6) with the help of a perturbation method in which b is treated as a nonzero and small parameter. The perturbation method used is similar to that of Peskin and Baw [5] as applied to the problem of turbulent flows containing suspended solid particles.

2. Solution of the spectral equation using the perturbation method

To solve Eq. (1.6) we define

$$\theta(\varkappa)=\int_0^{\varkappa}(\varkappa^1)^2\,E(\varkappa^1)\,d\varkappa^1. \tag{2.1}$$

Using $\theta(\varkappa)$ instead of $E(\varkappa)$ as dependent variable and differentiating Eq. (1.6) with respect to $\varkappa$, we obtain

$$\frac{\varepsilon}{2\theta^2}\frac{d\theta}{d\varkappa}=a\sqrt{\frac{1}{\varkappa^5}\frac{d\theta}{d\varkappa}}+\frac{ab}{2\theta}\sqrt{\frac{1}{\varkappa^5}\frac{d\theta}{d\varkappa}}+ +\frac{ab}{2\theta^2}\frac{d\theta}{d\varkappa}\int_{\varkappa}^{\infty}\sqrt{\frac{1}{(\varkappa^1)^5}\frac{d\theta}{d\varkappa^1}}\,d\varkappa^1. \tag{2.2}$$

Squaring and rearranging, Eq. (2.2) is transformed into

$$\frac{\theta^4}{\varkappa^5}=\left(\frac{\varepsilon}{2a}\right)^2\frac{d\theta}{d\varkappa}-\frac{b^2\theta^2}{4\varkappa^5}-\frac{b^2}{4}\frac{d\theta}{d\varkappa}\left[\int_{\varkappa}^{\infty}\sqrt{\frac{1}{(\varkappa^1)^5}\frac{d\theta}{d\varkappa^1}}\,d\varkappa^1\right]^2- -\frac{b\theta^3}{\varkappa^5}-\frac{b^2\theta}{2}\sqrt{\frac{1}{\varkappa^5}\frac{d\theta}{d\varkappa}}\int_{\varkappa}^{\infty}\sqrt{\frac{1}{(\varkappa^1)^5}\frac{d\theta}{d\varkappa^1}}\,d\varkappa^1- -b\theta^2\sqrt{\frac{1}{\varkappa^5}\frac{d\theta}{d\varkappa}}\int_{\varkappa}^{\infty}\sqrt{\frac{1}{(\varkappa^1)^5}\frac{d\theta}{d\varkappa^1}}\,d\varkappa^1. \tag{2.3}$$

Let us expand $\theta(\varkappa)$ and ε, respectively, as

$$\theta(\varkappa)=\theta_0(\varkappa)+b\theta_1(\varkappa)+b^2\,\theta_2(\varkappa)+\ldots \tag{2.4}$$

and

$$\varepsilon=\varepsilon_0+b\varepsilon_1+b^2\,\varepsilon^2+\ldots, \tag{2.5}$$

where b is considered to be a nonzero small parameter.

Then

$$\frac{d\theta}{d\varkappa} = \theta^1 = \theta_0^1 + b\theta_1^1 + b^2\,\theta_2^1 + \ldots. \tag{2.6}$$

Substituting Eqs. (2.4)–(2.6) in Eq. (2.3) and then arranging the terms on both sides in ascending power of b, we obtain

