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ON THE PROBABILITY OF RESPONSE OF A LINEAR OSCILLATOR TO A RANDOM PULSE TRAIN(*)

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Dynamic response of a linear oscillator to a Poisson distributed train of general pulses is considered. The complete expansion for the one-dimensional probability density function of the response is presented in explicit form. The coefficients of skewness and of excess are evaluated for the steady-state response to a stationary train of square pulses and their behaviour is analyzed. The truncated series is used to examine approximately the probability density function of the stationary response. The effect of the pulse duration and of the expected rate of pulses occurrence on the approximate probability density is discussed. Positive skewness and the departure of the response probability density from the Gaussian behaviour are explained.

1. Introduction

Dynamic response of structures to random trains of pulses has been the subject of interest for some time [1–4]. The theory of Poisson or more general stochastic point processes has been used to describe the occurrence of pulses in time.

Statistical moments of the response of a linear system to random trains of Dirac delta impulses can be evaluated by following the general procedure of averaging the pertinent multifold integrals equivalent to the products of convolution integrals which represent the response. Then use is made of the degeneracy properties of product density functions [3]. Likewise the statistical moments of the response to random trains of general pulses can be evaluated. It suffices to regard the response of a linear system as the random train of filtered pulses which are the responses to individual pulses of the excitation [4]. Equivalently, direct use can be made of existing theory of random pulses (with Poisson distributed occurrence times [5] or given by the general counting process [4]). Then, for example, the evaluation of the cumulants of the response of a linear system is straightforward; the response to the individual general pulse has to be substituted for the pulse shape function in the pertinent formulae [4].

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It is a more difficult task to evaluate the response probability density function. In the case of Poisson distributed Dirac delta impulses, the response represented in the phase plane is a Poisson-driven (non-diffusive) Markov process and its probability density function satisfies an integro-differential partial equation which is the generalized Fokker–Planck–Kolmogorov equation [6, 7]. A solution to this equation through the Fourier transform technique has been proposed [7–10], and this leads to the first-order partial differential equation for the characteristic function. However, it appears that effective evaluation of the response probability density is only possible in the case of a first-order system (filter) [7, 9]. Also, in the reference [11] the response probability density is obtained from the characteristic function.

Closed-form probability density functions have also been obtained [12] for some particular cases of the filtered pulse shape function by making use of the integral equation for the distribution function of the Poisson-driven shot noise.

In principle, the one-dimensional probability density function may be obtained from the characteristic function by the numerical inversion of the Fourier transform (e.g. [13]). However, it appears that straightforward numerical techniques for the inverse Fourier transforms are often time-consuming [14].

In view of the aforementioned difficulties, the approximate analytical solutions to the problem of probability density are sought. An approximate technique widely used in stochastic dynamics is the expansion of the unknown probability density with respect to the Gaussian density function known as a Gram–Charlier, or Edgeworth, expansion (cf. e.g., [15–18]). Of special interest is the paper by Roberts [15], where the Edgeworth form of the Gram–Charlier expansion is used to investigate the probability density of the oscillator response to a random train of Dirac delta impulses. Another approximate technique is the saddle point approximation approach [19] which is based on the expansion of the cumulant generating function about the saddle point. This method requires the numerical evaluation of some integrals. Although formulated for general pulses, this method is used for numerical analysis in the reference [19] only in the case of Dirac delta impulses.

In this paper the expansion for the one-dimensional probability density function is presented in an explicit form which allows to construct systematically both Gram–Charlier and Edgeworth series. Then the truncated series is used to evaluate approximately the probability density of the response of a linear oscillator to a Poisson – distributed train of general pulses. The skewness and excess coefficients are evaluated for the stationary response to square pulses and shown in figures. The effect of pulse duration and of the average rate of pulses occurrence on the approximate stationary probability density is discussed. The contribution of consecutive terms of expansion is also investigated.

2. Statement of the problem

Consider a linear oscillator subject to a random train of general pulses. The dynamic response is governed by the equation

$$\ddot{q}+2\alpha\omega_0\dot{q}+\omega_0^2 q=\sum_{i=1}^{N(t)}F_i s(t,t_i), \tag{2.1}$$

where $s(t, t_i)$ is the pulse shape function satisfying the conditions

$$s(t,t_i)=\begin{cases}s(t-t_i), & t_i<t<t_i+T,\\ 0 & t<t_i \quad \text{or} \quad t>t_i+T,\end{cases} \tag{2.2}$$

where T denotes the pulse duration. The number of pulse occurrences in the time interval $[0, t)$ is given by the Poisson process $N(t)$ with intensity $\nu(t)$ (the average rate of pulse occurrences). The magnitudes of pulses are given by the random variables F_i which are assumed to be mutually independent of the process $N(t)$.

