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FLOWS IN CONVERGING SLITS AND PIPES AS FLOWS WITH DOMINATING EXTENSIONS

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The concept of plane and axi-symmetric flows with dominating extensions (cf. [1, 2]) is applied to moderately converging flows in slits and pipes. It is shown that many properties of the flows considered essentially depend on the extensional viscosity function as well as its derivative with respect to the extension rate.

1. Introduction

In our recent papers [1, 2] the concepts of plane and axi-symmetric flows with dominating extensions (the so-called FDE approximations) were extensively developed. In the subsequent papers [3, 4] some applications of the above flows were presented for viscoelastic flows between two rotating cylinders as well as for viscoelastic boundary layers in stagnation point flows.

In the present contribution another application of flows with dominating extensions, i.e. the flows in converging plane slits and converging circular pipes frequently met in many practical situation is discussed in greater detail. To this end it is assumed that certain regions of plane slits and circular pipes are moderately converging to apply consequently thin-layer approximations, on the one hand, and to consider dominating extensional effects, on the other (cf. [5, 6]). By assumption of the model considered the straight parts of plane slits and circular pipes exhibit no extensional effects and can be related to the corresponding Poiseuille flows of purely viscous fluids.

2. Flows with dominating extensions

All the essential details concerned with the concept of plane and axi-symmetric flows with dominating extensions can be found elsewhere [1, 2]. At the present, we repeat only some fundamental relations necessary for understanding further considerations.

Let us consider plane velocity fields in the following form:

$$u^*(x, y) = qx + u(x, y),$$
(2.1)
$$v^*(x, y) = -qy + v(x, y),$$

where q denotes some extension gradient, and u, v are additional velocity components in the system of Cartesian coordinates (x, y). We assume, moreover, that the flow considered is realised in a thin layer of fluid, i.e. the characteristic dimension l in the x-direction is much greater than the layer thickness h, and $\varepsilon = h/l \ll 1$. It may happen for small vorticity flows that the diagonal terms of the corresponding velocity gradient are more meaningful as compared with the remaining terms (cf. [1, 2]).

The flows with dominating extensions (FDE) can be defined as such thin-layer flows for which the constitutive equations valid for purely extensional flows of an incompressible simple fluid, viz. (cf. [2]).

(2.2) $$\mathbf{T} = -p\mathbf{1} + \beta_1 \mathbf{A}_1 + \beta_2 \mathbf{A}_1^2, \quad \operatorname{tr} \mathbf{A}_1 = 0,$$

where p is a hydrostatic pressure, $\mathbf{A}_1$ — the first Rivlin–Ericksen kinematic tensor (cf. [7]), and the material functions β_1, β_2 depend on the invariants: $\operatorname{tr} \mathbf{A}_1^2$, $\operatorname{tr} \mathbf{A}_1^3$, may be used in a form linearly perturbed with respect to gradients of the additional velocity field.

Introducing the above constitutive equations into the equations of dynamic equilibrium and retaining the highest order terms with respect to ε, we arrive at the following governing equation (cf. [1]):

(2.3) $$\frac{\partial}{\partial y}\left[\frac{1}{2}\frac{d\beta_1}{dq}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2 + \beta_1 \frac{\partial^2 u}{\partial y^2}\right] = 0,$$

where $\beta_1(q)$ can be called the extensional viscosity function. Looking for an approximate solution of Eq. (2.3) in the form:

(2.4) $$u(x, y) = (a + x)\, w(y) - qx,$$

where $w(y)$ is a function of y only, we see that any solution of the simplified equation:

(2.5) $$\beta_1 w''(y) + \frac{d\beta_1}{dq} w'^2(y) = C = \text{const},$$

satisfies Eq. (2.3) for $dp^*/dx = C(a + x)$, where $p^* = -T^{22}$, i.e. for a parabolic dependence of pressure p^* on x. For other functions p^* we can also use Eq. (2.5) assuming that C weakly depends on x treated as a parameter.

In the case of axi-symmetric flows the velocity field is assumed in the form:

$$u^*(y,x) = qx + u(y,x), \qquad (2.6)$$
$$v^*(y,x) = -\frac{1}{2}qy + \frac{1}{y}u(y,x),$$

where q denotes some extension gradient, and u, v are additional velocity components in the system of cylindrical coordinates (y, ϑ, x). If, moreover, the characteristic dimension l in the x-direction is much greater than the radius h, and $\varepsilon = h/l \ll 1$, we arrive at the following governing equation instead of Eq. (2.3):

$$\frac{\partial}{\partial y}\left[\frac{1}{2}\left(\frac{d\beta}{dq} - \beta_2\right)\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)^2 + \beta\frac{\partial^2 w}{\partial y^2}\right] = 0, \qquad (2.7)$$

where

$$\beta(q) = \beta_1 + \beta_2 q, \qquad (2.8)$$

can also be called the extensional viscosity function (cf. [2]).

