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AXIALLY-SYMMETRIC SQUEEZED FILMS AS VISCOELASTIC FLOWS WITH DOMINATING EXTENSIONS

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Axially-symmetric continuous squeezing flows of visco elastic fluids with simulated approach velocities of the discs are considered as thin-layer flows with dominating extensions [1]. Certain approximate solutions are presented and possible effects of the extensional viscosity function on the load-bearing forces are discussed in greater detail.

1. Introduction

In our previous paper [1], we considered the concept of flows with dominating extensions (FDE) and its application to plane squeeze-film flows of viscoelastic fluids. This concept being inspired by the A. B. Metzner's idea of the extensional primary field (EPF) approximations [2], has led to interesting approximate solutions, especially in the case of the so-called "continuous squeezing flows" invented and studied experimentally by D. R. Oliver and H. Shahidullah [3, 4].

Although plane squeezing flows seem to be more important because of possible practical applications in various lubricating systems (bearings, gears, cams, etc.), axially-symmetric squeeze-film flows, i.e. compressive flows between two horizontal discs (cf. [3, 5]), are much easier for experimental investigations. This is a reason why in the present paper we briefly develop similar considerations in the case of axially-symmetric continuous squeezing flows. We also try to use certain results on the load-bearing forces for the extensional viscosity estimates.

2. Axially-symmetric continuous squeezing flows

Consider a test fluid contained between two horizontal discs of radii R, situated at the distance $2h$ and loaded with force P (Fig. 1). In the continuous squeezing flow, the fluid moves through a stationary lower disc,

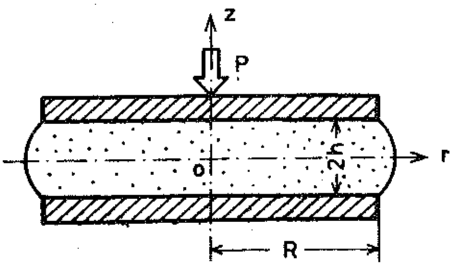

FIG. 1.

being extruded from numerous holes uniformly distributed over the lower surface (cf. [3, 4]). The force P is adjusted to a simulated constant approach velocity $-\dot{h}$ and a distance $2h$ between the discs.

Like in plane squeezing flows, we assume that

$$\varepsilon = \frac{h}{R} \ll 1, \qquad q = -\frac{\dot{h}}{h} = \text{const}, \tag{2.1}$$

i.e. that the fluid layer is sufficiently thin and the flow is a steady one.

The velocity field can be considered in the following form (cf. [1]):

$$\begin{aligned} u^* &= \frac{1}{2} qr + u(r, z), \\ v^* &= -qz + v(r, z), \end{aligned} \tag{2.2}$$

where u and v are additional velocity components in the system of cylindrical coordinates with the z-axis directed upwards (Fig. 1). Denoting

$$r = R\bar{r}, \qquad z = h\bar{z}, \qquad u = U\bar{u}, \qquad v = \varepsilon U\bar{v}, \tag{2.3}$$

where $U = qh$ is the characteristic velocity and overbars refer to dimensionless quantities, we have

$$\begin{aligned} &\frac{\partial u^*}{\partial r} = \frac{1}{2} q + \varepsilon q \frac{\partial \bar{u}}{\partial \bar{r}}, \qquad \frac{\partial u^*}{\partial z} = q \frac{\partial \bar{u}}{\partial \bar{z}}, \\ &\frac{u^*}{r} = \frac{1}{2} q + \varepsilon q \frac{\bar{u}}{\bar{r}}, \qquad \frac{\partial v^*}{\partial r} = \varepsilon^2 q \frac{\partial \bar{v}}{\partial \bar{r}}, \\ &\frac{\partial v^*}{\partial z} = -q + \varepsilon q \frac{\partial \bar{v}}{\partial \bar{z}}, \qquad \omega = \frac{1}{2} q \left(\frac{\partial \bar{v}}{\partial \bar{z}} - \varepsilon^2 \frac{\partial \bar{v}}{\partial \bar{r}} \right). \end{aligned} \tag{2.4}$$

The continuity equation gives the condition

$$\frac{\partial \bar{u}}{\partial \bar{r}} + \frac{\bar{u}}{\bar{r}} + \frac{\partial \bar{v}}{\partial \bar{z}} = 0. \tag{2.5}$$

The first parts of the diagonal components (2.4) may be much greater than the remaining parts if the dimensionless velocity gradients are of order $O(1)$ and the vorticity ω (or $\partial\bar{u}/\partial\bar{z}$) is sufficiently small. It is obvious that in the case of pure extensions ($\omega \equiv 0$), all the diagonal components are proportional to the extension rate q.

