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FINITE DEFORMATION OF NONLINEARLY ELASTIC RING

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The object of this paper is the analysis of compression of a ring made of incompressive material having nonlinear properties, described by the Mooney's equation. The case of compressing a ring without sliding between two plane rigid plates was discussed. The resolution of the problem for finite deformations using the hypothesis of plane sections was obtained by way of numerical minimizing the strain energy of a discrete model of a ring.

NOTATION

a_i, A_i parameters,
C_1, C_2 material constants,
D tensor of finite deformations,
E strain energy,
$e_{rr}, e_{\varphi\varphi}, e_{zz}, e_{zr}$ components of the strain tensor,
f deflection,
F compressive force,
i index,
I_1, I_2, I_3 invariants of tensor D,
n natural number,
$\mathbf{p}$ vector of stresses,
r, φ, z cylindrical coordinates,
r_0 radius of surface on which the radial displacements are zero,
r_1, r_2, h dimensions of the ring,
R, Z coordinates after deformation,
R_1, R_2 internal and external radii of the ring after deformation,
u, v, w components of the displacement,
U specific strain energy,
V ring volume,
W work of external forces,
ζ nondimensional coordinate,
λ_i principal components of tensor D,
π 3.14,
σ_i principal components of the actual stress tensor,
σ_p hydrostatic pressure,
$\bar{\sigma}_r, \bar{\sigma}_\varphi, \bar{\sigma}_z, \tau_{rz}$ components of the actual stress deviator.

1. Introduction

The object of this paper is the analysis of compression of a ring made of incompressive material having nonlinear properties, described by the Mooney's equation. A case was discussed of compression of a ring without sliding between two plane rigid plates. The ring had an internal radius r_1, external r_2 and thickness h (Fig. 1). The solution of the problem for finite deformations using the hypothesis of plane sections was obtained by numerical minimization of the strain energy.

A similar object was discussed analytically in the papers [1, 2] for some particular boundary conditions.

2. The mathematical model of the ring

The problem will be solved in cylindrical coordinates r, φ, z (Fig. 1). The components of the displacement vector of the material point in this system, considering cylindrical symmetry, are described by $u(r, z)$, $v = 0$, $w(r, z)$. Considering the symmetry about the plane $z = 0$

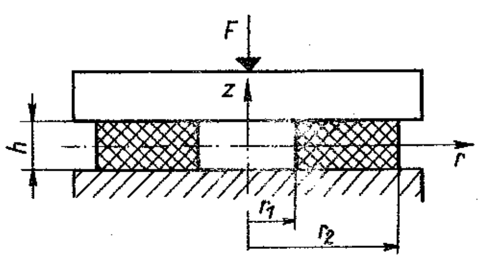

Fig. 1. The ring compressed between two rigid plates.

(2.1) $$u(r, z) = u(r, -z),$$

(2.2) $$w(r, z) = -w(r, -z),$$

and the contact without sliding with the rigid plates

(2.3) $$u\left(r, \frac{h}{2}\right) = 0,$$

(2.4) $$w\left(r, \frac{h}{2}\right) = -\frac{f}{2},$$

where f is the deflection of the whole ring.

On the surfaces not contacting with the plates

$$\mathbf{p}(r_1, z) = 0, \quad \mathbf{p}(r_2, z) = 0, \tag{2.5}$$

where $\mathbf{p}$ is the vector of stresses acting on an element of the external surface.

Taking into account the symmetry about the z-axis, we represent the tensor of finite deformations [3] in the form

$$D = \begin{pmatrix} 1+2e_{rr} & 0 & e_{zr} \\ 0 & 1+2e_{\varphi\varphi} & 0 \\ e_{zr} & 0 & 1+2e_{zz} \end{pmatrix}, \tag{2.6}$$

where

$$\begin{aligned} e_{rr} &= \frac{\partial u}{\partial r} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2\right], \\ e_{\varphi\varphi} &= \frac{u}{r} + \frac{1}{2}\left(\frac{u}{r}\right)^2, \\ e_{zz} &= \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right], \\ e_{zr} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial r}. \end{aligned} \tag{2.7}$$

Expressing the invariants of the tensor D with the help of elongations λ_i $(i = 1, 2, 3)$, we obtain