Looking for an approximate solution of Eq. (2.7) in the form (2.4), we have instead of Eq. (2.5)

$$\beta w''(y) + \left(\frac{d\beta}{dq} - \beta_2\right) w'^2(y) = C = \text{const}. \qquad (2.9)$$

Any solution of the above equation satisfies Eq. (2.7) for $dp^*/dx = C(x+x)$, i.e. for a parabolic dependence of pressure p^* on x; otherwise C depends on x as a parameter.

The nonlinear Eq. (2.5) as well as Eq. (2.9) are special Riccati equations for w' and can easily be solved for the appropriate boundary conditions. In what follows we shall require that for Eq. (2.5):

$$u^*(y = \pm h) = 0 \quad \text{or} \quad w(\pm h) = 0, \qquad (2.10)$$

$$\frac{\partial u^*}{\partial y}(y = 0) = 0 \quad \text{or} \quad w'(0) = 0, \qquad (2.11)$$

and similar conditions for Eq. (2.9).

3. Viscoelastic flows in converging plane slits

Let us consider a moderately converging region in a plane slit shown in Fig. 1. The thickness of a slit changes from $2h_1$ to $2h_0$, while x_0 and x_1 characterize positions of the cross-sections at which the flows considered become purely viscous Poiseuille flows. The latter quantities are determined at the end of this Section.

The solution of Eq. (2.5) satisfying the boundary conditions (2.11), (2.12) leads to

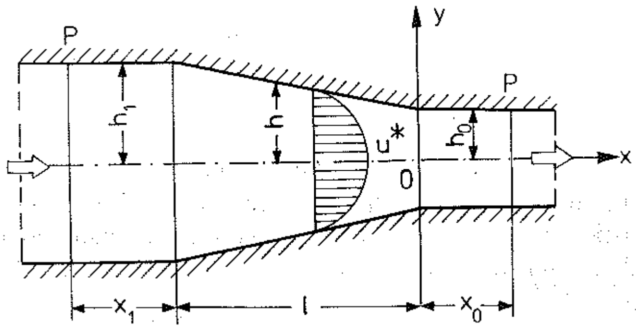

FIG. 1.

$$(3.1)\qquad u^* = \frac{a+x}{A}\left[\ln\cos\sqrt{\gamma\frac{y^2}{h^2}} - \ln\cos\sqrt{\gamma}\right] \quad \text{for} \quad \gamma > 0,$$

$$(3.2)\qquad u^* = \frac{a+x}{A}\left[\ln \mathrm{ch}\sqrt{-\gamma\frac{y^2}{h^2}} - \ln \mathrm{ch}\sqrt{-\gamma}\right] \quad \text{for} \quad \gamma < 0,$$

where

$$(3.3)\qquad A = \frac{1}{\beta_1}\frac{d\beta_1}{dq}, \qquad \gamma = -\frac{AC}{\beta_1}h^2 = \frac{HAQ}{2h(a+x)};$$

h as well as γ depend on x and Q denotes the constant volume discharge. H stands for a number parameter defined throughout the relation:

$$(3.4)\qquad \frac{dp^*}{dx} = C(a+x) = -\frac{H\beta_1(q)Q}{2h^3},$$

and more precisely determined a little further.

In the case of purely viscous fluids ($\beta_1 = \beta_0 = \text{const}$), we have, instead of Eqs. (3.1), (3.2),

$$(3.5)\qquad u^* = (a+x)\frac{C_N}{2\beta_0}(y^2-h^2) = \frac{1}{2\beta_0}\left(\frac{dp^*}{dx}\right)_N (y^2-h^2),$$

where

$$(3.6)\qquad C_N = -\frac{3Q\beta_0}{2h^3(a+x)};$$

thus for Newtonian fluids $H \equiv 3$.