The response $q(t)$ can be expressed as

$$q(t)=\sum_{i=1}^{N(t)}F_i z(t,t_i,T), \tag{2.3}$$

where $z(t, t_i, T)$ is the response at time t to the pulse which occurs at time t_i [4] The response $q(t)$ has the following integral representation (cf. [3]):

$$q(t)=\int_0^t z(t,\tau,T)F(\tau)dN(\tau). \tag{2.4}$$

The substitution of $z(t,\tau,T)=\int_\tau^t h(t-\theta)s(\theta-\tau)d\theta$ (where $h(t-\theta)$ is the impulse response function) into the expression (2.4) and elementary consideration of the integration domain (the integration is performed only over the domain where $0<\theta-\tau\leqslant T$) yield, as was shown by Kawczyński [20], the splitting of the function $z(t, \tau, T)$ into two parts:

$$z(t,\tau,T)=\begin{cases}z_1(t,\tau,T)=\int_\tau^t h(t-\theta)s(\theta-\tau)d\theta, & t-T\leqslant\tau\leqslant t,\\ z_2(t,\tau,T)=\int_\tau^{\tau+T} h(t-\theta)s(\theta-\tau)d\theta, & 0\leqslant\tau\leqslant t-T.\end{cases} \tag{2.5}$$

from which the obvious splitting of the integral (2.4) follows:

$$q(t)=\int_{t-T}^{t} z_1(t,\tau,T)F(\tau)dN(\tau)+\int_0^{t-T} z_2(t,\tau,T)F(\tau)dN(\tau). \tag{2.6}$$

This representation may be used to determine the expected value and variance of the response to random pulses (e.g. [21, 22]).

Cumulants of the response process are evaluated by making use of the expressions given by LIN [4], the response $z(t, \tau, T)$ being substituted for the pulse shape function. The n-th order cumulant $\varkappa_n$ is expressed as

$$\varkappa_n(t) = \int_0^t z^n(t-\tau, T)\nu(\tau)E[F^n(\tau)]\,d\tau, \tag{2.7}$$

and taking into account the splitting (2.5) of the function $z(t, \tau, T)$, one obtains

$$\varkappa_n(t) = \int_0^{t-T} z_2^n(t-\tau)\nu(\tau)E[F^n(\tau)]\,d\tau + \int_{t-T}^{t} z_1^n(t-\tau)\nu(\tau)E[F^n(\tau)]\,d\tau. \tag{2.8}$$

In order to evaluate the probability density function, the Edgeworth series will be used. Following the procedure due to LONGUET-HIGGINS [23] and shown also by OCHI [18], let us derive the explicit and systematic form of the expansion for the one-dimensional probability density.

The probability density function $f_t(q)$ is expressed as the inverse Fourier transform of the characteristic function $\Phi(i\omega)$, i.e.,

$$f_t(q) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(i\omega)e^{-i\omega q}\,d\omega. \tag{2.9}$$

Using the identity $\Phi(i\omega) = \exp[K(i\omega)]$ and expressing the cumulant generating function $K(i\omega)$ in terms of the cumulants, one obtains

$$\Phi(i\omega) = \exp\left\{\sum_{j=1}^{\infty}\frac{(i\omega)^j}{j!}\varkappa_j(t)\right\}. \tag{2.10}$$

where $\varkappa_j(t)$ is the j-th order cumulant. Substituting Eq. (2.10) into Eq. (2.9), introducing the standardized variable $\Xi = (q-\varkappa_1)/\sqrt{\varkappa_2}$, changing the variable $\omega = s/\sqrt{\varkappa_2}$ and expanding the exponential in the series, we obtain

$$f_t(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}(s^2+2i\xi s)\right\}\left\{1+\sum_{j=3}^{\infty}\frac{\lambda_j}{j!}(is)^j + \frac{1}{2!}\sum_{j,k=3}^{\infty}\frac{\lambda_j\lambda_k}{j!k!}(is)^{j+k} + \right.$$
$$\left. + \frac{1}{3!}\sum_{j,k,l=3}^{\infty}\frac{\lambda_j\lambda_k\lambda_l}{j!\,k!\,l!}(is)^{j+k+l} + \ldots\right\}ds, \tag{2.11}$$

where

$$\lambda_j = \varkappa_j/\varkappa_2^{j/2} \quad\text{and}\quad \xi = (q-\varkappa_1)/\sqrt{\varkappa_2}.$$

