ROZPRAWY INŻYNIERSKIE • ENGINEERING TRANSACTIONS • 36, 2, 299–310, 1988
Polska Akademia Nauk • Instytut Podstawowych Problemów Techniki

ERROR ESTIMATES FOR A SIMPLIFIED REISSNER THEORY OF PLATE BENDING

Z. RYCHTER (BIAŁYSTOK)

A simplified Reissner theory for plate bending is dealt with in which transverse displacement occurs as the single unknown variable. The two-dimensional theory used in conjunction with the plane stress theory serves to construct three-dimensional fields whose relative mean square error, under so-called „regular" edge conditions, is shown to be of order of the plate thickness cubed with respect to elasticity solutions.

1. Introduction

Reissner's theory [1-4] of plate bending is generally acknowledged for its variational consistency, simplicity and accuracy. The theory takes into account transverse shear strain and transverse normal stress effects, it is thus suitable for composite plates exhibiting sharply different in-plane and out-of-plane elastic properties.

There are three unknown kinematic variables in Reissner's theory: the mean transverse displacement w and two components of average cross section rotation. Alternately, the governing differential equations can be formulated in terms of w and a stress function [4]. A further possibility has been disclosed by Speare and Kemp [5] in their so-called simplified Reissner theory. By adopting $O(h^2)$ as a sufficient level of accuracy with error terms of order $O(h^4)$. Possible descriptions of averaged dislocation density and distribution by means of the diffusion equations (h being the thickness) they reduced the original Reissner equations to a sixth-order equation for only one unknown displacement w along with explicit expressions for internal forces in terms of w.

While an $O(h^4)$ error in the two-dimensional plate theory equations is fairly small, one would be much more interested in comparing predictions supplied by the Speare-Kemp theory with those of three-dimensional elasticity. This task is undertaken in the present work which sets out to do two things: a) to provide possibly most accurate three-dimensional displacement and stress distributions constructed from the two-dimensional plate theory, and b) to estimate the error in those fields in comparison to elasticity solutions in a global sense. To achieve those aims we use a slightly different simplified

Reissner theory from that of [5], the latter being readily obtainable from the former by eliminating transverse load terms. Also, we employ the plane stress theory, not considered in [5], to account for in-plane membrane deformation caused by transverse load via Poisson's effect. From such a combination of two-dimensional bending and stretching theories three-dimensional displacement and stress fields are constructed that involve several effects disregarded in the classical theory of bending: membrane strains, in-plane strains distributed nonlinearly over the thickness, transverse shear and normal strains, and stresses self-equilibrating over the thickness. In so doing we are guided by [6, 7] where the importance of such contributions is emphasized.

Referring to the PRAGER–SYNGE theorem [8, 9], we find that the three-dimensional fields obtained are capable of approximating the exact elasticity solutions with the relative mean square error proportional to the plate thickness cubed, $O(h^3)$. Known estimates in plate and shell theory involve global error bounds $O(h)$ — [10, 11] and $O(h^2)$ — [12-15], depending on which non-elementary contributions have been accounted for. There are also $O(h^3)$ error estimates in Reissner's theory [16, 17], but [16] permits no surface load and [17] assumes low transverse normal deformability so that the present $O(h^3)$ result is far more general. Its validity, however, is restricted to the so-called „regular" boundary conditions [10] on the cylindrical bounding surface, assumed also in [10-17]. Irregular edge data which do not conform to our three-dimensional fields give rise to an additional error that, understandably, cannot be expressed once and for all as a specific power of h, but varies from case to case. For more detail about the effect of irregular boundary conditions we refer to [10].