For slight variability of the extensional viscosity function $\beta_1(q)$, i.e. for reasonably small values of A and γ, Eqs. (3.1), (3.2) expanded into a series with respect to γ lead to

(3.7) $$u^* = \frac{HQ}{4h}\left[\left(1-\frac{y^2}{h^2}\right)+\frac{\gamma}{6}\left(1-\frac{y^4}{h^4}\right)+\dots\right],$$

and

(3.8) $$\left.\frac{\partial u^*}{\partial y}\right|_{y=\pm h} = -\frac{HQ}{2h^2}\left(1+\frac{\gamma}{3}\right).$$

Thus for Newtonian fluids we obtain

(3.9) $$u_N^* = \frac{3Q}{4h}\left(1-\frac{y^2}{h^2}\right).$$

The continuity of flow (constant volume discharge) expressed by integration of Eqs. (3.1) or (3.2) over the slit thickness $2h$, viz.

(3.10) $$Q = 2\frac{a+x}{A}\int_0^h\left[\ln\cos\sqrt{\gamma\frac{y^2}{h^2}}-\ln\cos\sqrt{\gamma}\right]dy = \text{const},$$

results in the following relations:

(3.11) $$\frac{\gamma}{H} = -\left[\ln\cos\sqrt{\gamma}+\frac{1}{\sqrt{\gamma}}L(\sqrt{\gamma})\right] \quad \text{for} \quad \gamma > 0,$$

(3.12) $$\frac{\gamma}{H} = \left[\ln\text{ch}\sqrt{-\gamma}-\frac{1}{\sqrt{-\gamma}}\bar{L}(\sqrt{-\gamma})\right] \quad \text{for} \quad \gamma < 0,$$

where

(3.13) $$L(x) = -\int_0^x \ln\cos z\,dz, \qquad \bar{L}(x) = \int_0^x \ln\text{ch}\,z\,dz,$$

denote the Lobachevsky and the modified Lobachevsky functions, respectively. The graphs of these functions have been presented in the paper [1].

According to Eqs. (3.11), (3.12) the volume discharge can also be expressed as

(3.14) $$Q = -\frac{4}{3}\frac{H}{\gamma}Vh\left[\ln\cos\sqrt{\gamma}+\frac{1}{\sqrt{\gamma}}L(\sqrt{\gamma})\right] \quad \text{for} \quad \gamma > 0,$$

(3.15) $$Q = \frac{4}{3}\frac{H}{\gamma}Vh\left[\ln\text{ch}\sqrt{-\gamma}-\frac{1}{\sqrt{-\gamma}}\bar{L}(\sqrt{-\gamma})\right] \quad \text{for} \quad \gamma < 0,$$

where V denotes the maximum velocity at the axis. For small values of the parameter γ we vrrive at

$$H = 3\left(1+\frac{\gamma}{5}+\dots\right)^{-1}, \tag{3.16}$$

independently of the sign of γ. The diagram of $H(\gamma)$ calculated from exact Eqs. (3.11), (3.12), simplified Eq. (3.16) and other approximate relations are shown in Fig. 2.

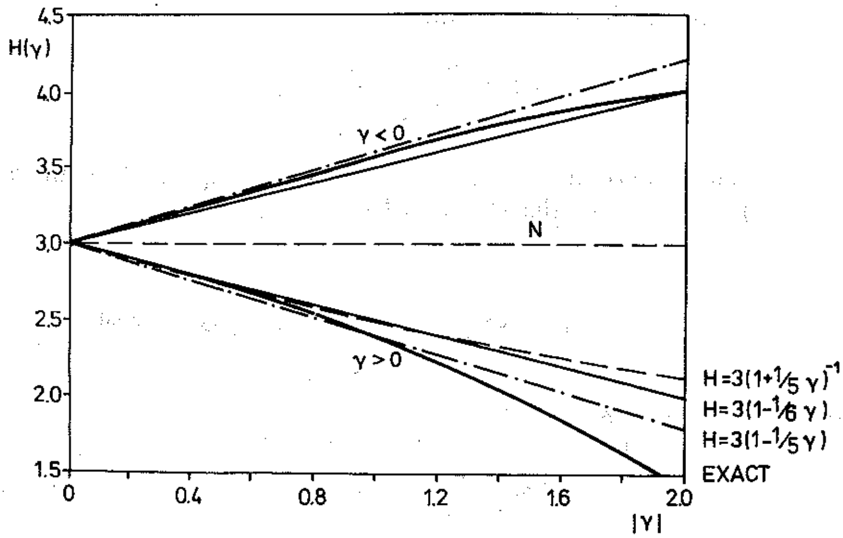

FIG. 2.

