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NATURAL VIBRATION FREQUENCIES OF TAPERED BEAMS

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In this paper the free vibrations frequencies of tapered Euler-Bernoulli beams are calculated, in the presence of an arbitrary number of rotationally and/or axially, elastically flexible constraints. The dynamic analysis is performed by means of the so-called cell discretization method (CDM), according to which the beam is reduced to a set of rigid bars, linked together by elastic sections, where the bending stiffness and the distributed mass of the bars is concentrated. The resulting stiffness matrix and mass matrix are easily deduced, and the generalized symmetric eingenvalue problem can be immediately solved. Various numerical comparisons allow us to show the potentialities of the proposed approach.

Key words: free vibrations, tapered beam, elastically restrained, CDM.

1. Introduction

The dynamic analysis of beams with continuously varying cross-section is a classical structural problem, which nowadays is becoming more and more important, even in mechanical engineering and in aeronautic engineering.

Numerous authors have approached the analysis assuming that the beam is sufficiently slender to be considered as an Euler-Bernoulli beam, and trying to analytically solve the resulting fourth-order differential equation with variable coefficients. Among the others, Craver and Jampala [1] examine the free vibration frequencies of a cantilever beam with variable cross-section and constraining springs; De Rosa and Auciello [2] give the exact free frequencies of a beam with linearly varying cross-section, in the presence of generic non-classically boundary conditions, so that all the usual boundary conditions can be treated as particular cases; Datta and Sill [3] give the general solution in terms of Bessel functions, and the first eingevalue for a beam with constant width and linearly varying height is found. In 1995 Abrate [4] solved the differential equation for various taper laws, and also performed a numerical comparison with the

Rayleigh–Ritz approach. Grossi *et al.* [5] employed both the classical Rayleigh–Ritz method and the optimized Rayleigh–Schmidt method to find the frequencies of beams with constant width and varying height, and also of beams with varying width and varying height. A lot of numerical results were given, for various non-classical boundary conditions. Mou *et al.* [6] employed the exact dynamic stiffness matrix (EDSM) to find the frequencies of circular and elliptic tapered beams, and of beams defined by a linear-tapered section, a uniform section and a non-linearly varying section. All the results are compared with a classical finite element analysis. The Rayleigh–Ritz approach is used by Zhou and Cheung [7] to find the first free vibration frequencies of three different tapered beams with various boundary conditions and truncation factors. Finally, free vibrations of Euler-Bernoulli beams of bilinearly varying thickness are studied in [8] using: a) the optimized Rayleigh–Ritz method, b) the differential quadrature technique and c) the finite element approach.

Tapered beams with more complex geometry and non-classical boundary conditions were studied by Auciello *et al.* in [9–10]; the beam is divided into two segments, and each segment has a different tapering law. The exact solutions are obtained in both the above-mentioned papers, solving the corresponding boundary value problem.

In [11–14] the dynamic stability problem of a non-prismatic beam is solved using the Chebyshev series approximation: the method is used to solve the problem of vibration for a Euler–Bernoulli and Timoshenko beams.

In this paper a numerical approach is adopted, to find the free vibration frequencies of Euler–Bernoulli multi-span beams with arbitrarily varying cross-sections, in the presence of elastically flexible supports. The analysis is performed reducing the beam to a set of rigid bars linked together by means of elastic sections (elastic cells), in which the stiffness and the mass of the beam is properly concentrated. In this way, the structure is reduced to a system with finite number of degrees of freedom, and the global stiffness matrix and the global mass matrix can be easily calculated. Obviously, the method can be dated back to the first manual attempts to solve the vibration problem [15–16 e.g.], but in this paper its feasibility to be computerized is clearly shown, using the powerful symbolic software *Mathematica* [17], and various numerical comparisons show the method's usefulness.

2. Formulation of the problem

Let us consider the beam in Fig. 1, with span L, Young modulus E and mass density ρ, resting on elastically flexible constraints at the ends, with rotational stiffness k_{RL} at left and k_{RR} at rigth, and axial stiffness k_{TL} at left and k_{TR} at rigth, respectively.

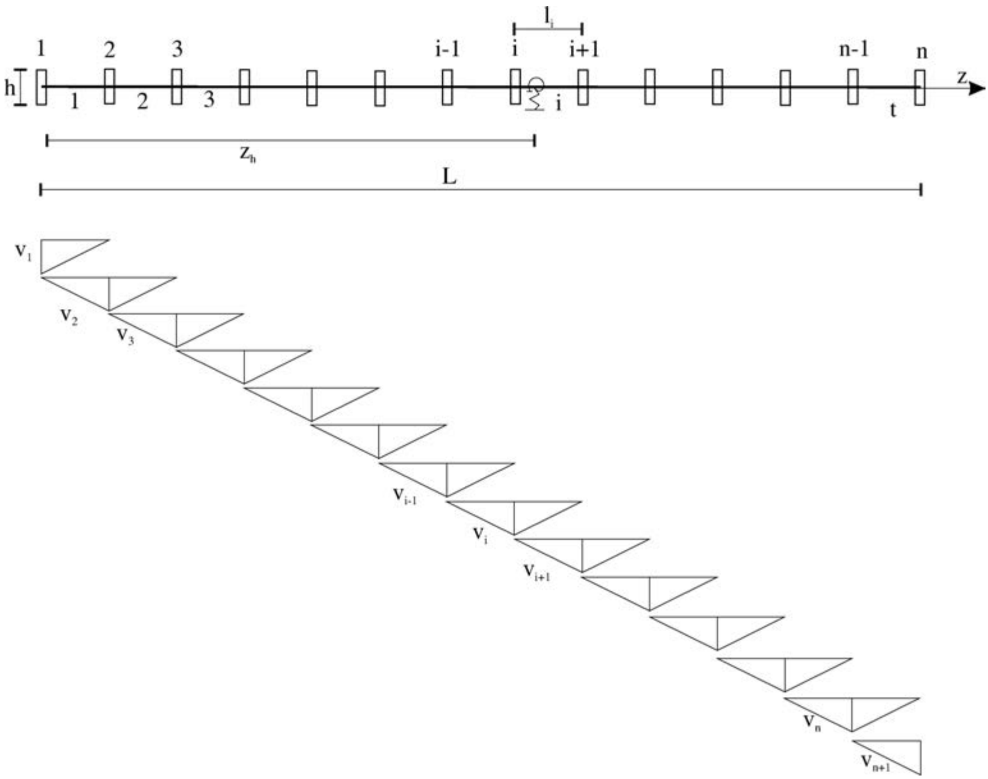

FIG. 1. Structural system.