2. Three-dimensional problem

Plates are three-dimensional bodies whose elastic, small-deflection behaviour may be described by the linear theory of elasticity which, in this work, will be treated as exact. Elasticity solutions, however, are in general difficult to find and for this reason it is customary to resort to approximations. One obvious way which we adopt for our purpose consists in searching for statically and kinematically admissible solutions whose closeness determines the quality of approximation (see e.g. [8-10]). The statically admissible stress field $\tilde{\sigma}(x_\alpha, x_3)$, $x_\alpha (\alpha = 1, 2)$ and x_3 being in-plane and thickness Cartesian coordinates, is in the present work required to meet the following traction boundary conditions:

$$\tilde{\sigma}_{\alpha 3}(x_\beta, x_3 = \pm h) = 0, \quad \tilde{\sigma}_{33}(x_\beta, x_3 = -h) = 0, \qquad (2.1)$$
$$\tilde{\sigma}_{33}(x_\beta, x_3 = h) = p(x_\beta),$$

at the upper ($x_3 = h$) and lower ($x_3 = -h$) faces of the plate, $2h$ being the constant thickness. It must also satisfy the equilibrium conditions

$$\tilde{\sigma}_{\alpha\beta,\beta} + \tilde{\sigma}_{\alpha 3,3} = 0, \quad \tilde{\sigma}_{\alpha 3,\alpha} + \tilde{\sigma}_{33,3} = 0, \tag{2.2}$$

with commas denoting partial differentiation with respect to x_α and x_3 and repeated indices implying summation over the range 1, 2. In defining $\tilde{\boldsymbol{\sigma}}$ we have assumed that there is only an arbitrary distributed transverse load $p(x_\alpha)$ on the upper face, this being true for most applications, and that body forces are unimportant.

The kinematically admissible solution consists of displacements $\hat{\mathbf{u}}(x_\alpha, x_3)$ and stresses $\hat{\boldsymbol{\sigma}}(x_\alpha, x_3)$, the latter being produced by the former via the constitutive relations

$$\hat{\sigma}_{\alpha\beta} = D[(1-\nu)\hat{u}_{(\alpha,\beta)} + \nu\delta_{\alpha\beta}\hat{u}_{\lambda,\lambda}] + C\delta_{\alpha\beta}\hat{\sigma}_{33}, \tag{2.3}$$

$$\hat{\sigma}_{\alpha 3} = G(\hat{u}_{\alpha,3} + \hat{u}_{3,\alpha}), \tag{2.4}$$

$$\hat{\sigma}_{33} = \frac{E_3}{1 - 2\nu_3 C}(\hat{u}_{3,3} + C\hat{u}_{\alpha,\alpha}), \tag{2.5}$$

where

$$D = \frac{E}{1-\nu^2}, \quad C = \frac{\nu_3}{1-\nu}\,\frac{E}{E_3}, \tag{2.6}$$

with $\delta_{\alpha\beta}$ denoting the Kronecker delta and a pair indices enclosed in parentheses indicating symmetrization. Here a homogeneous, linearly elastic, transversely isotropic material has been adopted with Young's moduli E and E_3, Poisson's ratios ν and ν_3 and the transverse shear modulus G. This type of elastic law is fairly simple but sufficiently general to allow for possibly large transverse shear and/or normal deformability ($G/E \ll 1$ and/or $E_3/E \ll 1$) characteristic of composite plates.

As already stated, by minimizing $\tilde{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}}$ one approaches the exact solution. In the present context the crucial point is that such three- dimensional fields $\tilde{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\sigma}}$ can be constructed from the two-dimensional plate theory presented in the next section.

3. TWO-DIMENSIONAL PLATE THEORY

The kinematic variables of Reissner's theory of plate bending involve the mean transverse displacement w and average rotations b_α:

$$w(x_\beta) = \frac{3}{4h}\int_{-h}^{h} \hat{u}_3(x_\beta,\ x_3)(1 - x_3^2/h^2)dx_3, \tag{3.1}$$

$$b_\alpha(x_\beta) = \frac{3}{2h^3}\int_{-h}^{h} \hat{u}_\alpha(x_\beta,\ x_3)x_3\, dx_3. \tag{3.2}$$

The corresponding static variables are the moments $M_{\alpha\beta}$ and transverse shear forces Q_α related to the components of stress as

$$M_{\alpha\beta}(x_\lambda) = \int_{-h}^{h} \tilde{\sigma}_{\alpha\beta}(x_\lambda,\ x_3)x_3\, dx_3, \tag{3.3}$$

$$Q_\alpha(x_\beta) = \int_{-h}^{h} \tilde{\sigma}_{\alpha 3}(x_\beta,\ x_3)\, dx_3. \tag{3.4}$$