On the basis of Eqs. (2.2), we arrive at the following Rivlin-Ericksen kinematic tensors (cf. [6]):

$$(2.6)\qquad [\mathbf{A}_1^*] = [\mathbf{A}_1]+[\mathbf{A}_1]' = \begin{bmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & -2q \end{bmatrix} + \begin{bmatrix} 2\dfrac{\partial u}{\partial r} & 0 & \dfrac{\partial u}{\partial z}+\dfrac{\partial v}{\partial r} \\ 0 & 2\dfrac{u}{r} & 0 \\ \dfrac{\partial u}{\partial z}+\dfrac{\partial v}{\partial r} & 0 & 2\dfrac{\partial v}{\partial z} \end{bmatrix},$$

$$[\mathbf{A}_1^{*2}] = [\mathbf{A}_1^2]+[\mathbf{A}_1^2]' = \begin{bmatrix} q^2 & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & 4q^2 \end{bmatrix} +$$

$$+\begin{bmatrix} 4q\dfrac{\partial u}{\partial r}+4\left(\dfrac{\partial u}{\partial r}\right)^2+\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial v}{\partial r}\right)^2 & 0 & -\left(q+\dfrac{u}{r}\right)\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial v}{\partial r}\right) \\ 0 & 4q\dfrac{u}{r}+4\dfrac{u^2}{r^2} & \\ -\left(q+\dfrac{u}{r}\right)\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial v}{\partial r}\right) & 0 & -8q\dfrac{\partial v}{\partial z}+4\left(\dfrac{\partial v}{\partial z}\right)^2+\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial v}{\partial r}\right)^2 \end{bmatrix},$$

and at the invariants

$$\operatorname{tr}\mathbf{A}_1^* = 0,$$

$$(2.7)\qquad \operatorname{tr}\mathbf{A}_1^{*2} = \operatorname{tr}\mathbf{A}_1^2+(\operatorname{tr}\mathbf{A}_1^2)' = 6q^2+12q\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)+$$
$$+8\left(\frac{\partial u}{\partial r}\right)^2+8\frac{u^2}{r^2}+8\frac{u}{r}\frac{\partial u}{\partial r}+2\left(\frac{\partial u}{\partial z}+\frac{\partial v}{\partial r}\right)^2,$$

$$\operatorname{tr}\mathbf{A}_1^{*3} = \operatorname{tr}\mathbf{A}_1^3+(\operatorname{tr}\mathbf{A}_1^3)' = -6q^3-18q^2\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)-$$
$$-24q\frac{u}{r}\frac{\partial u}{\partial r}-12q\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)^2-24\frac{u}{r}\left(\frac{\partial u}{\partial r}\right)^2-24\frac{\partial u}{\partial r}\frac{u^2}{r^2}-$$
$$-\left(3q+4\frac{u}{r}\right)\left(\frac{\partial u}{\partial z}+\frac{\partial v}{\partial r}\right)^2.$$

The constitutive equations of an incompressible simple fluid in any extensional (irrotational) steady flow, treated as a motion with constant stretch history [6, 7], can be written in the form

$$\mathbf{T} = -p\mathbf{1}+\beta_1(I_2, I_3)\mathbf{A}_1+\beta_2(I_2, I_3)\mathbf{A}_1^2, \tag{2.8}$$

where p denotes the hydrostatic pressure, and

$$I_2 = \operatorname{tr}\mathbf{A}_1^2, \quad I_3 = \operatorname{tr}\mathbf{A}_1^3. \tag{2.9}$$

are the corresponding invariants.

Taking into account the order arguments following Eq. (2.4), we can define the axially-symmetric "*flows with dominating extensions*" (FDE) as such thin-layer flows in which the constitutive equation (2.8), valid for pure extensional flows, may be used in a form linearly perturbed with respect to the extension rate q. This means that

$$\mathbf{T}^* = -p\mathbf{1}+\beta_1\mathbf{A}_1+\beta_2\mathbf{A}_1^2+\beta_1\mathbf{A}_1'+\beta_2(\mathbf{A}_1^2)'+\frac{d\beta_1}{dq}q'\mathbf{A}_1+\frac{d\beta_2}{dq}q'\mathbf{A}_1^2, \tag{2.10}$$

where the linear increment of q, denoted by q', reads

$$q' = \frac{\partial u}{\partial r}+\frac{u}{r}+\frac{2}{3q}\left(\frac{\partial u}{\partial r}\right)^2+\frac{2}{3q}\frac{u^2}{r^2}+\frac{2}{3q}\frac{u}{r}\frac{\partial u}{\partial r}+\frac{1}{6q}\left(\frac{\partial u}{\partial z}+\frac{\partial v}{\partial r}\right)^2, \tag{2.11}$$

when determined on the basis of Eq. $(2.7)_2$.