The unknown parameter a can be determined under the assumption that constant C (cf. Eq. (2.5)) is exactly the same at the beginning and the end of a slit, i.e. for $x = 0$ and $x = -l$ as well (Fig. 1). This assumption means that

$$\left(\frac{dp^*}{dx}\right)_0 \frac{1}{a} = \left(\frac{dp^*}{dx}\right)_{-l} \frac{1}{a-l}, \tag{3.17}$$

and we have finally

$$a = l\left[1-\left(\frac{h_0}{h_1}\right)^3 \frac{H_1}{H_0}\right]^{-1}, \tag{3.18}$$

or as consequence of the above formula:

$$\gamma_1 = \gamma_0\left(\frac{h_1}{h_0}\right)^2, \tag{3.19}$$

where the corresponding subscripts 0 and 1 refer to various quantities defined at the ends of a slit.

Since, on the other hand, at $x = x_0$ and $x = -(l+x_1)$ the corresponding pressure gradients (driving forces) for viscoelastic and purely viscous fluids must be the same, it results from Eq. (3.4) that

$$H_0 = H_1 = 3\frac{\beta_0}{\beta_1(q)}, \tag{3.20}$$

where $\beta_0 = \beta_1(0)$.

The so far unknown quantities x_0 and x_1 (Fig. 1) can be determined on the basis of the requirement that the corresponding maximum velocities (at the axis) for viscoelastic and purely viscous fluids are exactly the same. Thus we have from Eqs. (3.1), (3.2) and (3.9)

$$x_0 = -a - \frac{3}{4}\frac{QA}{h_0}\frac{1}{\ln\cos\sqrt{\gamma_0}}, \tag{3.21}$$

$$l+x_1 = a+\frac{3}{4}\frac{QA}{h_1}\frac{1}{\ln\cos\sqrt{\gamma_1}}. \tag{3.22}$$

Similar expressions valid for small values of γ can be written in the following form:

$$x_0 = \frac{AH_0^2 Q}{12h_0(3-H_0)} - a, \tag{3.23}$$

$$x_1 + l = a - \frac{AH_1^2 Q}{12h_1(3-H_1)}. \tag{3.24}$$

Knowledge of the extension gradient q, although not involved explicitly in the above formulae, is essential for proper determination of the extensional viscosity function $\beta(q)$. This gradient can be determined as the difference in the axial velocities at both ends of the converging region of a slit. Thus we obtain

$$q = \frac{5Q}{8h_0 l}\left[\frac{6+\gamma_0}{5+\gamma_0} - \frac{h_0}{h_1}\frac{6+\gamma_1}{5+\gamma_1}\right], \tag{3.25}$$

and for small values of γ, approximately,

$$q \simeq \frac{3Q}{4h_0 l}\left(1-\frac{h_0}{h_1}\right). \tag{3.26}$$

At the end of the present considerations we intend to clarify the procedure of eventual numerical calculations. First, knowing the gradient q, we can determine the corresponding values of the extensional viscosity function $\beta(q)$. Next, the values of the parameters H_0 and H_1 result from Eq. (3.20). On the basis of Eqs. (3.21)–(3.24) the quantities x_0 and x_1 can be calculated.

Let us also remind that the parameters H_0, H_1 are directly related to γ_0, γ_1 by Eqs. (3.11), (3.12) or (3.16).

4. Viscoelastic flows in converging circular pipes

Let as consider a moderately converging region of a circular pipe shown in Fig. 1. The radius of a pipe changes from h_1 to h_0, while x_0 and x_1 characterize positions of the cross-sections at which the flows considered become purely viscous Poiseuille flows.

The solution of Eq. (2.9), satisfying the boundary conditions of the type (2.10), (2.11), can be written again in the form (3.1), (3.2) with

$$A = \frac{1}{\beta}\left(\frac{d\beta}{dq} - \beta_2\right), \quad \gamma = -\frac{AC}{\beta}h^2 = \frac{HAQ}{\pi h^2(a+x)}, \tag{4.1}$$

where Q denotes the constant volume discharge. H stands again for a number parameter defined in the following way:

$$\frac{dp^*}{dx} = C(a+x) = -\frac{H\beta(q)Q}{\pi h^4(a+x)}. \tag{4.2}$$

In the case of a purely viscous fluid ($\beta = \beta_0 = \text{const}$) we have again Eq. (3.5) with

$$C_N = -\frac{4Q\beta_0}{\pi h^4(a+x)}; \tag{4.3}$$

thus for Newtonian fluids $H \equiv 4$.