Moreover, let us suppose that both the moment of inertia $I(z)$ and the cross-sectional area $A(z)$ vary with the abscissa z. As already said, the beam is reduced to a set of t rigid bars with length l_i, connected by $n = t + 1$ elastic cells. Whereas the possibility to adopt different lengths for each bar is invaluable in order to simulate rapidly varying geometries, nevertheless in the following we shall adopt the simplest choice, for which, $l_i = l, \ i = 1, ...t$. Moreover, the moment of inertia $I(z)$ and the cross-sectional area $A(z)$ will be evaluated at the cells abscissae, obtaining the concentrated stiffness $k_i = EI(z)/l$ and the concentrated masses $m_i = \rho A(z)l$. Both these quantities can be organized into the so-called unassembled stiffness matrix $\mathbf{k} = \text{diag}\{k_i\}, \ i = 1, ...n$ and the unassembled mass matrix $\mathbf{M} = \text{diag}\{m_i\}, \ i = 1, ...n$.

In this way, the structures is reduced to a classical holonomic system, with n degrees of freedom. The n vertical displacements v_i at the cells abscissae can be assumed as Lagrangian coordinates, and they will be organized into the n-dimensional vector $\mathbf{v}$; equivantely, the vector $\mathbf{v}$ can be viewed as a $(n \times 1)$-dimensional matrix. The $n - 1$ rotations of the rigid bars can be calculated as a function of the Lagrangian coordinates as follows:

$$\phi_i = \frac{v_{i+1} - v_i}{l} \tag{2.1}$$

or, in matrix form: $\phi = \mathbf{V}\mathbf{v}$ and $\mathbf{V}$ is a rectangular transfer matrix with $n-1$ rows and n columns.

The relative rotations between the two faces of the elastic cells are given by:

$$\psi_1 = \phi_1, \qquad \psi_i = \phi_i - \phi_{i-1}, \qquad \psi_n = -\phi_{n-1}, \tag{2.2}$$

or in matrix form $\psi = \Delta\phi$, and Δ is another rectangular transfer matrix with n rows and $n-1$ columns.

The bending strain energy L_e is concentrated at the cells, and is given by:

$$L_e = \frac{1}{2}\sum_{i=1}^{n} k_{ii}\,\psi_i^2 = \frac{1}{2}\boldsymbol{\psi}^T\mathbf{k}\boldsymbol{\psi}. \tag{2.3}$$

In order to obtain a quadratic form of the Lagrangian coordinates it is necessary to use Eqs. (2.1)–(2.2):

$$L_e = \frac{1}{2}\boldsymbol{\psi}^T\mathbf{k}\boldsymbol{\psi} = \frac{1}{2}\boldsymbol{\phi}^T\Delta^{\mathrm{T}}\mathbf{k}\,\Delta\boldsymbol{\phi} = \frac{1}{2}\mathbf{v}^T\left(\mathbf{V}\Delta^{\mathrm{T}}\mathbf{k}\Delta\mathbf{V}\right)\mathbf{v} \tag{2.4}$$

or else:

$$L_e = \frac{1}{2}\mathbf{v}^T\mathbf{K}\mathbf{v}, \tag{2.5}$$

where $\mathbf{K}$ is the assembled stiffness matrix.

The kinetic energy can be simply expressed as:

$$T = \frac{1}{2}\mathbf{v}^T\mathbf{M}\mathbf{v}. \tag{2.6}$$

The strain energy of the axially flexible constraints at the ends is given by:

$$L_{TL} = \frac{1}{2}k_{TL}v_1^2, \qquad L_{TR} = \frac{1}{2}k_{RL}v_n^2, \tag{2.7}$$

so that the assembled stiffness matrix must be modified as follows:

$$K[1,1] = K[1,1] + k_{TL}, \qquad K[n,n] = K[n,n] + k_{TR}. \tag{2.8}$$

The presence of axially flexible intermediate supports can be similarly dealt with. If the constraint is placed at the abscissa $z_h = z_i + l_h$, and if its axial stiffness is given by k_T, its vertical displacement is given by (cf. Fig. 2):

$$v_h = v_i + \frac{v_{i+1} - v_i}{l}l_h \tag{2.9}$$

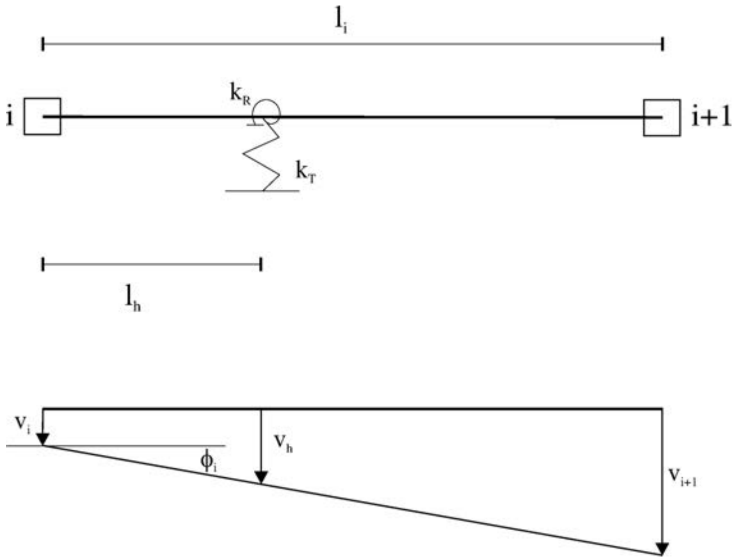

FIG. 2. Intermediate axially and rotationally flexible supports.

and its strain energy is equal to:

$$L_T = \frac{1}{2} k_T v_h^2. \tag{2.10}$$

The rotational stiffnesses of the constraints can be taken into account by summing up the corresponding flexibilities with the flexibilities of the rigid bars. For example, for the end constraints we have:

$$K[1,1] = \frac{K[1,1]\, k_{RL}}{k_{RL} + K[1,1]}, \qquad K[n,n] = \frac{K[n,n]\, k_{RR}}{k_{RR} + K[n,n]}. \tag{2.11}$$

The equation of motion can be written as:

$$\mathbf{M}\ddot{\mathbf{v}} + \mathbf{K}\mathbf{v} = \mathbf{0}. \tag{2.12}$$

The resulting generalized symmetric eingevalue problem can be easily solved, and the frequencies ω_i^2 can be obtained, together with the corresponding vibration modes.

3. Numerical comparisons

In order to show the method's potentialities, several numerical examples will be examined, using a general code developed in *Mathematica* [17]. In this paper we are not particularly interested in the convergence properties of the solutions, therefore all the examples will be performed by using a large number of cells, i.e. $n = 300$.

1. As a first numerical comparison, let us consider a tapered Euler-Bernoulli beam with cross-sectional area and moment of inertia given by the following laws:

$$A(z) = A_0\left((\alpha-1)\frac{z}{L}+1\right)^2, \qquad I(z) = I_0\left((\alpha-1)\frac{z}{L}+1\right)^4, \tag{3.1}$$

where $\alpha = \dfrac{h_1}{h_0} = \dfrac{b_1}{b_0}$, and A_0 and I_0 are the cross-sectional area and the moment of inertia of the section at left.