The field equations comprise the overall equilibrium conditions

$$M_{\alpha\beta,\beta} = Q_\alpha, \quad Q_{\alpha,\alpha} = -p, \tag{3.5}$$

and the constitutive relations

$$M_{\alpha\beta} = \frac{2}{3}h^3 D\,[(1-\nu)\, b_{(\alpha,\beta)} + \nu\delta_{\alpha\beta}\ b_{\lambda,\lambda}] + \frac{2}{3}h^2\, C\delta_{\alpha\beta}\, p, \tag{3.6}$$

$$Q_\alpha = \frac{5}{3}Gh\,(b_\alpha + w_{,\alpha}), \tag{3.7}$$

which should be used in conjunction with the boundary conditions prescribing

$$M_{\alpha\beta}\, n_\beta \text{ or } b_\alpha \quad \text{and} \quad Q_\alpha n_\alpha \text{ or } \quad w \quad \text{on } S, \tag{3.8}$$

S being the curve bounding the midplane and n_α the unit normal to S.

The relations (3.5) – (3.8) constitute the original Reissner's theory [1-4] and have been recorded here for comparison purposes. The transition to the simplified theory we wish to deal with is straightforward. It consists in resolving the constitutive equation (3.7) for the rotations

$$b_\alpha = -w_{,\alpha} + \frac{3}{5Gh}T_\alpha, \tag{3.9}$$

wherein the shear force Q_α has been replaced by an approximation of the form, familiar in the classical Kirchhoff plate theory,

$$T_\alpha = -\frac{2}{3}h^3 D \Delta w_{,\alpha}, \tag{3.10}$$

$\Delta(\,) = (\,)_{,\alpha\alpha}$ denoting the two-dimensional Laplacian. Now with Eqs. (3.9) and (3.10) the moments in Eq. (3.6) become

$$M_{\alpha\beta} = -\frac{2}{3}h^3 D\left(1+\frac{2D}{5G}h^2\Delta\right)[(1-\nu)w_{,\alpha\beta}+\nu\delta_{\alpha\beta}\Delta w] + \frac{2}{5}h^2 C\delta_{\alpha\beta}p, \tag{3.11}$$

Substitution of Eq. (3.11) into the condition $(3.5)_1$ yields the shear forces

$$Q_\alpha = -\frac{2}{3}h^3 D\left(1+\frac{2D}{5G}h^2\Delta\right)\Delta w_{,\alpha} + \frac{2}{5}h^2 C p_{,\alpha}, \tag{3.12}$$

whence T_α introduced in the relation (3.10) is now seen to represent indeed the major part of Q_α. Finally, inserting Eq. (3.12) into the condition $(3.5)_2$ produces the sixth-order governing differential equation in one unknown w:

$$\frac{2}{3}h^3 D\left(1+\frac{2D}{5G}h^2\Delta\right)\Delta\Delta w = p + \frac{2}{5}Ch^2\Delta p. \tag{3.13}$$

The simplified Reissner theory (3.9) – (3.13) may be further transformed by using Eq. (3.13) to eliminate p, $p_{,\alpha}$, and Δp from Eqs. (3.11) – (3.13). Accepting $O(h^2)$ as an adequate degree of accuracy with error terms of order $O(h^4)$, one obtains equations derived by SPEARE and KEMP [5], but for our present purposes the version in Eqs. (3.9) – (3.13) is more convenient.

While the single differential equation (3.13) of the modified Reissner theory is appealingly simple as compared with the original Reissner equations, there is a problem in formulating appropriate boundary conditions. Retaining the original conditions (3.8) of Reissner's theory would seem to be physically most sound, but they are not variationally consistent with the differential equation (3.13) when expressed through w with the help of Eqs. (3.9) – (3.12). On the other hand, mere knowledge of the differential governing equation (3.13) leads to some variationally consistent boundary conditions which, however, are found to be not so convincing physically as Reissner's conditions (3.8). We shall not pursue that question further, but this boundary condition problem indicates that the range of applicability of the simplified Reissner theory cannot be as wide as that of the original Reissner's version. It should also be noted that reducing the number of unknowns inevitably increases smoothness requirements, a serious drawback from the point of view of numerical analysis.