Thus we have the following non-vanishing stress components:

$$\begin{aligned}
T^{*11} &= -p+\beta_1 q+2\beta_1\frac{\partial u}{\partial r}+\beta_2 q^2+4\beta_2 q\frac{\partial u}{\partial r}+4\beta_2\left(\frac{\partial u}{\partial r}\right)^2+\\
&\qquad +\beta_2\left(\frac{\partial u}{\partial z}+\frac{\partial v}{\partial r}\right)^2+\frac{d\beta_1}{dq}q'q+\frac{d\beta_2}{dq}q'q^2,\\
T^{*22} &= -p+\beta_1 q+2\beta_1\frac{u}{r}+\beta_2 q^2+4\beta_2 q\frac{u}{r}+4\beta_2\left(\frac{u}{r}\right)^2+\\
&\qquad +\frac{d\beta_1}{dq}q'q+\frac{d\beta_2}{dq}q'q^2,\\
T^{*33} &= -p-2\beta_1 q+2\beta_1\frac{\partial v}{\partial z}+4\beta_2 q^2-8\beta_2 q\frac{\partial v}{\partial z}+4\beta_2\left(\frac{\partial v}{\partial z}\right)^2+\\
&\qquad +\beta_2\left(\frac{\partial u}{\partial z}+\frac{\partial v}{\partial r}\right)^2-2\frac{d\beta_1}{dq}q'q+4\frac{d\beta_2}{dq}q'q^2,\\
T^{*13} &= (\beta_1-\beta_2 q)\left(\frac{\partial u}{\partial z}+\frac{\partial v}{\partial r}\right)-\beta_2\frac{u}{r}\left(\frac{\partial u}{\partial z}+\frac{\partial v}{\partial r}\right).
\end{aligned} \tag{2.12}$$

The above components introduced into the dynamic equations of equilibrium in the form

$$\tag{2.13} \begin{aligned} \frac{\partial p}{\partial r} &= \frac{\partial T_E^{*11}}{\partial r} + \frac{1}{r}(T_E^{*11} - T_E^{*22}) + \frac{\partial T^{*13}}{\partial z}, \\ \frac{\partial p}{\partial z} &= \frac{\partial T_E^{*33}}{\partial z} + \frac{1}{r} T^{*13} + \frac{\partial T^{*13}}{\partial r}, \end{aligned}$$

where the subscript E denotes the extra-stress components ($T_E^{*ii} = T^{*ii} + p$, $i = 1, 2, 3$) and the inertia terms have been disregarded[(1)], lead to equations involving terms of different orders of magnitude with respect to $\varepsilon = h/R$. Expressing these equations in a dimensionless form by means of Eqs. (2.3) and

$$\tag{2.14} p = \frac{U\eta R}{h^2}\bar{p}; \quad q = \frac{U}{L}\bar{q}, \quad \beta_1 = \eta\bar{\beta}_1, \quad \beta_2 = \eta\frac{h}{U}\bar{\beta}_2,$$

where overbars denote dimensionless quantities, and η has the dimension of viscosity, we may retain only terms of the highest order of magnitude with respect to $\varepsilon = h/R$ (e.g. $O(\varepsilon^{-2})$). Such a procedure gives

$$\tag{2.15} \begin{aligned} \frac{\partial p'}{\partial r} &= (\beta_1 - \beta_2 q)\frac{\partial^2 u}{\partial z^2} + \frac{\beta_2}{r}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{1}{6}\frac{d\beta_1}{dq}\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2, \\ \frac{\partial p'}{\partial z} &= (\beta_1 - \beta_2 q)\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) - \frac{1}{3}\frac{d\beta_1}{dq}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{1}{2}\frac{d\beta_2}{dq}q\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)^2, \end{aligned}$$

where we have denoted

$$\tag{2.16} -p' = -p + \beta_2 q + \beta_2\left(\frac{\partial u}{\partial z}\right)^2 + \frac{1}{3}\frac{d\beta_2}{dq}q\left(\frac{\partial u}{\partial z}\right)^2.$$

Eliminating pressure p' from Eqs. (2.15) by consecutive differentiation with respect to z and r, we obtain

$$\tag{2.17} \frac{\partial}{\partial z}\left[\frac{1}{2}\left(\frac{d\beta_1}{dq} + \beta_2 - \frac{d\beta_2}{dq}q\right)\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2 + (\beta_1 - \beta_2 q)\frac{\partial^2 u}{\partial z^2}\right] = 0.$$

An alternative system of Eqs. (2.15) can be derived introducing the notation

$$\tag{2.18} p^* = p' - (\beta_1 - \beta_2 q)\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + \left(\frac{1}{3}\frac{d\beta_1}{dq} - \frac{1}{2}\frac{d\beta_2}{dq}q\right)\left(\frac{\partial u}{\partial z}\right)^2.$$

Thus we arrive at

(1) The fluid inertia effects, which may be of importance in viscoelastic squeezing flows (cf. [5]), can be taken into account in an approximate way (Sect. 4).

$$\frac{dp^*}{dr} = -(\beta_1 - \beta_2 q)\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + (\beta_1 - \beta_2 q)\frac{\partial^2 u}{\partial z^2} + \\ + \frac{\beta_2}{r}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{d\beta_1}{dq} - \frac{d\beta_2}{dq} q\right)\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2,$$

(2.19)

$$\frac{\partial p^*}{\partial z} = 0.$$

It is also noteworthy that Eqs. (2.15) or (2.19) can be obtained directly from the following simplified constitutive equations:

(2.20)

$$T^{*11} = -p' + \beta_1 q + \frac{1}{6}\frac{d\beta_1}{dq}\left(\frac{\partial u}{\partial z}\right)^2,$$

$$T^{*22} = -p' + \beta_1 q + \frac{1}{6}\frac{d\beta_1}{dq}\left(\frac{\partial u}{\partial z}\right)^2 - \beta_2\left(\frac{\partial u}{\partial z}\right)^2,$$

$$T^{*33} = -p' - 2\beta_1 q + 3\beta_2 q^2 - \frac{1}{3}\frac{d\beta_1}{dq}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{1}{2}\frac{d\beta_2}{dq} q\left(\frac{\partial u}{\partial z}\right)^2,$$

$$T^{*13} = (\beta_1 - \beta_2 q)\frac{\partial u}{\partial z},$$

taking into account that $q' = 1/6q\,(\partial u/\partial z)^2$ (cf. Eq. (2.11)).

It can easily be observed that the function $\beta_1 - \beta_2 q$ is simply related to the elongational viscosity function $\eta^*(q)$ widely used in theoretical considerations and studied experimentally (cf. [2, 6]). For one-dimensional extension (or compression) in the direction of the z-axis, we have

$$\eta^*(q) = -\frac{T^{*33}}{q} = \frac{T^{*11} - T^{*33}}{q} = 3(\beta_1 - \beta_2 q) = 3\beta(q), \tag{2.21}$$

since then $\partial u/\partial z \equiv 0$. In our further considerations we shall call $\beta(q)$ — the extensional viscosity function.

3. Certain approximate solutions

Equation (2.17) is a third-order nonlinear partial differential equation, an analytical solution of which, even for the simplest boundary conditions, is not known at all.

Bearing in mind a possibility of approximate solutions valid for thin layers, we assume

$$u = rw(z), \tag{3.1}$$

where $w(z)$ is a function of z only. Introducing the above velocity into Eq. (2.17), and taking into account Eq. $(2.19)_1$, we arrive at

$$r\left[\beta w''(z)+\left(\frac{d\beta}{dq}+2\beta_2\right)w'^2(z)\right]=\frac{dp^*}{dr}(r), \tag{3.2}$$

where primes denote differentiation with respect to z, and $\beta(q)$ is defined by Eq. (2.21). It is quite clear that any solution of the equation

$$\beta w''(z)+\left(\frac{d\beta}{dq}+2\beta_2\right)w'^2(z)=C=\text{const}, \tag{3.3}$$

also satisfies Eq. (3.2), when $dp^*/dr=Cr$, i.e. for a parabolic dependence of p^* on r. Other nonparabolic distributions of the thrust p^* can be approximated under the assumption that C depends on r treated as some parameter in solutions of Eq. (3.3).

The simplified Eq. (3.3) is a special Riccati equation for $w'(z)$, and its solution satisfying the condition $w'(0)=0$, can be presented as

$$\begin{aligned} w'(z)&=\frac{B}{\sqrt{-AB}}\,\mathrm{tg}\left(\sqrt{-AB}\,z\right) \quad &&\text{for}\quad AB<0,\\ w'(z)&=\frac{B}{\sqrt{AB}}\,\mathrm{th}\left(\sqrt{AB}\,z\right) &&\text{for}\quad AB>0, \end{aligned} \tag{3.4}$$

where

$$A=\frac{1}{\beta}\left(\frac{d\beta}{dq}+2\beta_2\right),\qquad B=\frac{C}{\beta}=\frac{1}{\beta r}\frac{dp^*}{dr}. \tag{3.5}$$

This leads to

$$\begin{aligned} w(z)&=\frac{1}{A}\ln\frac{\cos\left(\sqrt{-AB}\,z\right)}{\cos\left(\sqrt{-AB}\,h\right)}-\frac{1}{2}q \quad &&\text{for}\quad AB<0,\\ w(z)&=\frac{1}{A}\ln\frac{\mathrm{ch}\left(\sqrt{AB}\,z\right)}{\mathrm{ch}\left(\sqrt{AB}\,h\right)}-\frac{1}{2}q &&\text{for}\quad AB>0, \end{aligned} \tag{3.6}$$

if the following boundary condition is assumed:

$$u^*(\pm h)=0 \quad\text{or}\quad w(\pm h)=-\frac{1}{2}q. \tag{3.7}$$

The positive sign of A corresponds, in principle, to an increasing function $\beta(q)=\beta_1-\beta_2 q$, while the negative one — to a decreasing $\beta(q)$ if, moreover, $-d\beta/dq>2\beta_2$. The sign of B depends on the sign of C, i.e. on whether the thrust p^* is an increasing or decreasing function of r. Symmetry of the problem considered implies that p^* should attain its maximum at $r=0$; therefore p^* is a decreasing function of r, and thus $C<0$ or $B<0$.

It is noteworthy that for Newtonian fluids ($\beta = \beta_0 = \text{const}$), we arrive at

$$w(z) = \frac{C_N}{2\beta_0}(z^2 - h^2) - \frac{1}{2}q \quad \text{for} \quad A = 0, \tag{3.8}$$

where C_N denotes the Newtonian quantity.