$$\begin{aligned}
&\frac{\theta_0^4}{\varkappa^5} + b\,\frac{4\theta_0^3\,\theta_1}{\varkappa^5} + b\,\frac{(4\theta_0^3\,\theta_2 + 6\theta_0^2\,\theta_1^2)}{\varkappa^5} = \\
&= \frac{1}{4a^2}\,\varepsilon_0^2\,\frac{d\theta_0}{d\varkappa} + b\left[\frac{1}{4a^2}\left(\varepsilon_0^2\,\frac{d\theta_1}{d\varkappa} + 2\varepsilon_0\,\varepsilon_1\,\frac{d\theta_0}{d\varkappa}\right) - \right. \\
&\left. -\frac{\theta_0^3}{\varkappa^5} - \theta_0^2\sqrt{\frac{1}{\varkappa^5}\,\frac{d\theta_0}{d\varkappa}}\int_\varkappa^\infty \sqrt{\frac{1}{(\varkappa^1)^5}\,\frac{d\theta_0}{d\varkappa^1}}\,d\varkappa^1\right] + \\
&+ b^2\left[\frac{1}{4\mathrm{a}^2}\left(\varepsilon_0^2\,\frac{d\theta_2}{d\varkappa} + 2\varepsilon_0\,\varepsilon_1\,\frac{d\theta_1}{d\varkappa} + 2\varepsilon_0\,\varepsilon_2\,\frac{d\theta_0}{d\varkappa} + \varepsilon_1^2\,\frac{d\theta_0}{d\varkappa}\right) - \right. \\
&-\frac{\theta_0^2}{4\varkappa^5} - \frac{1}{4}\,\frac{d\theta_0}{d\varkappa}\left(\int_\varkappa^\infty \sqrt{\frac{1}{(\varkappa^1)^5}\,\frac{d\theta_0}{d\varkappa^1}}\,d\varkappa^1\right)^2 - \frac{3\theta_0^2\,\theta_1}{\varkappa^5} - \\
&-\frac{\theta_0}{2}\sqrt{\frac{1}{\varkappa^5}\,\frac{d\theta_0}{d\varkappa}}\int_\varkappa^\infty \sqrt{\frac{1}{(\varkappa^1)^5}\,\frac{d\theta_0}{d\varkappa^1}}\,d\varkappa^1 - \frac{\theta_0^2\,\theta_0^{1\,1/2}}{(2\varkappa^1)^{5/2}}\int_\varkappa^\infty \frac{1}{(\varkappa^1)^{5/2}}\,\frac{d\theta_1/d\varkappa^1}{(d\theta_0/d\varkappa^1)^{1/2}}\,d\varkappa^1 - \\
&\left. -\frac{1}{\varkappa^{5/2}}\left(\frac{\theta_0^2\,\theta_1^1}{(2\theta^1)^{1/2}} + 2\theta_0\,\theta_1\,\theta_0^{1\,1/2}\right)\int_\varkappa^\infty \sqrt{\frac{1}{(\varkappa^1)^5}\,\frac{d\theta_0}{d\varkappa^1}}\,d\varkappa^1\right].
\end{aligned} \tag{2.7}$$

Equating the coefficients of b^0 and b^1 from both sides fo Eq. 2.7 we obtain, respectively,

$$b^0\!: \quad \frac{\theta_0^4}{\varkappa^5} = \frac{1}{4a^2}\,\varepsilon_0^2\,\frac{d\theta_0}{d\varkappa}, \tag{2.8}$$

$$\begin{aligned}
b^1\!: \quad &\frac{4\theta_0^3\,\theta_1}{\varkappa^5} = \frac{1}{4a^2}\,\varepsilon_0^2\,\frac{d\theta_1}{d\varkappa} + \frac{\varepsilon_0\,\varepsilon_1}{2a^2}\,\frac{d\theta_0}{d\varkappa} - \frac{\theta_0^3}{\varkappa^5}; \\
&-\theta_0^2\sqrt{\frac{1}{\varkappa^5}\,\frac{d\theta_0}{d\varkappa}}\int_\varkappa^\infty \sqrt{\frac{1}{(\varkappa^1)^5}\,\frac{d\theta_0}{(4\varkappa^1)^9}}\,d\varkappa^1.
\end{aligned} \tag{2.9}$$

When $\varkappa \to \infty$, Eq. (6) gives

$$\text{(2.10)} \quad \varepsilon_0 + b\varepsilon_1 + b^2 \varepsilon_2 + \ldots = \varepsilon = 2\nu\theta(\infty) = 2\nu\left[\theta_0(\infty) + b\theta_1(\infty) + b^2\theta_2(\infty) + \ldots\right].$$

From Eq. (2.8) the basic solution for θ_0 is obtained as

$$\text{(2.11)} \quad \theta_0^3 = \frac{1}{\frac{3}{4}\left(\frac{2a}{\varepsilon_0}\right)^2 \varkappa^{-4} + \text{const}}.$$

Since $\theta_0 = \varepsilon_0/2\nu$ when $\varkappa \to \infty$, the constant in the solution of θ_0 must equal to $(2\nu/\varepsilon_0)$. If, further, we introduce $\varkappa_d = (\varepsilon_0/\nu^3)^{1/4} = 1/\eta$, where η is the Kolmogoroff length scale, then,

$$\text{(2.12)} \quad \theta_0^3 = \frac{\varepsilon_0^3}{3a^2\nu^3} \cdot \frac{\left(\frac{\varkappa}{\varkappa_d}\right)^4}{1 + \frac{8}{3a^2}\left(\frac{\varkappa}{\varkappa_d}\right)^4}.$$