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ I_2 &= \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \\ I_3 &= \lambda_1^2\lambda_2^2\lambda_3^2, \end{aligned} \tag{2.8}$$

where the elongations λ_i, defined as relations of the length of the elementary line element after deformation to its initial length in principal directions, are linked with the components of the tensor D by the formulae

$$\begin{aligned} \lambda_1^2 &= 1 + e_{rr} + e_{zz} + \sqrt{(e_{rr} - e_{zz})^2 + e_{zr}^2}, \\ \lambda_2^2 &= 1 + 2e_{\varphi\varphi}, \\ \lambda_3^2 &= 1 + e_{rr} + e_{zz} - \sqrt{(e_{rr} - e_{zz})^2 + e_{zr}^2}. \end{aligned} \tag{2.9}$$

The value of the third invariant for rubber-like materials can be accepted:

$$I_3 = 1. \tag{2.10}$$

The above relationship expresses the postulate of incompressibility of the material. It can be presented with the help of components of displacements in the form

$$\left[\left(1+\frac{\partial u}{\partial r}\right)\left(1+\frac{\partial w}{\partial z}\right)-\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\right]\left(1+\frac{u}{r}\right)=1. \tag{2.11}$$

The displacements u and w are mutually depending.

If the approximating functions of the displacements depend on the unknown parameters a_i $(i=1,2,...,n)$

$$\begin{aligned} u &= u(r,z;a_1,a_2,...,a_n),\\ w &= w(r,z;a_1,a_2,...,a_n), \end{aligned} \tag{2.12}$$

then the strain energy of the ring

$$E=\int_V U\,dV, \tag{2.13}$$

where U — specific strain energy, V — ring volume; depends also on the parameters

$$E=E(a_1,a_2,...,a_n). \tag{2.14}$$

According to the principle of conservation of energy, the work of external forces acting on the system is equal to the strain energy E:

$$W=E, \tag{2.15}$$

and

$$\delta W=\delta E. \tag{2.16}$$

For the variations of the parameters a_i selected in such a manner that

$$\delta w\big|_{z=\frac{h}{2}}=0, \tag{2.17}$$

and in regard of the relation (2.5)

$$\delta W=0, \tag{2.18}$$

thus

$$\delta E=0. \tag{2.19}$$

From the above results it follows that among all possible states defined by parameters a_i selected so as to keep the work of external forces constant, the body in stable form of equilibrium assumes this state for which the strain energy is minimum. The solution of the problem consists in defining the state of deformation of a nonlinearly elastic body in the case of finite deformations is limited to finding a system of parameters a_i which, for a given deflection f of the ring, minimize the strain energy E.

Assuming that the specific strain energy for rubber-like materials is described by the Mooney's formulae [4],

$$U = C_1 (I_1 - 3) + C_2 (I_2 - 3), \tag{2.20}$$

where C_1, C_2 are material constants.

From Eqs. (2.13) and (2.20) it follows that the increment of strain energy of the ring

$$dE = \int_V (C_1\, dI_1 + C_2\, dI_2)\, dV. \tag{2.21}$$

Making use of the energy conservation principle, we define the compressive force

$$F = \frac{dE}{df}. \tag{2.22}$$

The increment of specific strain energy is connected with the principal components of the strain tensor and actual stress tensor (Cauchy) σ_i by the term

$$dU = \frac{\sigma_1}{\lambda_1} d\lambda_1 + \frac{\sigma_2}{\lambda_2} d\lambda_2 + \frac{\sigma_3}{\lambda_3} d\lambda_3. \tag{2.23}$$

For the Mooney body the components of the stress deviator

$$\bar{\sigma}_i = \sigma_i - \sigma_p = 2\left[C_1\left(\lambda_i^2 - \frac{I_1}{3}\right) - C_2\left(\frac{1}{\lambda_i^2} - \frac{I_2}{3}\right)\right], \tag{2.24}$$

where σ_p is the hydrostatic pressure defined by the conditions of equilibrium.