The continuity equation or the requirement that volume discharge Q is preserved during the flow considered, leads to

$$Q = -\pi R^2 \dot{h} = 2\int_0^h u^*|_{r=R}\, 2\pi R\, dz = qR + 2\int_0^h w(z)\, 2\pi R^2\, dz, \tag{3.9}$$

from which, in the Newtonian case, we obtain

$$C_N = \frac{3}{4}\frac{\beta_0 \dot{h}}{h^3} < 0 \quad \text{or} \quad \frac{dp^*}{dr} = C_N r. \tag{3.10}$$

It is quite reasonable to assume that in general

$$C = -\frac{H}{2}\frac{\beta(q)\, q}{h^2} < 0, \tag{3.11}$$

where H denotes a dimensionless number (3/2 for Newtonian fluids).

Thus, for viscoelastic fluids, Eqs. (3.6) introduced into Eq. (3.9) lead to the relationships

$$\begin{aligned} \frac{|\gamma|}{2H} &= -\left[\ln\cos\sqrt{|\gamma|} + \frac{1}{\sqrt{|\gamma|}}L\left(\sqrt{|\gamma|}\right)\right] \quad \text{for} \quad A > 0, \\ \frac{|\gamma|}{2H} &= \left[\ln \operatorname{ch}\sqrt{|\gamma|} - \frac{1}{\sqrt{|\gamma|}}\bar{L}\left(\sqrt{|\gamma|}\right)\right] \quad \text{for} \quad A < 0, \end{aligned} \tag{3.12}$$

where

$$\gamma = \frac{1}{2}AqH = \frac{1}{2\beta}\left(\frac{d\beta}{dq} + 2\beta_2\right)qH, \tag{3.13}$$

and

$$L(x) = -\int_0^x \ln\cos t\, dt, \qquad \bar{L}(x) = \int_0^x \ln \operatorname{ch} t\, dt, \tag{3.14}$$

are the Lobachevsky and modified Lobachevsky functions, respectively. These functions can easily be tabelarised (cf. [8]); their diagrams have been shown in our previous paper [1].

By way of illustration, in Fig. 2 we show the relation between $\sqrt{|\gamma|}$ and H (or C), resulting from Eqs. (3.12). It is seen that for positive as well as negative parameters A, the number H does not differ significantly from 3/2, when $\sqrt{|\gamma|}$ is not too large. This fact can be used in further numerical calculations.

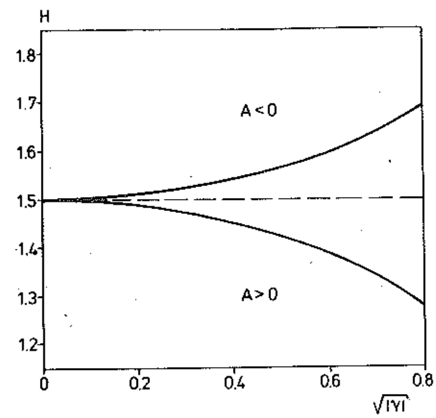

FIG. 2.

4. LOAD-BEARING FORCES AND BOUNDARY CONDITIONS AT THE FREE SURFACE

For any successful determination of the total force acting on the top disc, or the corresponding load-bearing force, one should assume something more on the boundary conditions at the free surface or at the edge. The problem would become very complex, if we wanted to take into account an unknown shape of the free surface, full surface tension effects and relations between the outer pressure and the wetting angle at the edge (cf. [9]). On the other hand, the obtained solutions are very sensitive to the boundary conditions considered at the free surface, as it will be seen below.

For sake of simplicity, we assume that the condition of vanishing total tension vector at the free surface, may be replaced either by

$$T^{*11}|_{r=R} = 0\,(= -p_a), \tag{4.1}$$

or by (cf. [5])

$$\int_0^h T^{*11}|_{r=R}\,dz = \int_0^h (-p' + T_E^{*11})|_{r=R}\,dz = 0\,(= -p_a), \tag{4.2}$$

if p_a denotes an atmospheric pressure, and the wetting angle at the wedge is assumed close to 90°. An effect of the surface tension can be taken into account by means of the following modification:

$$T^{*11}|_{r=R} = -p_a - HS\,\frac{\dot{h}R}{h^3}, \tag{4.3}$$

where S denotes the surface-tension coefficient.

An effect of fluid inertia can be introduced into the problem, considering a force resulting from the total mass balance. The value obtained by D. R. OLIVER [10] amounts to

$$F_I = 0.048\,\frac{\varrho \dot{h}^2 R^3}{h^2}, \tag{4.4}$$

where ϱ is a density of the fluid. Thus the load-bearing force P can be expressed as

$$P = F + F_I, \tag{4.5}$$

where F denotes the force exerted by the fluid on the top disc, calculated without taking into account inertia effects.