The beam is constrained at both ends with elastically flexible constraints, defined by the following non-dimensional quantities:

$$R_1 = \frac{k_{RL}L}{EI_0}, \qquad R_2 = \frac{k_{RR}L}{EI_1}, \qquad T_1 = \frac{k_{TL}L^3}{EI_0}, \qquad T_2 = \frac{k_{TR}L^3}{EI_1}. \tag{3.2}$$

This structure has been already solved in [2] using an exact approach, and the first five non-dimensional frequencies $p_i = \sqrt{\sqrt{\dfrac{\rho A_0 \omega_i^2 L^4}{EI_0}}}$ are reported in Table 1. With this discretization level, the discrepancies are negligible.

Table 1. Numerical comparison between the first five non-dimensional frequency coefficients p_i for $T_1 = T_2 \to \infty$, $\alpha = 2$.

R_1	R_2	p_1	p_2	p_3	p_4	p_5
0	0	3.7300	7.6302	11.4217	15.2083	18.9954
		3.7300	7.6301	11.4212	15.2072	18.9932
0	0.01	3.7345	7.6317	11.4226	15.2089	18.9959
		3.7345	7.6316	11.4221	15.2078	18.9937
0	0.1	3.7737	7.6447	11.4306	15.2147	19.0004
		3.7737	7.6446	11.4301	15.2136	19.9982
0	1	4.0635	7.7619	11.5054	15.2695	19.0436
		4.0635	7.7618	11.5049	15.2684	19.0114
0	10	4.7549	8.2846	11.9277	15.6221	19.3456
		4.7549	8.2845	11.9272	15.6209	19.3432
1	0	3.7984	7.6803	11.4604	15.2397	19.0218
		3.7984	7.6802	11.4600	15.2386	19.0195
1	0.1	3.8409	7.6946	11.4693	15.2461	19.0267
		3.8409	7.6945	11.4688	15.2450	19.0245
1	1	3.1249	7.8105	11.5436	15.3007	19.0698
		3.1249	7.8104	11.5431	15.2995	19.0676

2. The free vibration frequencies of cantilever tapered beams have been studied by ABRATE [4] using a Rayleigh–Ritz approach and an n-term approximation.

The non-dimensional frequencies $\Omega_i = \omega_i \sqrt{\dfrac{\rho A_0 L^4}{E I_0}}$ are given in Table 2, for the following variation law:

$$\frac{A}{A_0} = \frac{I}{I_0} = 1 + \alpha z. \tag{3.3}$$

Table 2. First four non-dimensional frequency coefficients Ω_i for $\alpha = 0$ and $\alpha = -1/2$.

α	N	Mode	Abrate [4]	Hodges [19]	Thomson [18]	CDM
0	10	1	3.5160152	–	3.5160	3.5160
$-1/2$	10	1	4.3151703	4.3151703	–	4.3151575
		2	23.519257	–	–	23.518686
		3	63.199197	–	–	63.195723
		4	122.43963	–	–	122.42584

In the same table, the exact values for a constant beam are reported from Thomson [18], as well as the particular case $\alpha = -\dfrac{1}{2}$, which was studied by Hodges [19] using a finite element transfer matrix approach.

The non-dimensional frequencies Ω_i are given in Table 3, for the following quadratic variation law:

$$\frac{A}{A_0} = \frac{I}{I_0} = 1 + z + z^2, \tag{3.4}$$

the Rayleigh–Ritz results have been obtained using 20 trial functions, and the results show some discrepancies within the sixth decimal place.

Table 3. As in Table 2, but $A/A_0 = I/I_0 = 1 + z + z^2$.

Mode	Abrate [4]	Hodges [19]	CDM
1	2.4707858401571	2.4707858401571	2.4707660120
2	19.844681725047	–	19.844038124
3	59.7740637	–	59.770332125
4	119.040848	–	119.02840258

3. A numerical comparison is illustrated in Table 4, between the results given by our approach and the results given by Grossi *et al.* [5], using a classical Rayleigh–Ritz method and a more sophisticated Rayleigh–Schmidt procedure.

Table 4. Numerical comparison between the results in [5] and CDM.

$\sqrt{\lambda_1}$				
R_1	$b_2/b_1=1$	$b_2/b_1=.5$	$b_2/b_1=1$	$b_2/b_1=.5$
	$T_2=0.00$		$T_2=0.10$	
0.0	–	–	0.32193	0.30080
	–	–	0.32172	0.30049
	–	–	0.32172	0.30046
0.1	0.90219	1.00180	0.90603	1.00401
	0.90200	1.00150	0.90574	1.00361
	0.90197	1.00145	0.90570	1.00355
10	1.95338	2.15046	1.95429	2.15123
	1.94044	2.13050	1.94110	2.13095
	1.93828	2.12654	1.93890	2.12696
100	2.05048	2.25019	2.05136	2.25095
	2.03481	2.22614	2.03544	2.2269
	2.03200	2.22101	2.03259	2.22141
∞	2.06219	2.26179	2.06306	2.26254
	2.04655	2.23784	2.04718	2.23828
	2.04367	2.23258	2.04427	2.23299
	$T_2=10$		$T_2=\infty$	
0.0	1.06415	1.02179	2.36301	2.34082
	1.01514	0.95216	2.32154	2.27992
	1.00992	0.94413	2.31286	2.26429
0.1	1.19009	1.21458	2.39812	2.38694
	1.15137	1.16844	2.35653	2.32640
	1.14723	1.16320	2.34785	2.31092
10	2.04639	2.23125	3.10163	3.21538
	2.00323	2.17527	3.03750	3.12459
	1.99724	2.16623	3.02511	3.10289
100	2.13995	2.32944	3.27145	3.39476
	2.09525	2.27026	3.19917	3.29240
	2.08847	2.25981	3.18515	3.26755
∞	2.15052	2.33995	3.29341	3.41670
	2.10664	2.28179	3.22144	3.31473
	2.09989	2.27131	3.20739	3.28980

The example refers to a tapered beam resting on elastically flexible ends with axial stiffnesses T_1 and T_2 and rotational stiffnesses R_1 and R_2, respectively. The cross-sectional area and the moment of inertia vary according to the following laws:

$$A(z) = b(z)h(z) = A_1 \left(1 + c_2 \frac{z}{L}\right) \left(1 + c_1 \frac{z}{L}\right), \tag{3.5}$$

$$I(z) = \frac{b(z)h(z)^3}{12} = I_1 \left(1 + c_2 \frac{z}{L}\right) \left(1 + c_1 \frac{z}{L}\right)^3, \tag{3.6}$$

where $c_1 = \dfrac{h_2}{h_1} - 1$, $c_2 = \dfrac{b_2}{b_1} - 1$ and $A_1 = b_1 h_1$, $I_1 = \dfrac{b_1 h_1^3}{12}$ are the area and the moment of inertia of the initial section.