The above equations account only for the flexure of the plate. This effect is clearly dominant but at the level of accuracy we intend to achieve, it must be supplemented by membrane deformations; they arise due to Poisson's effect. Therefore we introduce average in-plane displacements of the form

(3.14) $$v_\alpha(x_\beta) = \int_{-h}^{h} \hat{u}_\alpha(x_\beta,\ x_3)\,dx_3,$$

with accompanying membrane resultant forces

(3.15) $$N_{\alpha\beta}(x_\lambda) = \int_{-h}^{h} \tilde{\sigma}_{\alpha\beta}(x_\lambda,\ x_3)\,dx_3.$$

These variables are subject to the equilibrium equations

(3.16) $$N_{\alpha\beta,\beta} = 0,$$

constitutive relations

(3.17) $$N_{\alpha\beta} = 2hD\,[(1-\nu)\,v_{(\alpha,\beta)} + \nu\,\delta_{\alpha\beta} v_{\lambda,\lambda}] + hC\delta_{\alpha\beta}p$$

and boundary conditions which specify

(3.18) $$N_{\alpha\beta}\,n_\beta \quad \text{or} \quad v_\alpha \quad \text{on } S.$$

The two-dimensional bending and stretching equations as stated above may appear at this stage to be somewhat arbitrary. In fact their particular form adopted here ensures the existance of very accurate three-dimensional fields to be presented in the next section.

4. Three-dimensional fields

Let us begin with the following three-dimensional displacement distribution, with the notation $z = x_3/h$,

(4.1) $$\hat{u}_\alpha(x_\beta,\ z) = v_\alpha(x_\beta) + zhb_\alpha(x_\beta) + \left(z^2 - \frac{1}{3}\right)k_\alpha(x_\beta) + \left(z^3 - \frac{3}{5}z\right)f_\alpha(x_\beta),$$

(4.2) $$\hat{u}_3(x_\beta,\ z) = w(x_\beta) + zs(x_\beta) + \frac{1}{10}(5z^2-1)\,g\,(x_\beta) + \frac{1}{3}(z^3 - z)\,d\,(x_\beta) + \\ + \frac{1}{20}\left(5z^4 - 6z^2 + \frac{27}{35}\right)r\,(x_\beta) + \left(z^4 - 6z^2 - 8z + \frac{39}{35}\right)q\,(x_\beta),$$

which reduces to identities the relations (3.1), (3.2) and (3.14) defining the two-dimensional kinematic variables w, b_α and v_α of the plate theory. Assuming further that

$$s = hCv_{\alpha,\alpha}, \quad g = -h^2 Cb_{\alpha,\alpha}, \quad d = -hCk_{\alpha,\alpha},$$

$$r = -hCf_{\alpha,\alpha}, \quad q = -\frac{1-2\nu_3 C}{16E_3}hp, \tag{4.3}$$

$$k_\alpha = -\frac{h}{2}s_{,\alpha}, \quad f_\alpha = -\frac{h}{6}g_{,\alpha} - \frac{1}{4G}T_\alpha,$$

and keeping in mind Eqs. (3.9), (3.10) and (3.13), it is seen that $\hat{\mathbf{u}}$ in Eqs. (4.1) and (4.2) is specified entirely in terms of w, b_α and v_α.

The stress field $\hat{\boldsymbol{\sigma}}$ corresponding to $\hat{\mathbf{u}}$ is, from the constitutive equations (2.3) – (2.5) with Eqs. (4.1) – (4.3) and (3.9),

$$\begin{aligned}\hat{\sigma}_{\alpha\beta} = {} & D[(1-\nu)v_{(\alpha,\beta)} + \nu\delta_{\alpha\beta}v_{\lambda,\lambda}] + zhD[(1-\nu)b_{(\alpha,\beta)} + \nu\delta_{\alpha\beta}b_{\lambda,\lambda}] + \\ & + \left(z^2 - \frac{1}{3}\right)D[(1-\nu)k_{(\alpha,\beta)} + \nu\delta_{\alpha\beta}k_{\lambda,\lambda}] + \\ & + \left(z^3 - \frac{3}{5}z\right)D[(1-\nu)f_{(\alpha,\beta)} + \nu\delta_{\alpha\beta}f_{\lambda,\lambda}] + C\delta_{\alpha\beta}\hat{\sigma}_{33},\end{aligned} \tag{4.4}$$