For slight variability of the extensional viscosity function $\beta(q)$, i.e. for reasonably small values of A and/or γ, Eqs. (3.1), (3.2) expanded into a series with respect to γ lead to

$$u^* = \frac{HQ}{2\pi h^2}\left[\left(1-\frac{y^2}{h^2}\right) + \frac{\gamma}{6}\left(1-\frac{y^4}{h^4}\right) + \dots\right], \tag{4.4}$$

and

$$\left.\frac{\partial u^*}{\partial y}\right|_{y=h} = -\frac{HQ}{\pi h^3}\left(1+\frac{\gamma}{3}\right). \tag{4.5}$$

Thus for Newtonian fluids we obtain

$$u_N^* = \frac{2Q}{\pi h^2}\left(-\frac{y^2}{h^2}\right). \tag{4.6}$$

The continuity of flow (constant volume discharge), i.e. integration of

Eqs. (3.1) or (3.2) over the area of pipe cross-section with the radius h, viz.

$$Q = \frac{a+x}{A} \int_0^h \left[\ln\cos\sqrt{\gamma\frac{y^2}{h^2}} - \ln\cos\sqrt{\gamma} \right] 2\pi y\, dy = \text{const}, \tag{4.7}$$

gives

$$\frac{\gamma}{H} = -\left[\ln\cos\sqrt{\gamma} + \frac{2}{\gamma} K(\sqrt{\gamma}) \right] \quad \text{for} \quad \gamma > 0, \tag{4.8}$$

$$\frac{\gamma}{H} = \left[\ln\operatorname{ch}\sqrt{-\gamma} - \frac{2}{\gamma} \bar{K}(\sqrt{-\gamma}) \right] \quad \text{for} \quad \gamma < 0, \tag{4.9}$$

where

$$K(x) = -\int_0^x z \ln\cos z\, dz, \qquad \bar{K}(x) = \int_0^x z \ln\operatorname{ch} z\, dz, \tag{4.10}$$

can be called modified Lobachevsky functions of the second kind (cf. Eq. (3.13)).

According to Eqs. (4.8), (4.9), the volume discharge can also be expressed as

$$Q = -\frac{1}{2}\frac{H}{\gamma} V\pi h^2 \left[\ln\cos\sqrt{\gamma} + \frac{2}{\gamma} K(\sqrt{\gamma}) \right] \quad \text{for} \quad \gamma > 0, \tag{4.11}$$

$$Q = \frac{1}{2}\frac{H}{\gamma} V\pi h^2 \left[\ln\operatorname{ch}\sqrt{-\gamma} - \frac{2}{\gamma} \bar{K}(\sqrt{-\gamma}) \right] \quad \text{for} \quad \gamma < 0, \tag{4.12}$$

where V denotes the maximum velocity at the axis. For small values of the parameter γ, we arrive at

$$H = 4\left(1 + \frac{2}{9}\gamma + \ldots\right)^{-1}, \tag{4.13}$$

independently of the sign of γ. The illustrative diagrams of $H(\gamma)$ are shown in Fig. 3.

The unknown quantities a, H_0, H_1, x_0 and x_1 can be determined in a way similar to that discussed in Sect. 3. The parameter a also results from Eq. (3.18). Instead of Eq. (3.20) we obtain from Eq. (4.4):

$$H_0 = H_1 = 4\frac{\beta_0}{\beta(q)}. \tag{4.14}$$

The distances x_0 and x_1 can be calculated either from

$$x_0 = -a - \frac{2QA}{\pi h_0^2} \frac{1}{\ln\cos\sqrt{\gamma_0}}, \tag{4.15}$$