The first non-dimensional frequency $\sqrt{\lambda_1} = \sqrt{\sqrt{\dfrac{\rho A_1 \omega_i^2 L^4}{E I_1}}}$ is given in the Table 4 for $R_2 = 0$, $T_1 = \infty$, $\dfrac{h_2}{h_1} = 0.25$, and for various R_1 values. The first $\sqrt{\lambda_1}$ value has been obtained using the Rayleigh–Ritz method, the second value is obtained by the optimized Rayleigh–Schmidt method, and finally the last value has been obtained using the CDM. As expected, our values are nearer to the Rayleigh–Schmidt results.

4. The free vibration frequencies of tapered beams with circular or elliptic cross-sections have been studied by Mou *et al.* [6], using the exact dynamic stiffness matrix (EDSM). The variation laws of cross-sectional area and moment of inertia are given by:

$$A(z) = A_0 \left(\frac{z}{L}\right)^n, \qquad I(z) = I_0 \left(\frac{z}{L}\right)^m, \tag{3.7}$$

where A_0 and I_0 are the area and the moment of inertia of the largest cross-section, and m, n, are positive numbers.

Two particular cases are dealt with in some detail:

a) Circular cross-section with $n = 2p$, $m = 4p$ and $0.1 < p < 1$.

The first two non-dimensional frequencies $\lambda_i = \sqrt{\sqrt{\dfrac{\rho A_0 \omega_i^2 L^4}{E I_0}}}$ are given in Table 5 according to the EDSM, FEM and CDM, respectively, for a truncation factor $c = 0.4$.

b) Elliptic cross-section $n = p_1 + p_2$, $m = p_1 + 3p_2$, $c = 0.3$ and $p_1 = 0.3, 0.7$, $0.1 < p_2 < 1$.

As in Table 5, three sets of results are reported in Table 6, and in both the cases the CDM is nearer to the EDSM results than to the FEM results.

Table 5. Numerical comparison between the results in [6] and CDM. Circular cross-section.

$c = 0.4$						
	EDSM		FEM		CDM	
p	1st	2nd	1st	2nd	1st	2nd
0.1	3.19015	7.77380	3.21524	7.82701	3.19014	7.77367
0.2	3.25449	7.72284	3.27964	7.78458	3.25448	7.72272
0.3	3.31808	7.67068	3.34343	7.74074	3.31806	7.67056
0.4	3.38074	7.61739	3.40649	7.69554	3.38074	7.61727
0.5	3.44013	7.56198	3.46866	7.64903	3.44238	7.56291
0.6	3.50282	7.50765	3.52984	7.60125	3.50285	7.50755
0.7	3.56203	7.45133	3.58987	7.55227	3.56201	7.45123
0.8	3.61971	7.39411	3.64862	7.50211	3.61971	7.39402
0.9	3.67580	7.33606	3.70594	7.45084	3.67580	7.33597
1.0	3.73014	7.27722	3.76168	7.39850	3.73015	7.27714

Table 6. Numerical comparison between the results in [6] and CDM. Elliptic cross-section.

					EDSM		FEM		CDM	
c	p_1	p_2	m	n	1st	2nd	1st	2nd	1st	2nd
0.3	0.3	0.1	0.6	0.4	2.84831	6.73501	2.86186	6.77346	2.84830	6.73490
		0.2	0.9	0.5	2.87311	6.64825	2.88832	6.69790	2.87308	6.64810
		0.3	1.2	0.6	2.89672	6.56054	2.91376	6.62116	2.89672	6.56040
		0.4	1.5	0.7	2.91913	6.47194	2.93813	6.54326	2.91913	6.47182
		0.5	1.8	0.8	2.94029	6.38252	3.96133	6.46426	2.94023	6.38239
		0.6	2.1	0.9	2.95981	6.29231	2.98329	6.38421	2.95995	6.29220
		0.7	2.4	1.0	2.97812	6.20140	3.00393	6.30315	2.97819	6.20129
		0.8	2.7	1.1	2.99487	6.10984	3.023315	6.22114	2.99487	6.10973
		0.9	3.0	1.2	3.00852	6.01735	3.04087	6.13824	3.00990	6.01761
		1.0	3.3	1.3	3.02317	5.92507	3.05699	6.05451	3.02317	6.92498
	0.7	0.1	1.0	0.8	3.04548	6.90106	3.04583	6.91263	3.04541	6.88970
		0.2	1.3	0.9	3.06963	6.80113	3.07191	6.83524	3.06957	6.80099
		0.3	1.6	1.0	3.09245	6.71148	3.09686	6.75666	3.09246	6.71135
		0.4	1.9	1.1	3.11389	6.62094	3.12061	6.67692	3.11399	6.62082
		0.5	2.2	1.2	3.13410	6.52956	3.14308	6.59607	3.13410	6.52945
		0.6	2.5	1.3	3.15268	6.43741	3.16418	6.51415	3.15268	6.43730
		0.7	2.8	1.4	3.16966	6.34453	3.18383	6.43122	3.16966	6.34443
		0.8	3.1	1.5	3.1894	6.25100	3.20193	6.34732	3.18495	6.25091
		0.9	3.4	1.6	3.19843	6.15689	3.21839	6.26251	3.19844	6.15680
		1.0	3.7	1.7	3.21003	6.06266	3.23311	6.17686	3.21004	6.06217

5. The same structure has been studied by ZHOU *et al.* [7] for the particular case $n = 2$ and $m = 4$. The non-dimensional frequency coefficients $\Omega_i = \sqrt{\rho A_0 \omega_i^2 L^4 / EI_0}$ are given for various values of the truncation factor α, see Table 7, as obtained by the following five approaches:

a) Orthogonally generated polynomials as trial functions in the Rayleigh–Ritz energy approach [7], and 8 terms.

b) Generated polynomials as trial functions in the Rayleigh–Ritz method [20].

c) Exact solution [21].

d) Frobenius method [22].

e) CDM.

Table 7. Numerical comparison between the results in [7] and CDM.

α	Ref.	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0.2	(a)	6.1664	18.385	39.834	71.245	112.89
	(b)	6.1964	18.386	39.837	71.288	113.33
	(c)	6.1964	18.385	39.834	71.242	112.83
	(d)	6.1914	18.386	39.834	–	–
	(e)	6.1964	18.385	39.834	71.235	112.81
0.5	(a)	4.6252	19.548	48.579	91.816	149.43
	(c)	4.6252	19.548	48.579	91.813	149.39
	(d)	4.6252	19.548	48.579	–	–
	(e)	4.6252	19.548	48.577	91.806	149.37
0.8	(a)	3.8551	21.057	56.630	109.76	180.66
	(c)	3.8551	21.057	56.630	109.76	180.61
	(e)	3.8551	21.056	56.627	109.75	180.58

6. Let us consider now a set of assembled tapered beams, as given for example by MOU *et al.* [6]. The structure is given by a linearly tapered beam, an uniform beam and a non-uniform tapered beams assembled together. The first three non-dimensional frequencies are given in Table 8, and even in this case we observe the excellent agreement with the EDSM results.