$$\begin{aligned}\hat{\sigma}_{\alpha 3} = {} & \frac{3}{4h}(1-z^2)T_\alpha + \frac{1}{3}(z^3 - z)Gd_{,\alpha} + \\ & + \frac{1}{20}\left(5z^4 - 6z^2 + \frac{27}{35}\right)Gr_{,\alpha} + \left(z^4 - 6z^2 - 8z + \frac{39}{35}\right)Gq_{,\alpha},\end{aligned} \tag{4.5}$$

$$\hat{\sigma}_{33} = (2 + 3z - z^3)\frac{p}{4}. \tag{4.6}$$

The statically admissible stress field $\tilde{\boldsymbol{\sigma}}$, which should be close to the kinematically admissible one $\hat{\boldsymbol{\sigma}}$, is taken in the form

$$\begin{aligned}\tilde{\sigma}_{\alpha\beta} = {} & \frac{1}{2h}N_{\alpha\beta} + \frac{3z}{2h^3}M_{\alpha\beta} + \frac{1}{20}(3z - 5z^3)\delta_{\alpha\beta}Cp + \\ & + \left(z^2 - \frac{1}{3}\right)D[(1-\nu)k_{(\alpha,\beta)} + \nu\delta_{\alpha\beta}k_{\lambda,\lambda}] + \\ & + \left(z^3 - \frac{3}{5}z\right)D[(1-\nu)f_{(\alpha,\beta)} + \nu\delta_{\alpha\beta}f_{\lambda,\lambda}],\end{aligned} \tag{4.7}$$

$$(4.8)\quad \tilde{\sigma}_{\alpha 3} = \frac{3}{4h}(1-z^2)Q_\alpha + \frac{1}{80}(5z^4 - 6z^2 + 1)Chp_{,\alpha} - \\ -\frac{h}{3}(z^3 - z)D[(1-\nu)k_{(\alpha,\beta)\beta} + \nu k_{\beta,\beta\alpha}] - \\ -\frac{h}{20}(5z^4 - 6z^2 + 1)D[(1-\nu)f_{(\alpha,\beta)\beta} + \nu f_{\beta,\beta\alpha}],$$

$$(4.9)\quad \tilde{\sigma}_{33} = (2+3z-z^3)\frac{p}{4} - \frac{1}{80}(z^5 - 2z^3 - z)Ch^2\Delta p + \\ + \frac{1}{12}(z^4 - 2z^2 + 1)Dh^2\Delta k_{\alpha,\alpha} + \frac{1}{20}(z^5 - 2z^3 + z)Dh^2\Delta f_{\alpha,\alpha}.$$

It is evident that this three-dimensional stress distribution is given in terms of the two-dimensional internal forces $M_{\alpha\beta}$, Q_α and $N_{\alpha\beta}$ and the load p, and that the relations (3.3), (3.4) and (3.15) defining the internal forces are identically satisfied upon substitution from Eqs. (4.7) – (4.9). Moreover, in view of Eqs. (3.5) and (3.16) the above stress field fulfills the traction boundary conditions (2.1) at the faces and the equilibrium equations (2.2) in the body of the plate. Thus it remains only to show that $\tilde{\boldsymbol{\sigma}}$ is near to $\hat{\boldsymbol{\sigma}}$.

Comparison between Eqs. (4.4) — (4.6) and Eqs. (4.7) – (4.9) yields, with the help of Eqs. (3.6) and (3.17),

$$(4.10)\quad \tilde{\sigma}_{\alpha\beta} - \hat{\sigma}_{\alpha\beta} = 0,$$

$$(4.11)\quad \tilde{\sigma}_{\alpha 3} - \hat{\sigma}_{\alpha 3} = O\left[\frac{1}{h}(Q_\alpha - T_\alpha);\quad hp_{,\alpha};\quad hD(\Delta k_\alpha + k_{\beta,\beta\alpha}); \\ hD(\Delta f_\alpha + f_{\beta,\beta\alpha});\quad G(d+r+q)_{,\alpha}\right],$$

$$(4.12)\quad \tilde{\sigma}_{33} - \hat{\sigma}_{33} = O(h^2 C\Delta p;\quad h^2 D\Delta f_{\alpha,\alpha};\quad h^2 D\Delta k_{\alpha,\alpha}).$$