3. The choice of displacement functions

The incompressibility condition (2.11) can be presented in the form

$$\frac{\partial}{\partial r}(r+u)^2 \frac{\partial}{\partial z}(z+w) - \frac{\partial}{\partial r}(z+w)\frac{\partial}{\partial z}(r+u)^2 = 2r. \tag{3.1}$$

For rings of low height h axially compressed we assume the hypothesis of plane sections; this means that plane sections perpendicular to the axis z before deformation remain after deformation plane and perpendicular to the axis z. Hence the axial displacement $w = w(z)$ and

$$\frac{\partial w}{\partial r} = 0. \tag{3.2}$$

Then, from Eq. (3.1) the relationship between the displacements $u(r, z)$ and $w(z)$ results in the form

(3.3) $$u(r,z)=\left[\frac{r^2}{1+\dfrac{dw(z)}{dz}}+\varphi(z)\right]^{\frac{1}{2}}-r,$$

where $\varphi(z)$ is a function of an integration constant nature.

In the ring there exists a surface on which the radial displacement is equal to zero. We describe this surface by the equation

(3.4) $$r(z)=r_0(1+A_0)\left[1+A_2(1-\zeta^2)+A_4(1-\zeta^2)^2+\ldots\right]^{\frac{1}{2}},$$

where A_i $(i=0,2,4,\ldots)$ are parameters,

$$\zeta=\frac{2z}{h},\; r_0=r_1\left[\frac{\ln\left(\dfrac{r_2}{r_1}\right)^2}{\left(\dfrac{r_2}{r_1}\right)^2-1}+\frac{1}{3}\left(\frac{h}{r_2}\right)^2\right]^{\frac{1}{2}}$$

is the radius of the surface on which the radial displacements are zero in the solution of the linear problem of compression of a ring, based also on the hypothesis of plane sections.

Thus the function

(3.5) $$\varphi(z)=\frac{\dfrac{dw(z)}{dz}}{1+\dfrac{dw(z)}{dz}}\left[1+A_2(1-\zeta^2)+A_4(1-\zeta^2)^2+\ldots\right](1+A_0)^2r_0^2.$$

The function $w(z)$ describing the axial displacement has to fulfill the conditions (2.2) and (2.4) and also the condition resulting from Eqs. (3.1), (2.3) and the hypothesis of plane sections:

(3.6) $$\left.\frac{\partial w}{\partial z}\right|_{z=\frac{h}{2}}=0.$$

We assume this function to have the form

(3.7) $$w(z)=\frac{1}{4}f\zeta\left[\zeta^2-3+A_1(1-\zeta^2)^2+A_3(1-\zeta^2)^3+\ldots\right],$$

where A_i $(i=1,3,5,\ldots)$ are the parameters.

This form assures simultaneously the fulfilment of the conditions (2.1) and (2.3) of the radial displacements.

The parameters A_i $(i=0,1,2,\ldots)$ appearing in the functions of displacements approach zero for $f\to 0$. In the case of finite deformations their values will be defined by the law of strain energy minimum.

4. Discontinuity of the continuous ring model

The discussed ring in the above description was treated as a continuous body. For computer calculations the continuous model is replaced by a discrete model. In these calculations the discrete model presented in Fig. 2a was used. The continuous body was replaced by a finite number of points (nodes), from which any represented the surrounding space. In accordance with the relationships referred in the previous chapters, the value of the specific strain energy for the accepted parameters A_i was calculated for every node. The strain energy of the ring was defined by Simpson's method of numerical integration.

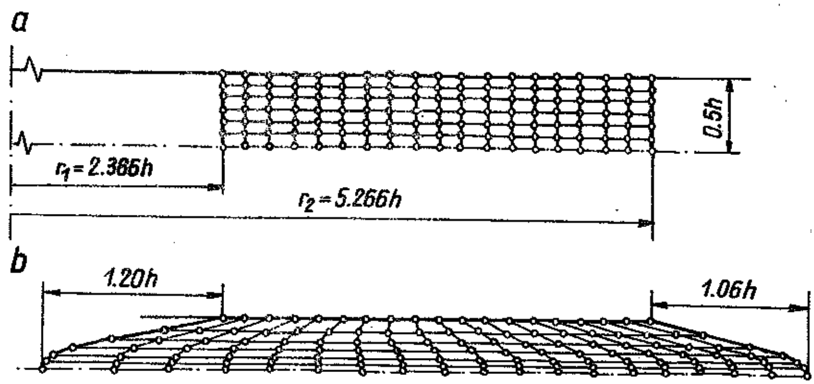

FIG. 2. a) Discontinuous model of an axial section of a ring.
b) Deformed section for nondimensional deflection 0.333.