7. Another interesting case is examined by LAURA *et al.* in [8]. The structure has rectangular cross-section and constant width. In the first span the height is supposed to vary according to the following linear law:

$$h(z) = h_0 \left(1 - \alpha \frac{z}{L}\right), \qquad 0 \le z \le L_1, \tag{3.8}$$

whereas in the second midspan the height has a constant value, given by:

$$h(z) = h_0 \left(1 - \alpha \frac{L_1}{L}\right), \qquad L_1 \le z \le L. \tag{3.9}$$

Table 8. Numerical comparison between the results in [6] and CDM. Three-segment beam with a linear segment, a constant segment and non-linear segment.

	EDSM			FEM			CDM		
p	1st	2nd	3rd	1st	2nd	3rd	1st	2nd	3rd
0.1	0.98852	2.37379	3.83817	0.89936	2.15550	3.52085	0.98851	2.37373	3.83795
0.2	1.01947	2.40456	3.84090	0.92184	2.18313	3.53005	1.01946	2.40450	3.84070
0.3	1.04900	2.43651	3.84342	0.94309	2.21167	3.53950	1.04899	2.43646	3.84323
0.4	1.07703	2.46952	3.84568	0.96310	2.24096	3.54919	1.07702	2.46948	3.84551
0.5	1.10239	2.50353	3.84717	0.98185	2.27079	3.55908	1.10351	2.50338	3.84747
0.6	1.28844	2.53801	3.84915	0.99936	2.30097	3.56913	1.12843	2.53798	3.84902
0.7	1.15180	2.57311	3.85015	1.01566	2.33129	3.57925	1.15179	2.57309	3.86004
0.8	1.17364	2.60850	3.85045	1.03080	2.36155	3.58930	1.17364	2.60849	3.85040
0.9	1.19402	2.64396	3.85059	1.04484	2.39154	3.59912	1.19401	2.64395	3.85055
1.0	1.21300	2.67923	3.85084	1.05784	2.42108	3.60849	1.21299	2.67927	3.85080

The first three non-dimensional frequencies Ω_i are calculated as in the Example 5, and A_0 and I_0 are the area and the moment of inertia of the initial section. The simply supported beam and the clamped-clamped beam are examined in the Tables 9–10, where the results obtained by the Differential Quadrature Method

Table 9. Numerical comparison between four different discretization methods, for simply supported two-segment beam. The first three non-dimensional frequencies are given for various values of α and $\gamma = L_1/L$.

		$\alpha = 0.1$			$\alpha = 0.2$			$\alpha = 0.3$		
γ		Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
0.25	(1)	9.629	38.56	86.84	9.387	37.64	84.86	9.145	36.72	82.87
	(2)	9.777	–	–	9.681	–	–	9.584	–	–
	(3)	9.627	–	–	9.388	–	–	9.143	–	–
	(4)	9.628	38.56	86.85	9.387	37.64	84.87	9.145	36.72	82.89
0.5	(1)	9.447	37.99	85.42	9.018	36.49	81.00	8.583	34.97	78.54
	(2)	9.733	–	–	9.577	–	–	9.404	–	–
	(3)	9.447	–	–	9.037	–	–	8.612	–	–
	(4)	9.446	37.99	85.43	9.018	36.49	82.01	8.583	34.97	78.55
0.75	(1)	9.374	37.56	84.59	8.863	35.62	80.29	8.331	33.64	75.91
	(2)	9.525	–	–	9.163	–	–	8.773	–	–
	(3)	9.382	–	–	8.870	–	–	8.338	–	–
	(4)	9.374	37.56	84.60	8.862	35.62	80.20	8.331	33.64	75.92

Table 10. Numerical comparison between four different discretization method, for clamped-clamped two-segment beam. The first three non-dimensional frequencies are given for various values of α and $\gamma = L_1/L$.

		$\alpha = 0.1$			$\alpha = 0.2$			$\alpha = 0.3$		
γ		Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
0.25	(1)	22.000	60.46	118.38	21.625			21.250		
	(2)	22.059	–	–	21.729	–	–	21.383	–	–
	(3)	22.005	–	–	21.635	–	–	21.266	–	–
	(4)	22.000	60.46	118.42	21.625	59.25	115.92	21.250	58.04	113.41
0.5	(1)	21.675	59.55	116.48	20.971	57.41	112.03	20.262	55.25	107.52
	(2)	21.979	–	–	21.567	–	–	21.134	–	–
	(3)	21.681	–	–	20.985	–	–	20.287	–	–
	(4)	21.675	59.56	116.50	20.971	57.42	112.04	20.261	55.25	107.54
0.75	(1)	21.432	58.89	115.31	20.471	56.06	109.65	19.488	53.16	103.85
	(2)	21.507	–	–	20.641	–	–	19.778	–	–
	(3)	21.435	–	–	20.476	–	–	19.497	–	–
	(4)	21.432	58.90	115.35	20.471	56.06	109.68	19.488	53.17	103.88

(DQM), the optimized Rayleigh–Ritz method and the Finite Element Method (FEM) are compared with the CDM results. Even in this case, our results give an excellent lower bound.

8. A similar structure has been studied in [10], where the free vibration frequencies of a two-beam structure on flexible supports are exactly calculated. The first beam constant has a cross-section, the second beam is defined by the following taper law:

$$A(z) = A_1 \eta^n, \qquad I(z) = I_1 \eta^{n+2}, \tag{3.10}$$

with:

$$\eta \left[1 + \frac{\alpha - 1}{L(1-\beta)} z \right], \tag{3.11}$$

and β is a multiplying factor of the span of the first beam, $\alpha = \dfrac{h_2}{h_1}$, $\dfrac{b_2}{b_1} = 1$ and A_1, I_1 are the cross-sectional area and the moment of inertia of the initial section.

For a clamped-clamped beam, the first five free non-dimensional vibration frequencies $p_i = \sqrt{\sqrt{\dfrac{\rho A_1 \omega_i^2 L^4}{E I_1}}}$ are given in Tables 11–12 for various β and α values, as obtained using an exact approach and our discretization method.

Table 11. Numerical comparison between the results in [9] and CDM. Two-segment beam β=0 and β=0.2.