These expressions may be made appreciably more legible using the following estimates:

$$(4.13)\quad \begin{gathered} p = O(h^3),\quad Q_\alpha - T_\alpha = O(h^5),\quad \mathrm{f}_\alpha = O(h^3), \\ r = O(h^4),\quad q = O(h^4), \end{gathered}$$

and

$$(4.14)\quad k_\alpha = O(h^2),\quad d = O(h^3).$$

The relations (4.13) and (4.14) are based on the observation that all the quantities involved depend on the elastic moduli and the thickness h (which are constants) and either on the deflection w, in the case of the estimates (4.13), or on the in-plane displacements v_α, in the case of the relations (4.14); for brevity, only h-dependence has been exposed. Specifically, the estimate $(4.13)_1$ results from Eq. (3.13); the estimate $(4.13)_2$ from Eqs. (3.10), (3.12), $(4.13)_1$; the estimate $(4.13)_3$ from Eq. $(4.3)_7$ with Eqs. (3.9), (3.10) and $(4.3)_2$; the estimate $(4.13)_4$ from Eqs. $(4.3)_4$ and $(4.13)_3$; the estimate $(4.13)_5$ from Eqs. $(4.3)_5$ and $(4.13)_1$; the relation $(4.14)_1$ from Eqs. $(4.3)_{1,6}$; the relation $(4.14)_2$ from Eqs. $(4.3)_3$ and $(4.14)_1$. Now comparison of the error stress $\tilde{\sigma} - \hat{\sigma}$ in Eqs. (4.10) – (4.12) with the kinematically admissible stress $\hat{\sigma}$ in Eqs. (4.4) – (4.6) yields, after using Eqs. (3.9), (3.10), (4.13) and (4.14),

$$\tilde{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}} = O(\hat{\sigma} h^3/L^3), \tag{4.15}$$

where L denotes the corresponding characteristic wavelength which depends on the displacements v_α, w and secures dimensional correctness of Eq. (4.15). As we see, $\tilde{\boldsymbol{\sigma}}$ is locally very close to $\hat{\boldsymbol{\sigma}}$. This property also ensures that $\tilde{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\sigma}}$ are good approximations to the exact solutions of elasticity in a global sense; this is shown in the section ot follow.

It should be emphasized that several non-classical contributions must be incorporated for that high accuracy. The in-plane and transverse displacement components (4.1) and (4.2) are accordingly third- and fourth-degree polynomials in the thickness coordinate, their respective elementary-theory distributions being linear and constant. Further, the statically admissible stress field (4.7) – (4.9) involves terms self-equilibrating over the thickness. Also, because of asymmetry of the surface load with respect to the middle plane, membrane deformation and stresses, produced owing to Poisson's effect, are to be accounted for. All of these effects are emphasized in [7] in the context of a different plate theory based on alternate kinematic variables. In [7] no distinction is made between statically and kinematically admissible fields, and non-elementary terms appearing in our $\tilde{\sigma}_{\alpha 3}$ and $\tilde{\sigma}_{33}$ distributions (4.8) and (4.9) are not provided.

5. Mean square error estimates

Stresses will now be regarded as points in function space endowed with the norm

$$\|\boldsymbol{\sigma}\|^2 = \int_{-h}^{h}\int_F \Big[\Big((1+\nu)\sigma_{\alpha\beta}\sigma_{\alpha\beta} - \nu\sigma_{\alpha\alpha}\sigma_{\beta\beta} - \frac{2\nu_3 E}{E_3}\sigma_{\alpha\alpha}\sigma_{33} + \frac{E}{G}\sigma_{\alpha 3}\sigma_{\alpha 3} + \frac{E}{E_3}\sigma_{33}\sigma_{33}\Big]dF dx_3, \tag{5.1}$$