In our further considerations, we first apply the condition (4.1), which, on the basis of Eqs. (2.20), leads to

$$T^{*33}|_{r=R} = -3\,(\beta_1 - \beta_2\, q)\, q - \frac{1}{2}\left(\frac{d\beta_1}{dq} - \frac{d\beta_2}{dq}\, q\right) R^2 w'^2. \tag{4.6}$$

The force exerted by the fluid on the upper disc can be calculated from

$$F = -\int_0^R T^{*33}|_{z=\pm h}\, 2\pi r\, dr = (-T^{*33})\Big|_{\substack{r=R\\ z=\pm h}} \pi R^2 + \int_0^R \frac{\partial T^{*33}}{\partial r}\Big|_{z=\pm h} \pi r^2\, dr, \tag{4.7}$$

where integration by parts has been used. Bearing in mind Eqs. (4.6) and (2.13), we arrive at

$$F = 3\beta q \pi R^2 + \frac{1}{2}\left(\frac{d\beta_1}{dq} - \frac{d\beta_2}{dq}\, q\right) \pi R^4 w'^2(h) + \tag{4.8}$$

$$+ \int_0^R \left[-\frac{\partial T^{*13}}{\partial z} - \frac{1}{r}(T^{*11} - T^{*22}) - \frac{\partial}{\partial r}(T^{*11} - T^{*33})\right]_{z=\pm h} \pi r^2\, dr,$$

and, after introducing Eqs. (2.20) and (3.11), at

$$F = -\frac{1}{4}\, C\pi R^4 \left(1 + 2\,\frac{\dfrac{d\beta}{dq} + \beta_2}{\dfrac{d\beta}{dq} + 2\beta_2}\, \mathrm{tg}^2 \sqrt{\frac{1}{2\beta}\left(\frac{d\beta}{dq} + 2\beta_2\right) q H} + \frac{24}{H}\frac{h^2}{R^2}\right), \tag{4.9}$$

for $d\beta/dq+2\beta_2>0$. Disregarding the last term on the right-hand side of Eq. (4.9), since $\varepsilon=h/R\ll 1$, we finally obtain

$$(4.10)\qquad F=-\frac{H}{8}\frac{\beta\pi R^4\dot{h}}{h^3}\left[1+2\left(1-\frac{\beta_2}{\dfrac{d\beta}{dq}+2\beta_2}\right)\mathrm{tg}^2\sqrt{\frac{1}{2\beta}\left(\frac{d\beta}{dq}+2\beta_2\right)qH}\right]\qquad \text{for}\quad \frac{d\beta}{dq}+2\beta_2>0,$$

and

$$(4.11)\qquad F=-\frac{H}{8}\frac{\beta\pi R^4\dot{h}}{h^3}\left[1-2\left(1-\frac{\beta_2}{\dfrac{d\beta}{dq}+2\beta_2}\right)\mathrm{th}^2\sqrt{-\frac{1}{2\beta}\left(\frac{d\beta}{dq}+2\beta_2\right)qH}\right]\qquad \text{for}\quad \frac{d\beta}{dq}+2\beta_2<0.$$

In the case of Newtonian fluids, a similar procedure leads to the well-know relationship, viz.

$$(4.12)\qquad F_N=-\frac{3}{16}\frac{\beta_0\,\pi R^4\dot{h}}{h^3}.$$

It is easily seen from Eqs. (4.9) and (4.10) that an increasing extensional viscosity function $\beta(q)$, if moreover $d\beta/dq+2\beta_2>0$, may lead to apparent load-enhancement effects as compared with the Newtonian case. This fact is in a qualitative agreement with the experimental observations made by D. R. OLIVER and M. SHAHIDULLAH [3].

In many practical situations, in which the extension rate q is small enough and the extensional viscosity $\beta(q)$ does not increase too fast with q, we may use the following simplified expression:

$$(4.13)\qquad F=\frac{H}{8}\frac{\pi R^4 q}{h^2}\left[\beta+\left(\frac{d\beta}{dq}+\beta_2\right)qH\right],$$

valid for $d\beta/dq>0$ and $d\beta/dq<0$ as well.

For further comparisons, we shall apply the integral condition (4.2). On using Eqs. (2.18) and $(2.20)_1$ and bearing in mind that the thrust p^* does not depend on z, we arrive at

$$(4.14)\qquad p^*|_{r=R}=-\frac{2\beta}{A}\left[\ln\cos\sqrt{|\gamma|}+\frac{1}{\sqrt{|\gamma|}}L\left(\sqrt{|\gamma|}\right)\right]+$$

$$+\beta_1 q-\frac{1}{2}CR^2\frac{\dfrac{d\beta}{dq}+\beta_2}{\dfrac{d\beta}{dq}+2\beta_2}\frac{1}{\sqrt{|\gamma|}}\left(\mathrm{tg}\sqrt{|\gamma|}-\sqrt{|\gamma|}\right),\qquad \text{for}\quad A>0,$$

where the same notations as those used in Eqs. (3.12) have been applied. On the other hand, integration of Eq. (2.19)$_1$ leads to the following expression for p^*:

$$(4.15)\quad p^* = \frac{1}{2}\beta w''(h)(r^2-R^2)+\frac{1}{2}\beta_2 w'^2(h)(r^2-R^2)+ +\frac{1}{2}\left(\frac{d\beta}{dq}+\beta_2\right)w'^2(h)(r^2-R^2)-\frac{2\beta}{A}\left[\ln\cos\sqrt{|\gamma|}+\frac{1}{\sqrt{|\gamma|}}L(\sqrt{|\gamma|})\right]+ +\beta_1 q-\frac{1}{2}CR^2\frac{\frac{d\beta}{dq}+\beta_2}{\frac{d\beta}{dq}+2\beta_2}\left(\frac{\operatorname{tg}\sqrt{|\gamma|}}{\sqrt{|\gamma|}}-1\right)\quad \text{for}\quad A>0.$$

Thus the force exerted by the fluid on the top disc can be calculated from

$$(4.16)\quad F = -\int_0^R (-p'+T_E^{*33})_{z=\pm h}\,2\pi r\,dr = = \int_0^R \left(p^*-\left(\frac{1}{3}\frac{d\beta_1}{dq}-\frac{1}{2}\frac{d\beta_2}{dq}q\right)w'^2(h)r^2\right)2\pi r\,dr.$$

Introducing Eq. (4.15) into Eq. (4.16) and performing necessary integrations, we arrive at

$$(4.17)\quad F = -\frac{1}{4}C\pi R^4\left[1+2\frac{\frac{d\beta}{dq}+\beta_2}{\frac{d\beta}{dq}+2\beta_2}\left(\frac{\operatorname{tg}\sqrt{|\gamma|}}{\sqrt{|\gamma|}}-1\right)\right]\quad \text{for}\quad A>0,$$

where the terms proportional to $\varepsilon^2 = h^2/R^2$ have been disregarded.

Repeating the above procedure also for $A<0$, and taking into account Eq. (3.11), we obtain finally

$$(4.18)\quad F = -\frac{H}{8}\frac{\beta\pi R^4 h}{h^3}\left[1+2\left(1-\frac{\beta_2}{\frac{d\beta}{dq}+2\beta_2}\right)\left(\frac{\operatorname{tg}\sqrt{\frac{1}{2\beta}\left(\frac{d\beta}{dq}+2\beta_2\right)qH}}{\sqrt{\frac{1}{2\beta}\left(\frac{d\beta}{dq}+2\beta_2\right)qH}}-1\right)\right] \quad \text{for}\quad \frac{d\beta}{dq}+2\beta_2>0,$$

and

$$(4.19)\quad F=-\frac{H}{8}\frac{\beta\pi R^4\dot{h}}{h^3}\left[1-2\left(1-\frac{\beta_2}{\frac{d\beta}{dq}+2\beta_2}\right)\times\right.$$

$$\left.\times\left(1-\frac{\operatorname{th}\sqrt{-\frac{1}{2\beta}\left(\frac{d\beta}{dq}+2\beta_2\right)qH}}{\sqrt{-\frac{1}{2\beta}\left(\frac{d\beta}{dq}+2\beta_2\right)qH}}\right)\right]\quad\text{for}\quad\frac{d\beta}{dq}+2\beta_2<0.$$

The above expressions differ essentially from those described by Eqs. (4.10) and (4.11). Although an increasing viscosity function $\beta(q)$, if moreover $d\beta/dq+2\beta_2>0$, leads to apparent load-enhancement effects, they are less pronounced. This fact is also seen from the following simplified equation:

$$(4.20)\qquad F=\frac{H}{8}\frac{\pi R^4 q}{h^2}\left[\beta+\frac{1}{3}\left(\frac{d\beta}{dq}+\beta_2\right)qH\right],$$

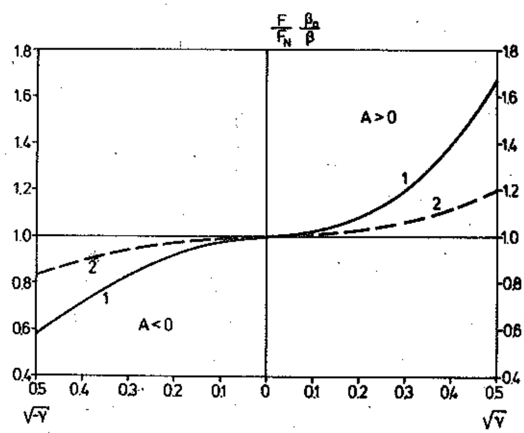

FIG. 3.

valid for sufficiently small extension rates q and not too fast varying extensional viscosities $\beta(q)$.

By way of illustration, Fig. 3 shows the dependence of $F\beta_0/F_N\beta$ on $\sqrt{\gamma}$ or $\sqrt{-\gamma}$ under the assumption that $\beta_2=0$, and the extensional viscosity is a linear function of extension rate q, viz.

$$(4.21)\qquad \beta(q)=\beta_1=\beta_0+\beta_{01}q,\qquad \beta_0=\text{const},\qquad \beta_{01}=\text{const}.$$

The solid line (*1*) corresponds to the force described by Eq. (4.10), while the broken line (*2*) — to that described by Eq. (4.18). It is also quite clear that a load-enhancement effect is much weaker in the second case, i.e. for the integral boundary condition (4.2) at the free surface.