α	$\beta = 0$					$\beta = 0.2$				
	p_1	p_2	p_3	p_4	p_5	p_1	p_2	p_3	p_4	p_5
1	4.73	7.8532	10.9956	14.1372	17.2788	–	–	–	–	–
	4.73	7.8529	10.9949	14.1358	17.2764	–	–	–	–	–
1.25	5.0098	8.3172	11.6449	14.9718	18.2988	4.9828	8.2468	11.5303	14.8165	18.1036
	5.0097	8.3168	11.6442	14.9703	18.2962	4.9827	8.2464	11.5290	14.8136	18.0985
1.43	5.1933	8.6210	12.0699	15.5179	–	–	–	–	–	–
	5.1946	8.6230	12.0724	15.5206	–	–	–	–	–	–
1.5	5.2636	8.7374	12.2325	15.7268	19.2213	5.2104	8.5986	12.0071	15.4214	18.8356
	5.2634	8.7370	12.2317	15.7253	19.2186	5.2103	8.5982	12.0057	15.4183	18.8307
1.54	5.3007	8.7988	12.3184	15.8373	–	–	–	–	–	–
	5.3021	8.8009	12.3210	15.8401	–	–	–	–	–	–
1.66	5.4215	8.9985	12.5975	16.1958	–	–	–	–	–	–
	5.4152	8.9879	12.5824	16.1759	–	–	–	–	–	–
1.75	5.4976	9.1242	12.7732	16.4215	20.0700	5.4186	8.9189	12.4404	15.9700	19.4973
	5.4975	9.1239	12.7724	16.4198	20.1671	5.4185	8.9185	12.4390	15.9669	19.4924
2	5.7159	9.4848	13.2769	17.0684	20.7145	5.6112	9.1246	12.8398	16.4741	20.1029
	5.7157	9.4844	13.2760	17.0666	20.8570	5.6111	9.2142	12.8384	16.4709	20.0982
2.25	5.9213	9.8238	13.7502	17.6761	21.6024	5.7910	9.4904	13.2118	16.9418	20.6627
	5.9211	9.8233	13.7492	17.6742	21.5992	5.7910	9.4899	13.2103	16.9386	20.6581
2.5	6.1159	10.1447	14.1981	18.2512	22.3047	5.9601	9.7498	13.5609	17.3789	21.1841
	6.1157	10.1412	14.1971	18.2492	22.3012	5.9600	9.7493	13.5594	17.3758	21.1796
2.75	6.3012	10.4501	14.6243	18.7983	22.9727	6.1199	9.9954	13.8907	17.7899	21.6727
	6.3010	10.4496	14.6232	18.7961	22.3691	6.1199	9.9950	13.8891	17.7867	21.6683
3	6.4785	10.7421	15.0317	19.3211	23.6112	6.2719	10.2293	14.2038	18.1780	22.1329
	6.4783	10.7416	15.0305	19.3189	23.6074	6.2719	10.2288	14.2022	18.1749	22.1286
4	7.1242	11.8048	16.5134	21.2222	25.9321	6.8185	11.0756	15.3250	19.5488	23.7544
	7.1240	11.8041	16.5119	21.2194	25.9275	6.8185	11.0751	15.3232	19.5459	23.7501
5	7.6947	12.7427	17.8202	22.8984	27.9780	7.2960	11.8213	16.2894	20.7025	25.1251
	7.6944	12.7419	17.8183	22.8951	27.9724	7.2960	11.8206	16.2876	20.6999	25.1205
10	9.9421	16.4342	22.9582	29.4844	36.0136	9.2302	14.7957	19.7536	24.8107	30.1851
	9.9412	16.4322	22.9544	29.4779	36.0034	9.2301	14.7949	19.7524	24.8078	30.1771

Table 12. Numerical comparison between the results in [9] and CDM. Two-segment beam β=0.4 and β=0.6.

α	$\beta = 0.4$					$\beta = 0.8$				
	p_1	p_2	p_3	p_4	p_5	p_1	p_2	p_3	p_4	p_5
1.25	4.9557	8.1761	11.4107	14.6498	17.9006	4.9355	8.1007	11.3025	14.4934	17.7072
	4.9556	8.1758	11.4100	14.6481	17.8979	4.9354	8.1003	11.3017	14.4917	17.7042
1.5	5.1583	8.4647	11.7714	15.0992	18.4409	5.1286	8.3130	11.5711	14.8030	18.0716
	5.1582	8.4643	11.7707	15.0976	18.4383	5.1285	8.3126	11.5702	14.8013	18.0684
1.75	5.3450	8.7264	12.0917	15.5022	18.9192	5.3117	8.5015	11.8086	15.0804	18.3887
	5.3449	8.7261	12.0910	15.5005	18.9166	5.3116	8.5011	11.8077	15.0786	18.3854
2	5.5203	8.9661	12.3812	15.8689	19.3490	5.4852	8.6737	12.0202	15.3335	18.6705
	5.5203	8.9657	12.3804	15.8671	19.3464	5.4851	8.6733	12.0192	15.3318	18.6671
2.25	5.6874	9.1869	12.6465	16.2057	19.7399	5.6491	8.8350	12.2099	15.5672	18.9256
	5.6873	9.1866	12.6457	16.2040	19.7373	5.6490	8.8346	12.2089	15.5654	18.9221
2.5	5.8481	9.3913	12.8926	16.5173	20.0994	5.8032	8.9891	12.3813	15.7842	19.1603
	5.8480	9.3910	12.8918	16.5155	20.0968	5.8031	8.9886	12.3803	15.7823	19.1567
2.75	6.0040	9.5812	12.1233	16.8069	20.4332	5.9473	9.1383	12.5373	15.9862	19.3793
	6.0039	9.5809	12.1225	16.8052	20.4304	5.9472	9.1379	12.5362	15.9843	19.3757
3	6.1559	9.7581	13.3414	17.0773	20.7456	6.0814	9.2844	12.6805	16.1745	19.5859
	6.1558	9.7578	13.3405	17.0755	20.7428	6.0813	9.2839	12.6793	16.1725	19.5822
4	6.7646	10.3592	14.1231	18.0024	21.8410	6.5221	9.8517	13.1664	16.8069	20.3242
	6.7345	10.3589	14.1221	18.0007	21.8377	6.5219	9.8513	13.1650	16.8046	20.3230
5	7.2772	10.8334	14.8061	18.7435	22.7672	6.8326	10.3908	13.5838	17.2834	20.9536
	7.2771	10.8331	14.8050	18.7417	22.7637	6.8323	10.3903	13.5824	17.2807	20.9495
10	9.4280	12.4761	17.2360	21.3717	25.8612	7.4616	12.0636	15.7599	22.7572	26.8834
	9.4279	12.4755	17.2349	21.3692	25.8572	7.4610	12.0623	15.7585	18.7567	22.7506

9. An interesting two-beams structure has been studied in [9], where the first beam is defined by the following taper ratio:

$$A(z) = A_1 \left[1 + \frac{\alpha_1 - 1}{\beta L} z\right]^n,$$

$$I(z) = I_1 \left[1 + \frac{\alpha_1 - 1}{\beta L} z\right]^{n+2}, \tag{3.12}$$

$$0 \leq z \leq \beta L,$$

whereas for the second beam we have:

$$A(z) = A_1 \left[\frac{\alpha_1\alpha_2 - \alpha_1}{L(1-\beta)}(z - L) + \alpha_1\alpha_2 \right]^n ,$$

(3.13)

$$I(z) = I_1 \left[\frac{\alpha_1\alpha_2 - \alpha_1}{L(1-\beta)}(z - L) + \alpha_1\alpha_2 \right]^{n+2}$$

and $\beta L \leq z \leq L$.