F being the region of the midplane; one easily identifies Eq. (5.1) as the elastic energy functional which must be positive definite to represent a norm. The Prager–Synge hypersphere theorem [8, 9] asserts that the exact stress field σ may be approached by means of $(\tilde{\boldsymbol{\sigma}} + \hat{\boldsymbol{\sigma}})/2$, $\tilde{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\sigma}}$ being statically and kinematically admissible fields, respectively, with the relative mean square error e,

$$(5.2) \qquad \|\boldsymbol{\sigma} - \frac{1}{2}(\tilde{\boldsymbol{\sigma}} + \hat{\boldsymbol{\sigma}})\| / \|\hat{\boldsymbol{\sigma}}\| = e,$$

which is computable from

$$(5.3) \qquad e = \|\tilde{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}}\| / 2\|\hat{\boldsymbol{\sigma}}\|.$$

We have already seen that locally $\tilde{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}}$ is $O(h^3)$ relative to $\hat{\boldsymbol{\sigma}}$ and so will their norms be, evaluated from Eq. (5.1), for a norm represents a homogeneous functional. Thus from Eq. (5.3) with Eq. (4.15) it can be concluded that

$$(5.4) \qquad e = h^3/L_*^3 + O(h^n), \qquad n > 3.$$

Here L_* is the mean square wavelength which characterizes the deformation pattern of the middle surface through the displacements w and v_α; it also depends on the elastic moduli. An explicit formula for L_* is too complex to be worth recording the more so that in each particular problem the error e may be directly calculated from Eq. (5.3). In an isotropic plate under uniform load, L_* is expected to be of the order of the plate surface dimensions. It then follows that the error (5.4) will be reasonably small even in moderately thick plates, a conjecture numerically confirmed in [7] for a similar theory.

The inequality (see [9, 12])

$$(5.5) \qquad \|\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}\| / \|\hat{\boldsymbol{\sigma}}\| \leqslant 2e$$

shows with Eq. (5.4) that the statically admissible stress $\tilde{\boldsymbol{\sigma}}$ is also a good approximation to $\boldsymbol{\sigma}$. Moreover, although the global error estimate (5.5) appears worse than that in Eq. (5.2), it is expected that locally $\tilde{\boldsymbol{\sigma}}$ rather than $(\tilde{\boldsymbol{\sigma}} + \hat{\boldsymbol{\sigma}})/2$ will be closer to $\boldsymbol{\sigma}$, because the former field satisfies the traction boundary conditions on the faces while the latter does not.

Another inequality,

$$(5.6) \qquad \|\boldsymbol{\sigma}(\mathbf{u}) - \hat{\boldsymbol{\sigma}}(\hat{\mathbf{u}})\| / \|\hat{\boldsymbol{\sigma}}\| \leqslant 2e,$$

implies that to within rigid body displacement our kinematically admissible displacement field $\hat{\mathbf{u}}$ approaches the exact field $\mathbf{u}$ again with an error $O(h^3)$.

The main novelty of our $O(h^3)$ error estimates in the Reissner theory compared with their predecessors [16, 17] is that they permit surface lateral load and impose no restriction on transverse rigidity. Although we were concerned with a simplified Reissner theory, the conclusions apply for the original Reissner theory, this subject being treated in detail elsewhere. Finally, it should be remembered that the error (5.4) corresponds to certain idealized, „regular" boundary conditions on the cylindrical edge surface of the plate. Irregular edge data, i.e. distributed through the thickness in a different fashion from our u and fields in Eqs. (4.1), (4.2) and (4.7) – (4.9), produce an additional error to that in Eq. (5.4). KOITER [10] shows how to handle irregular boundary conditions when there is a straight edge and points out that an error as high as $O(h^{1/2})$ should be expected if irregular strains have the same magnitude as regular ones.

6. CONCLUDING REMARKS

This work has studied the accuracy of a simplified Reissner theory for the bending of elastic, homogeneous plates subjected to transverse load on one face. The two-dimensional bending theory used in conjunction with the plane stress theory has been proved to afford three-dimensional displacement and stress distributions bearing a relative mean square error of order of the plate thickness cubed with respect to the elasticity solutions, provided the edge conditions are regular. As a consequence, the theory may be expected to furnish reliable results for not-so-thin plates, plates under loads varying rapidly over the faces and anisotropic or composite plates exhibiting increased transverse deformability.