The distribution of the specific strain energy, assuring the fulfilment of equilibrium conditions in a body, and the true values of strain energy were obtained by optimization of the parameters A_i.

Repeating the described procedure for the following increasing deflections, the increments of energy were calculated and, in accordance with the relation (2.22), the forces F compressing the ring.

The hydrostatic pressure $\sigma_p(r, z)$ which is needed to define the total stresses σ_i from Eq. (2.24) was calculated by integrating the system of equations

$$(4.1)\qquad \begin{aligned} \frac{\partial \sigma_p}{\partial R} &= -\frac{\partial \bar{\sigma}_r}{\partial R} - \frac{\partial \tau_{rz}}{\partial Z} - \frac{\bar{\sigma}_r - \bar{\sigma}_\varphi}{R}, \\ \frac{\partial \sigma_p}{\partial Z} &= -\frac{\partial \bar{\sigma}_z}{\partial Z} - \frac{\partial \tau_{rz}}{\partial R} - \frac{\tau_{rz}}{R}, \end{aligned}$$

using the method of trapezoids. The system (4.1) results from equilibrium conditions for a deformed body. In this system

$$R = r+u(r,z), \quad Z = z+w(z), \tag{4.2}$$

are coordinates of material points after deformation.

Since the displacements u, w are described in coordinates before deformation, the elongations λ_i and the components of the stress deviator (2.24) are also functions of the coordinates r, z. Taking this into account when calculating the right sides of the equations (4.1), the formulae

$$\begin{aligned} \frac{\partial(\cdot)}{\partial R} &= \frac{R}{r}\left[\left(1+\frac{\partial w}{\partial z}\right)\frac{\partial(\cdot)}{\partial r} - \frac{\partial w}{\partial r}\frac{\partial(\cdot)}{\partial z}\right], \\ \frac{\partial(\cdot)}{\partial Z} &= \frac{R}{r}\left[\left(1+\frac{\partial u}{\partial r}\right)\frac{\partial(\cdot)}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial(\cdot)}{\partial r}\right], \end{aligned} \tag{4.3}$$

were used.

The integration constant of the system (4.1) was defined from the condition

$$F = 2\pi \int_{R_1}^{R_2} [\bar{\sigma}_z(R,0)+\sigma_p(R,0)]\, R\, dR, \tag{4.4}$$

where

$$R_i = r_i + u(r_i, 0), \qquad i = 1, 2. \tag{4.5}$$

This condition, in the coordinates before deformation, takes the form.

$$F = 2\pi \int_{r_1}^{r_2} [\bar{\sigma}_z(r,0)+\sigma_p(r,0)]\left(1+\frac{u}{r}\right)\left(1+\frac{\partial u}{\partial r}\right) r\, dr. \tag{4.6}$$

5. Example

Computer calculations were performed for a ring having proportions presented in Fig. 2. As regards symmetry, half of the axial section of the ring was discreted. The ratio of the material constants $C_2 : C_1$ was assumed to be 0.03039. In the displacement functions only the parameters A_0, A_1, A_2 were taken into account. The values of those parameters were obtained by minimizing the functional (2.13) by the optimization method of the golden section in conjunctive directions for ten deflections.

Figure 2b represents a deformed net of nodes in the case of nondimensional deflection 0.333, and Fig. 3 the distribution of nondimensional specific strain energy in an axial section of a ring. Figure 4 represents the relationship of a nondimensional force to a nondimensional deflection.