The structure is supposed to be clamped at left, and resting on an elastically flexible end at right.

The first three free non-dimensional frequencies p_i, as in Table 11, are given in Tables 13–14 for various β, α and various materials. Even in this last case, our results present an excellent lower bound.

10. The numerical example which is presented below was taken from Ref. [11]: in this paper, the problem of vibration of beam with rectangular cross-section, where the base is constant and the height is variable, was studied. In this case, the variation laws of cross-sectional area and moment of inertia are given by

$$A(z) = A_0 \left(\frac{z}{L}(\alpha - 1) + 1 \right) ,$$

(3.14)

$$I(z) = I_0 \left(\frac{z}{L}(\alpha - 1) + 1 \right)^3 ,$$

where A_0 and I_0 are the cross-sectional area and the moment of inertia of the initial beam, respectively, and $\alpha = h_2/h_1 = 0.5$, where h_1 and h_2 are the initial and final beam's cross-section height, respectively.

By using the data of the numerical example, p. 461 of the paper [11], the vibration frequencies are determined:

(3.15) $$f_i = \frac{\omega_i}{2\pi}.$$

In particular, in Table 15 the first seven vibration frequencies for a simply supported beam (Example (a)) and the first five vibration frequencies for a cantilever beam (Example (b)) are reported.

The problem of vibration frequencies is solved using the presented method and the Chebyshev series approximation: the obtained results show an excellent agreement.

In Appendix 1 the numerical program, using "Mathematica" code, is reported. The data refer to this particular case, as can be noted by the cross-sectional areas and moment of inertia expressions which are identical to those of Formula (3.14).

Table 13. Numerical comparison between the results in [10] and CDM. Two-segment beam with the first constant segment and the second variable segment. Wedge beam.

$\alpha_1 = \alpha_2 = 1.5$				
	β	p_1	p_2	p_3
Single material $\varepsilon = 1$ $\nu = 1$	0.2	2.960984	6.623893	10.653342
		2.961004	6.623736	10.652826
	0.4	2.955257	6.339284	10.116494
		2.955256	6.339196	10.116103
	0.6	2.831723	6.092949	9.714487
		2.831713	6.092845	9.714025
	0.8	2.610069	5.874837	9.440699
		2.609593	5.874592	9.439987
Aluminium $\varepsilon = 3$ $\nu = 2.88889$	0.2	3.341329	7.266658	10.982127
		3.341349	7.626415	10.981648
	0.4	3.638804	6.379341	10.365182
		3.638802	6.379263	10.364734
	0.6	3.352323	6.436236	9.652822
		3.352308	6.436130	9.652349
	0.8	2.809352	6.321979	9.990769
		2.809358	6.321652	9.989850
Steel-Aluminium $\varepsilon = 0.33333$ $\nu = 0.34615$	0.2	2.448228	6.152010	10.232177
		2.448235	6.151898	10.231652
	0.4	2.310298	5.987174	10.093364
		2.310299	5.987064	10.092935
	0.6	2.255394	5.732798	9.537642
		2.255445	5.732686	9.537146
	0.8	2.258520	5.874837	9.440699
		2.258414	5.453477	9.047567
Tungsten-Aluminium $\varepsilon = 0.2$ $\nu = 0.15$	0.2	2.223087	6.381117	10.567467
		2.224018	6.380980	10.566875
	0.4	2.050444	5.865405	10.710794
		2.051179	5.865266	10.710309
	0.6	2.006496	5.539217	9.568246
		2.006565	5.539101	9.567671
	0.8	2.060400	5.308516	8.883911
		2.060452	5.308366	8.883313
Aluminium-Tungsten $\varepsilon = 5$ $\nu = 6.66666$	0.2	3.223087	7.139275	10.91989
		3.221416	7.138936	10.619406
	0.4	3.797734	6.077439	9.997253
		3.797706	6.077367	9.996747
	0.6	3.527982	6.354572	9.292952
		3.527960	6.354485	9.292463
	0.8	2.858571	6.472948	10.206816
		2.858605	6.472582	10.205816

Table 14. Numerical comparison between the results in [10] and CDM. Two-segment beam with the first constant segment and the second variable segment. Cone beam.

$\alpha_1 = \alpha_2 = 1.5$				
	β	p_1	p_2	p_3
Single material $\varepsilon = 1$ $\nu = 1$	0.2	3.241992	6.890704	10.872190
		3.241997	6.890536	10.871682
	0.4	3.295517	6.585544	10.284771
		3.295519	6.585466	10.284399
	0.6	3.135124	6.355970	9.883609
		3.135132	6.355880	9.883170
	0.8	2.826031	6.098033	9.655944
		2.826067	6.097798	9.655200
Aluminium $\varepsilon = 3$ $\nu = 2.88889$	0.2	3.601942	7.570925	11.289006
		3.601924	7.570652	11.288512
	0.4	4.013659	6.625189	10.579703
		4.013651	6.625127	10.579268
	0.6	3.653699	6.759004	9.818821
		3.653686	6.758905	9.8183760
	0.8	3.005513	6.525742	10.241486
		3.005487	6.525412	10.240488
Steel-Aluminium $\varepsilon = 0.33333$ $\nu = 0.34615$	0.2	2.719027	6.366531	10.431558
		2.719020	6.363403	10.431012
	0.4	2.591143	6.257891	10.208067
		2.591070	6.257762	10.207648
	0.6	2.5200643	5.982285	9.7002030
		7.520663	5.982184	9.6997084
	0.8	2.484248	5.647080	9.2249650
		2.484001	5.646931	9.224344
Tungsten-Aluminium $\varepsilon = 0.2$ $\nu = 0.15$	0.2	2.485453	6.604193	10.794795
		2.485391	6.604093	10.794207
	0.4	2.303951	6.171426	10.786371
		2.303799	6.171287	10.785913
	0.6	2.247777	5.799427	9.727984
		2.247773	5.799312	9.727416
	0.8	2.280355	5.486273	9.059468
		2.280369	5.486060	9.058860
Aluminium-Tungsten $\varepsilon = 5$ $\nu = 6.66666$	0.2	3.444017	7.392360	10.993261
		3.444000	7.392008	10.992690
	0.4	4.142424	6.344458	10.200137
		4.142405	6.344406	10.199647
	0.6	3.816945	6.688564	9.456933
		3.816922	6.688486	9.456470
	0.8	3.048076	6.658406	10.461855
		3.048054	6.658042	10.460753

Table 15. Numerical comparison between the results in [11] and CDM. Non-prismatic beam. Example (a) – a simply supported beam. Example (b) – a cantilever beam.