REFERENCES

1. E. REISSNER, *On the theory of bending of elastic plates*, J. Math. Phys., **23**, 184-191, 1944.
2. E. REISSNER, *The effect of transverse shear deformations on the bending of elastic plates*, J. Appl. Mech., **12**, A 69-A 77, 1945.
3. E. REISSNER, *On the bending of elastic plates*, Quart. Appl. Math., 4, 268-278, 1947.
4. E. REISSNER, *On the theory of transverse bending of elastic plates*, Int. J. Sol. Struct., **12**, 545-554, 1976.
5. P. R. S. SPEARE and K. O. KEMP, *A simplified Reissner theory for plate bending*, Int. J. Sol. Struct., **13**, 1073-1079, 1977.
6. J. N. GOODIER, *On the problem of beam and plate in the theory of elasticity*, Trans. Roy. Soc. Canada, **32**, 65-68, 1938.
7. L. W. REHFIELD and R. R. VALISETTY, *A simple, refined theory for bending and stretching of homogeneous plates*, AIAA J., **22**, 90-95, 1984.
8. W. PRAGER and J. L. SYNGE, *Approximations in elasticity based on the concept of function space*, Quart. Appl. Math., **5**, 241-269, 1947.
9. J. L. SYNGE, *The hypercircle in mathematical physics*, University Press, Cambridge 1957.
10. W. T. KOITER, *On the foundations of the linear theory of thin elastic shells*, Proc. Kon. Ned. Ak. Wet., **B 73**, 169-195, 1970.
11. R. P. NORDGREN, *A bound on the error in plate theory*, Quart. Appl. Math., **28**, 587-595, 1971.
12. D. A. DANIELSON, *Improved error estimates in the linear theory of thin elastic shells*, Proc. Kon. Ned. Ak. Wet., **B 74**, 294-300, 1971.

13. R. P. NORDGREN, *A bound on the error in Reissner's theory of plates*, Quart. Appl. Math., **29**, 551-556, 1972.
14. J. G. SIMMONDS, *An improved estimate for the error in the classical linear theory of plate bending*, Quart. Appl. Math., **29**, 439-447, 1971.
15. P. LADEVEZE, *Justification de la théorie linéaire des coques élastiques*, J. Mécanique, **15**, 813-856, 1976.
16. V. L. BERDICHEVSKII, *An energy inequality in the theory of plate bending*, Prikl. Mat. Mekh., **37**, 941-944, 1973 [in Russian].
17. Z. RYCHTER, *An error bound in Reissner's theory of transversely isotropic plates*, ZAMP, **37**, 919-923, 1986.

STRESZCZENIE

OCENA BŁĘDU UPROSZCZONEJ TEORII REISSNERA ZGINANIA PŁYT

Przedmiotem pracy jest uproszczona teoria Reissnera zginania płyt, w której ugięcie jest jedyną niewiadomą wielkością. Rozważana teoria dwuwymiarowa w połączeniu z płaską teorią sprężystości służy do skonstruowania pól trójwymiarowych, których średniokwadratowy błąd względny, w przypadku tak zwanych „regularnych" warunków brzegowych, jest wielkością rzędu grubości płyty w trzeciej potędze w porównaniu z rozwiązaniami teorii sprężystości.

РЕЗЮМЕ

РАСЧЕТ ОШИБКИ ДЛЯ УПРОЩЕННОЙ ТЕОРИИ РЕЙССНЕРА ИЗГИБА ПЛАСТИН

Рассматривается упрощенная теория Рейсснера изгиба пластин, в которой поперечный прогиб является единственной неизвестной величиной. Эта двумерная теория, в сочетании с плоской теорией упругости, используется для конструкции трехмерных полей, которых среднеквадратическая ошибка, предполагая так называемые ,,регулярные" граничные условия, является величиной порядка толщины пластины в третьей степени по сравнению с решениями пространственной теории упругости.

TECHNICAL UNIVERSITY OF BIAŁYSTOK.

Received April 21, 1987.