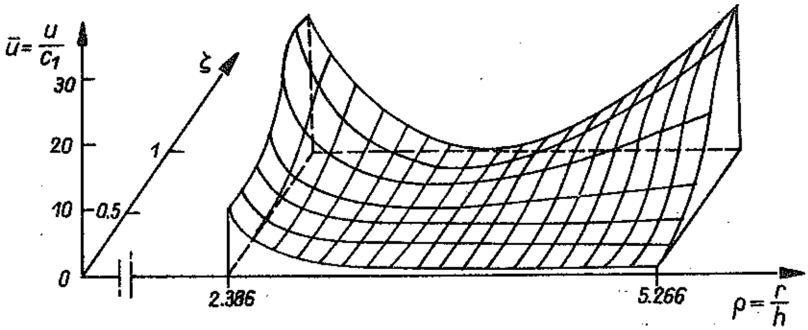

FIG. 3. The specific strain energy in the axial section for the nondimensional deflection 0.333.

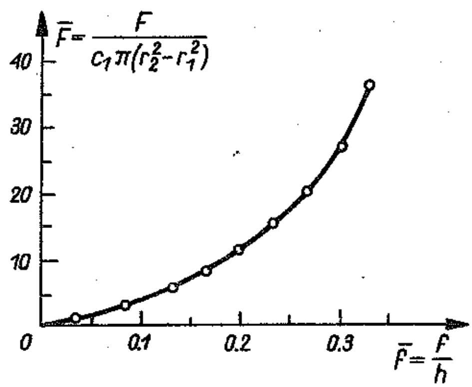

FIG. 4. Relationship between the compressive force and the deflection.

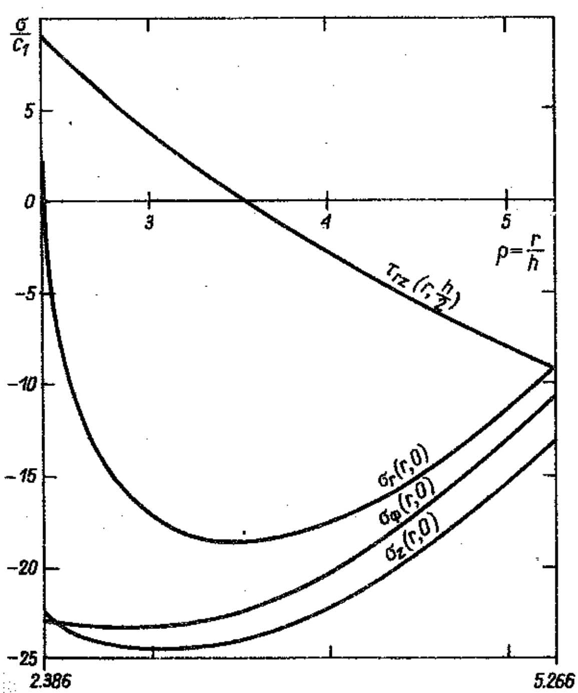

FIG. 5. Stresses for the nondimensional deflection 0.333.

Figure 5 represents the principal stresses in the symmetrical section of the ring ($z = 0$) and the shear stresses on the surface of contact with the plate ($z = h/2$). As can be seen, the radial stresses are not equal to zero on the surfaces $r = r_1$ and $r = r_2$. This is connected with the impossibility to fulfill the condition (2.5) according to the hypothesis of plane sections. Consequently, the obtained solution in regions near r_1 and r_2 has to be considered as approximated.

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Streszczenie

SKOŃCZONE DEFORMACJE NIELINIOWO SPRĘŻYSTEGO PIERŚCIENIA

W pracy analizowano zagadnienie ściskania pierścienia wykonanego z materiału nieściśliwego, mającego nieliniowe własności opisane równaniem Mooney'a. Rozważano przypadek ściskania bez poślizgów pierścienia między dwoma sztywnymi płytami. Rozwiązanie problemu dla dużych deformacji, z wykorzystaniem hipotezy płaskich przekrojów, otrzymano na drodze numerycznej minimalizacji energii odkształcenia zdyskretyzowanego modelu pierścienia.

Резюме

КОНЕЧНЫЕ ДЕФОРМАЦИИ НЕЛИНЕЙНОГО УПРУГОГО КОЛЬЦА

В работе рассмотреня задача сжатия кольца из несжимаемого материала, нелинейные свойства которого подчинены уравнения Мунея. Рассмотрен случай сжатия, без скольжения, между двумя жёсткими плитами. Предпологались большие деформации с учётом гипотеза плоских сечений. Решение было получено путём численной минимизации энергии деформации дискретной модели кольца.

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