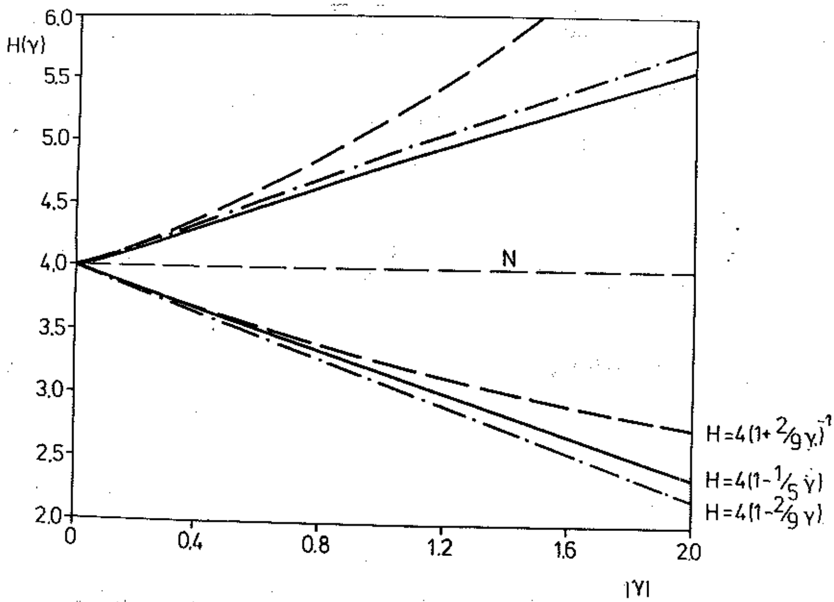

FIG. 3.

$$l+x_1 = a+\frac{2QA}{\pi h_1^2}\frac{1}{\ln\cos\sqrt{\gamma_1}}, \tag{4.16}$$

or, for small values of γ, from

$$x_0 = \frac{AH_0^2 Q}{6\pi h_0^2(4-H_0)}-a, \tag{4.17}$$

$$x_1+l = a-\frac{AH_1^2 Q}{6\pi h_1^2(4-H_1)}. \tag{4.18}$$

Defining the extension gradient q in a way similar to that presented in Sect. 3, we arrive at

$$q = \frac{3Q}{\pi h_0^2 l}\left[\frac{6+\gamma_0}{9+2\gamma_0}-\frac{h_0^2}{h_1^2}\frac{6+\gamma_1}{9+2\gamma_1}\right], \tag{4.19}$$

and for small values of γ, approximately at

$$q \simeq \frac{3Q}{2\pi h_0^2 l}\left(1-\frac{h_0^2}{h_1^2}\right). \tag{4.20}$$

The possible procedure of numerical calculations is exactly the same as that mentioned in the previous section. To this end, however, some information on the dependence of β_1 and β_2 on the extension gradient q is necessary.

5. Final remarks

On the basis of the considerations presented we can formulate the following remarks:

1) Flows in converging regions of plane slits and circular pipes can be treated as flows with dominating extensions. This approach seems to be especially useful for the case of moderately converging slits and pipes.

2) The method proposed is very sensitive for non-vanishing values of the extension gradients; in the regions of straight slits or pipes the flows are viscometric and the fluid behaves like a purely viscous one.

3) The regions in which viscoelastic (related to the extensional viscosity) effects are important are usually longer than the corresponding converging parts of lits and pipes. This is in agreement with general experimental observations (cf. [5]).

4) It results from Eqs. (3.7), (3.8) and (4.4), (4.5) that for an increasing extensional viscosity the velocity profiles in converging regions of slits and pipes are remarkably "flattened" as compared with those for purely viscous fluids; a decreasing extensional viscosity exerts quite opposite effect.

5) The flows considered depend not only on the extensional viscosity functions themselves, but also on the corresponding rates of increase or decrease, i.e. on their derivatives with respect to the extension gradients.

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Streszczenie

PRZEPŁYWY W ZBIEŻNYCH SZCZELINACH I RURACH JAKO PRZEPŁYWY Z DOMINUJĄCYMI ROZCIĄGANIAMI

Koncepcja płaskich i osiowo-symetrycznych przepływów z dominującymi rozciąganiami (por. [1, 2]) została zastosowana do umiarkowanie zbieżnych przepływów w szczelinach i rurach. Pokazano, że liczne własności rozważanych przepływów istotnie zależą zarówno od funkcji lepkości na rozciąganie jak i jej pochodnej względem szybkości rozciągania.

Резюме

ТЕЧЕНИЯ В СХОДЯЩИХСЯ ЩЕЛЬАХ И ТРУБАХ КАК ТЕЧЕНИЯ С ДОМИНИРУЮЩИМИ РАСТЯЖЕНИЯМИ

Концепция плоских и осесиметричных течений с доминирующими растяжениями (ср. [1, 2]) применена к умеренно сходящимся течениям в щельах и трубах. Показано, что многие свойства рассматриваемых течений существенно зависят так от функции вязкости на растяжение, как и от ее производной по отношению к скорости растяжения.

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