In our previous paper [1], we expressed some doubts whether, in more general cases, any measurement of the load-bearing forces may be useful for determination of the extensional viscosity function $\beta(q)$ since these forces depend not only on $\beta(q)$ itself, but also on its derivatives $d\beta/dq$. In certain particular cases, however, some very crude estimates can be made.

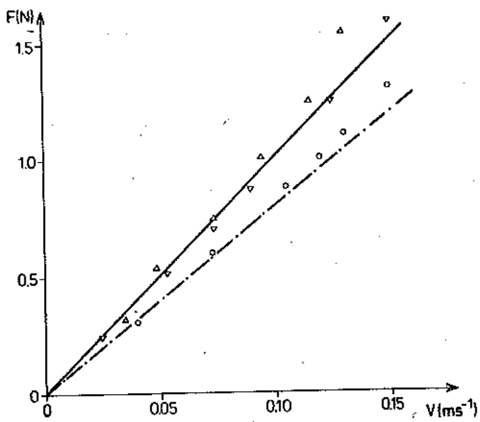

FIG. 4.

Figure 4 shows the experimental data obtained by D. R. OLIVER and M. SHAHIDULLAH [3] for the Esso motor oil thickened with a vinyl acetate/N-vinylpyrrolidone/C_8—C_{18} alkyl fumarate (VAVPAF). At 75°C its viscosity at shear rate 10^4 s^{-1} was 0.0263 Pas, density — 0.846 g/cm³, and flow behaviour index 0.886. The experimental points were measured in the continuous squeezing flow system without any distributor (circles) and with a distributor (triangles). The radius of the upper disc was 5.5 mm and the true corrected gaps ($2h$) between the discs varied from 240 to 240.5 μm.

The above experimental results can also be interpreted according to Eq. (4.13) under the assumption that $H = 3/2$ and $\beta_2 q \simeq 0$ as compared with β_1 described by the linear relationship (4.21). This can be done since for the maximum $q = 1.25 \cdot 10^3$ s^{-1}, the parameter $\gamma = \frac{1}{2}$ AqH (cf. Eq. (3.13)) is really very small. Under such conditions Eqs. (4.5) and (4.13) lead to

$$P = F_1 + F_N \left(1 + 2.5 \frac{\beta_{01}}{\beta_0 h} V\right), \tag{4.22}$$

where the fluid inertia force F_I is defined by Eq. (4.4), and $V = -\dot{h}$ denotes the simulated approach velocity of the discs. In Fig. 4, the broken line corresponds to purely Newtonian behaviour with $\beta_0 = 2.56\times10^{-2}$ Pas, while the solid one is adjusted to the experimental points obtained with a distributor (triangles), by means of Eq. (4.22) with $\beta_{01}/\beta_0 = 5.76\times10^{-5}$ s. The experimental points obtained without any distributor (circles) describe approximately the case of purely viscous fluid with inertia effects, like in the paper [3].

Thus we may conclude that the behaviour of the VAVPAF thickened oil in the continuous squeezing flow considered could be described, in an approximate way, by the relation (4.22), if the elongational viscosity $\eta^*(q)$ increased linearly in agreement with the formula

$$\eta^*(q) = 3\beta(q) = 7.68\times10^{-2}(1+5.76\times10^{-5}q)\ \mathrm{Pas}. \tag{4.23}$$

It can easily be deduced from Eq. (4.20) that, in the case of the integral boundary condition (4.2), the resulting value of β_{01}/β_0 is 5/3 times higher.

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Streszczenie

OSIOWO-SYMETRYCZNE WYCISKANIE WARSTWY JAKO LEPKOSPRĘŻYSTY PRZEPŁYW Z DOMINUJĄCYM ROZCIĄGANIEM

Osiowo-symetryczne, ciągłe przepływy wyciskające cieczy lepkosprężystych, z symulowanymi prędkościami zbliżania się tarcz, rozważono jako przepływy w cienkich warstwach z dominującym rozciąganiem [1]. Przedstawiono niektóre przybliżone rozwiązania oraz przedyskutowano szczegółowo możliwy wpływ funkcji lepkości przy rozciąganiu na siły nośne.

Резюме

ОСЕСИММЕТРИЧЕСКОЕ ВЫДАВЛИВАНИЕ СЛОЯ
КАК ВЯЗКОУПРУГОЕ ТЕЧЕНИЕ С ДОМИНИРУЮЩИМ РАСТЯЖЕНИЕМ

Осесимметричные сплошные выдавливающие течения вязкоупругих жидкостей, с имитирующими скоростями сближения дисков, рассматриваются как течения в тонких слоях с доминирующим растяжением [1]. Представлены некоторые приближенные решения, а также обсуждено подробно возможное влияние функции вязкости при растяжении на несущие силы.

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