From Eq. (2.8) we get

$$\text{(2.13)} \quad \theta_1^1 - 4\frac{\theta_0^1}{\theta_0}\theta_1 + 2\frac{\varepsilon_1}{\varepsilon_0}\theta_0^1 - \frac{\theta_0^1}{\theta_0} - \left(\frac{2a}{\varepsilon_0}\right)^2 \theta_0^2 \sqrt{\frac{1}{\varkappa^5}\frac{d\theta_0}{d\varkappa}} \int_{\varkappa}^{\infty} \sqrt{\frac{1}{(\varkappa^1)^5}\frac{d\theta_0}{d\varkappa^1}}\, d\varkappa^1 = 0.$$

The solution of Eq. (2.13) is given by

$$\text{(2.14)} \quad \theta_1 = c\theta_0^4 + 2\frac{\varepsilon_1}{\varepsilon_0}\left(\frac{2a}{\varepsilon_0}\right)^2 \cdot \frac{\theta_0^4}{4\varkappa^4} - \frac{1}{4} - \left(\frac{2a}{\varepsilon_0}\right)^4 \frac{\theta_0^4}{4} \int_{\varkappa}^{\infty} \frac{\theta_0^2}{(\varkappa^1)^5}\, d\varkappa^1 - \left(\frac{2a}{\varepsilon_0}\right)^4 \theta_0^4 \int_{\varkappa}^{\infty} \frac{\theta_0^2}{(4\varkappa^1)^9}\, d\varkappa^1.$$

Since $\theta_1 = \varepsilon_1/2\nu$ when $\varkappa \to \infty$, the integration constant c is given by

$$c = \left(\frac{\varepsilon_1}{2\nu} + \frac{1}{4}\right)\left(\frac{2\nu}{\varepsilon_0}\right)^4.$$

Putting the value of c in Eq. (2.13), we obtain

$$\theta_1(\varkappa)=\left(\frac{\varepsilon_1}{2\nu}+\frac{1}{4}\right)\left(\frac{2\nu}{\varepsilon_0}\right)^4\theta_0^4+2\frac{\varepsilon_1}{\varepsilon_0}\left(\frac{2a}{\varepsilon_0}\right)^2\frac{\theta_0^4}{4\varkappa^4}-\frac{1}{4}-\left(\frac{2a}{\varepsilon_0}\right)^4\frac{\theta_0^2}{4\varkappa^4}\int_{\varkappa}^{\infty}\frac{\theta_0^2}{(\varkappa^1)^5}\,d\varkappa^1-\left(\frac{2a}{\varepsilon_0}\right)^4\theta_0^4\int_{\varkappa}^{\infty}\frac{\theta_0^2}{(4\varkappa^1)^9}\,d\varkappa^1. \tag{2.15}$$

In order to find the expression for $E(\varkappa)$, we use the relations (2.1) and (2.4) and obtain

$$E(\varkappa)=\frac{1}{\varkappa^2}\frac{d\theta}{d\varkappa}=\frac{1}{\varkappa^2}\frac{d\theta_0}{d\varkappa}+b\cdot\frac{1}{\varkappa^2}\frac{d\theta_1}{d\varkappa}+b^2\frac{d\theta_2}{d\varkappa}+\dots. \tag{2.16}$$

For simplicity we will retain terms up to the order of b during the calculation of $E(\varkappa)$ and discuss their behaviour in the wave number ranges, e.g. $\varkappa \ll \varkappa_d$ and $\varkappa \gg \varkappa_d$.

Case I. When $\varkappa/\varkappa_d \ll 1$.

In this case, utilizing the relations (2.12) and (2.15), we obtain after some simple but lengthy calculations

$$E(\varkappa)=\frac{1}{\varkappa^2}\frac{d\theta_0}{d\varkappa}+b\cdot\frac{1}{\varkappa^2}\frac{d\theta_1}{d\varkappa}=\left(\frac{8}{9a}\right)^{2/3}(\varepsilon_0\nu^5)^{1/4}\left[1+\frac{2b}{3}\frac{\varepsilon_1}{\varepsilon_0}\right]\left(\frac{\varkappa}{\varkappa_d}\right)^{-5/3}+O\left(\frac{\varkappa}{\varkappa_d}\right)^{7/3}. \tag{2.17}$$

Case II. When $\varkappa/\varkappa_d \gg 1$.