Example (a)	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
This paper	188.44	757.97	1703.89	3027.50	4728.90	6808.07	9264.89	12099.20
[11]	188.44	757.99	1703.97	3027.75	4729.39	6808.80	9286.04	12103.70
Example (b)								
This paper	85.66	455.80	121.55	2350.21	3864.60	5746.44		
[11]	85.66	455.80	1215.48	2349.93	3862.32	5752.45		

11. Finally, in a recent paper [23] the free vibration frequency of an isotropic beam have been found, for a variable cross-section with an exponential law:

$$A(z) = A_0 e^{\delta z}, \qquad I(z) = I_0 e^{\delta z}, \tag{3.16}$$

where δ is the non-uniformity parameter.

In Table 16 the free vibration frequencies given in Table 1, p. 82 of the paper [23], have been reproduced using CDM. The agreement is very good, both for simply supported beams and for clamped-clamped beams. On the contrary, the discrepancies for the first two free frequencies in cantilever beams are noticeable, both for $\delta = -1, -2$ and for $\delta = 1, 2$, so that we have reproduced the calculations, as described in [19], and the newly calculated results show an excellent agreement with the CDM.

Consequently, it seems that the values given in [23] are misprinted.

4. Conclusions

The free vibration frequencies of tapered beams are studied, for arbitrary variation laws of cross-sectional area and moments of inertia, in the presence of rotationally and axially flexible supports. The beam is viewed as a set of rigid bars linked together at discrete sections, in which stiffness and mass are concentrated, and the resulting system with finite number of degrees of freedom is so simple to analyze to permit a careful discretization, using a large number of rigid bars (in our case, 300 bars). Several examples are treated in some details, comparing exact and approximate results from the literature, and the proposed approach always gives excellent results.

Table 16. Numerical comparison between the results in [22].

$\|\delta\|$	Mode number	Natural frequencies									
		SS		CC		CF					
		C.D.M.	[22]	C.D.M.	[22]	C.D.M.	[22]	C.D.M.	[22]	C.D.M.	[22]
0	1	9.86960	9.86960	22.37319	22.37327			3.51602	3.51602		
	2	39.47829	39.47841	61.67226	61.67281			22.03439	22.03449		
	3	88.82578	88.82643	120.90151	120.90338			61.69665	61.69721		
	4	157.91159	157.91367	199.85470	199.85945			120.90003	120.90191		
	5	246.73503	246.74011	298.54551	298.55552			199.85478	199.85953		
						$\delta < 0$		exact	excat	$\delta > 0$	
1	1	9.77291	9.77291	22.51158	22.51167	4.73491	4.72298	4.73491	2.56534	2.56534	2.85833
	2	39.57024	39.57036	61.85913	61.85968	24.20173	24.20168	24.20181	20.03838	20.03827	20.03917
	3	88.96986	88.97052	121.10610	121.10799	63.86395	63.86448			59.87027	59.87084
	4	158.08211	158.08418	200.06937	200.07411	123.09607	123.09790			119.09669	119.09862
	5	246.92142	246.92650	298.76659	298.77661	202.06410	202.06876			198.06480	198.06964
2	1	9.48725	9.48725	22.93763	22.93771	6.26264	6.25877	6.26264	1.84057	1.84053	2.90893
	2	39.85219	39.85231	62.42217	62.42272	26.58351	26.58350	26.58359	18.17212	18.17202	18.17520
	3	89.40455	89. 40520	121.72084	121.72272	66.37398	66.37449			58.38808	58.38868
	4	158.59481	158.59689	200.71386	200.71860	125.68293	125.68471			117.69019	117.69217
	5	247.48121	247. 48629	299.43011	299.44012	204.69073	204.69531			196.69732	196.70224

Appendix 1

```
Cell1[n_, span_, h1_, h2_, b_, young_, ρ_, kTL_, kTR_, kRL_, kRR_] :=
    Module[
        {i, j, t, α, I0, A0, z, m, inerz, are, k, V, Δ, K, M, FREQUENCIES},
        t = (span)/(n - 1); α = h2/h1; I0 = b * h1^3/12; A0 = b * h1;
    z = Table[0, {i, 1, n}]; m = Table[0, {i, 1, n}];
    inerz = Table[0, {i, 1, n}]; are = Table[0, {i, 1, n}];
k = Table[0, {i, 1, n}, {j, 1, n}]; V = Table[0, {i, 1, n - 1}, {j, 1, n}];
    Δ = Table[0, {i, 1, n}, {j, 1, n - 1}]; K = Table[0, {i, 1, n}, {j, 1, n}];
    M = Table[0, {i, 1, n}, {j, 1, n}]; FREQUENCIES = Table[0, {i, 1, n}, {j, 1, n}];
    z[[1]] = 0; z[[n]] = span; Do[z[[i]] = (i - 1) * t, {i, 2, n - 1}];
    Do[are[[i]] = A0 * (z[[i]]/span (α - 1) + 1), {i, 1, n}];
Do[inerz[[i]] = I0 * (z[[i]]/span (α - 1) + 1)^3, {i, 1, n}];
    m[[1]] = ρ * are[[1]] * t/2; m[[n]] = ρ * are[[n]] * t/2;
    Do[m[[i]] = ρ * are[[i]] * t, {i, 2, n - 1}];
k[[1, 1]] = young * inerz[[1]]/(t/2); k[[n, n]] = young * inerz[[n]]/(t/2);
    k[[1, 1]] = k[[1, 1]]/(1 + k[[1, 1]]/kRL);
k[[n, n]] = k[[n, n]]/(1 + k[[n, n]]/kRR);
    Do[k[[i, i]] = young * inerz[[i]]/t, {i, 2, n - 1}];
Do[V[[i, i]] = -1/t; V[[i, i + 1]] = 1/t, {i, 1, n - 1}];
    Do[Δ[[i, i]] = 1; Δ[[i + 1, i]] = -1, {i, 1, n - 1}];
Do[M[[i, i]] = 1/m[[i]], {i, 1, n}];
    K = Transpose[V].Transpose[Δ].k.Δ.V;
K[[1, 1]] = K[[1, 1]] + kTL; K[[n, n]] = K[[n, n]] + kTR;
FREQUENCIES = Sqrt[Chop[N[Eigenvalues[M.K]]]]/(2 π);
                                  Return[FREQUENCIES]];
```

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