In this case, as in case I, we utilize Eqs. (2.12) and (2.15) and obtain after some calculation

$$E(\varkappa)=\left[\left(\frac{a}{2}\right)^2(\varepsilon_0\nu^5)^{1/4}+b\left\{\frac{a}{2}^2\left(\frac{\nu^5}{\varepsilon_0^3}\right)^{1/4}(2\varepsilon_1+\nu)\right\}\right]\left(\frac{\varkappa}{\varkappa_d}\right)^{-7}+O\left(\frac{\varkappa}{\varkappa_d}\right)^{-11}, \tag{2.18}$$

when $\varkappa\to\infty$.

3. Discussions

In the case when the interaction between the vorticities of the main and turbulent motion is weak (i.e. the vorticity of the main motion is comparatively smaller than the vorticity of the turbulence) and b as defined by Eq. (1.7) can be treated as a small nonzero parameter, the energy spectrum $E(\varkappa)$ is found approximately to decay as $(\varkappa/\varkappa_d)^{-5/3}$ and $(\varkappa/\varkappa_d)^{-7}$

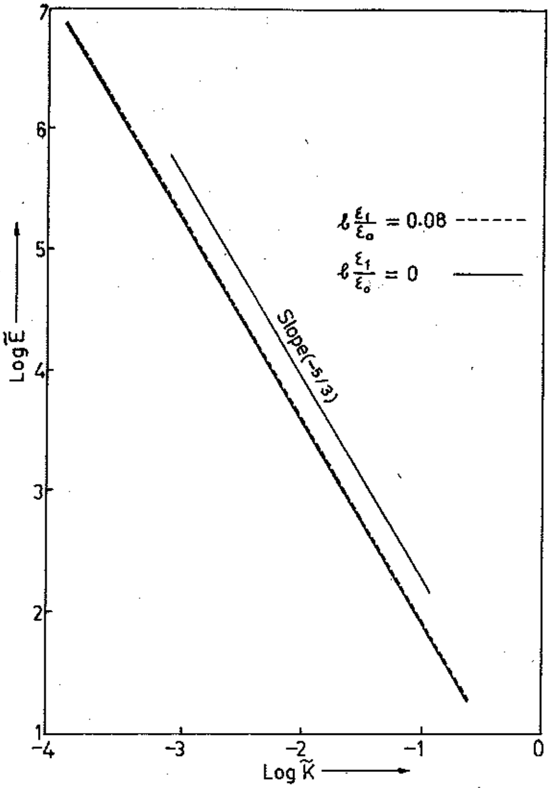

FIG. 1. Nondimensional energy spectra ($b\varepsilon_1/\varepsilon_0 = 0$, $b\varepsilon_1/\varepsilon_0 = 0.08$).

respectively, in the ranges $\varkappa \ll \varkappa_d$ and $\varkappa \gg \varkappa_d$. It is noticeable from the relations (2.17) and (2.18) that in both cases the multiplying factors (of $(\varkappa/\varkappa_d)^{-5/3}$ and $(\varkappa/\varkappa_d)^{-7}$) are modified in contrast to those in ordinary homogeneous and isotropic turbulence as they involve the effect of nonuniformities of the fields of mean temperature and mean velocity. Scaling the wave number $\varkappa$ and $E(\varkappa)$ as $\tilde{\varkappa} = \varkappa/\varkappa_d$ and $\tilde{E} = E(\varkappa)/(\varepsilon_0 \nu^5)^{1/4}$ respectively, we plot Log $\tilde{E}$ against Log $\tilde{\varkappa}$ from Eq. (2.17) for two values of the nondimensional parameter $b\varepsilon_1/\varepsilon_0$, viz. $b\varepsilon_1/\varepsilon_0 = 0$ and $b\varepsilon_1/\varepsilon_0 = 0.08$ (Fig. 1). The straight lines ([6], Fig. 1) with the slope $-5/3$ indicate that the interaction between the vorticity of the basic motion and that of turbulent motion is slight for a small parametric value of $b\varepsilon_1/\varepsilon_0$. For $F(\varkappa)$ we have used Heisenberg's expression which is believed to be more applicable to the case of energy transfer between widely differing wave numbers than to the case of energy transfer between wave numbers of the same order of magnitude. It is therefore desirable to use a more physically plausible assumption and perform the calculation of energy spectrum for the present case which is planned to be incorporated in future work.

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