By making use of the relationship

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}(s^2+2i\xi s)\right\}(is)^n\,ds = e^{-\xi^2/2}H_n(\xi), \tag{2.12}$$

where $H_n(\xi)$ is the n-th order Hermite polynomial, the following expansion for the probability density is arrived at:

$$f_t(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2}\left\{1 + \sum_{j=3}^{\infty} \frac{\lambda_j}{j!} H_j(\xi) + \frac{1}{2!} \sum_{j,k=3}^{\infty} \frac{\lambda_j \lambda_k}{j!k!} H_{j+k}(\xi) + \right.$$
$$\left. + \frac{1}{3!} \sum_{j,k,l=3}^{\infty} \frac{\lambda_j \lambda_k \lambda_l}{j!\,k!\,l!} H_{j+k+l}(\xi) + \ldots \right\}. \tag{2.13}$$

In each term the sum of the indices of the coefficients (cumulants) is equal to the order of the pertinent Hermite polynomial. This explicit form (cf. e.g., [18], [23]) allows to derive systematically the expansion up to any degree required.

All, except for the unity, terms of the expansion (2.13) are non-Gaussian terms, i.e., they account for the departure of the probability density $f_t(\xi)$ from the Gaussian behaviour. The basic measure of this departure are the coefficients of skewness λ_3 and of excess λ_4. Therefore it is important to evaluate these coefficients and to examine their behaviour.

3. Skewness and excess coefficients of the stationary response to a random train of square pulses

Let us confine our attention to the steady-state response to the stationary train of square pulses, i.e., when $s(t) = 1$ for $t_i < t < t_i + T$, the Poisson process is homogeneous: $\nu(t) = \nu = \text{const}$ and the statistical moments of the random variables F_i are constant as well: $E[F^n(t)] = E[F^n] = \text{const}$.

The expression (2.8) for the cumulants takes, after change of the variable $u = t - \tau$, the form

$$\varkappa_n = \nu E[F^n] \int_T^{\infty} z_2^n(u)\,du + \nu E[F^n] \int_0^T z_1^n(u)\,du. \tag{3.1}$$

Evaluating the integrals (2.5) in the case of square pulses yields

$$z_1(u) = \omega_0^{-2}\left[1 - e^{-\alpha\omega_0 u}\left(\frac{\alpha}{\sqrt{1-\alpha^2}} \sin\zeta u + \cos\zeta u\right)\right]. \tag{3.2}$$

$$z_2(u) = \omega_0^{-2} s(\alpha, \omega_0, T) e^{-\alpha\omega_0 u} \sin\zeta u + \omega_0^{-2} c(\alpha, \omega_0, T) e^{-\alpha\omega_0 u} \cos\zeta u, \tag{3.3}$$

where

$$s(\alpha, \omega_0, T) = \frac{-\alpha}{\sqrt{1-\alpha^2}} + \left(\frac{\alpha}{\sqrt{1-\alpha^2}} \cos\zeta T + \sin\zeta T\right) e^{\alpha\omega_0 T}, \tag{3.4}$$

$$c(\alpha, \omega_0, T) = -1 + \left(\frac{-\alpha}{\sqrt{1-\alpha^2}} \sin\zeta T + \cos\zeta T\right) e^{\alpha\omega_0 T}. \tag{3.5}$$

The same results for $z_1(u)$ and $z_2(u)$ are given in Ref. [21], in which the variance of the response to square pulses is also evaluated.

The expressions for the variance $\sigma_q^2 = \varkappa_2$, the third-order cumulant $\varkappa_3$ and the fourth-order cumulant $\varkappa_4$ are given, respectively, by

$$(3.6)\quad \sigma_q^2 = \nu E[F^2]\omega_0^{-4}\Big\{T - \frac{2\alpha}{\sqrt{1-\alpha^2}} s_1(T) - 2c_1(T) + \frac{\alpha^2}{1-\alpha^2} s_2(T) + \frac{2\alpha}{\sqrt{1-\alpha^2}} D_{11}(T) + c_2(T) + s^2(\alpha, \omega_0, T) s_2(\infty) + 2s(\alpha, \omega_0, T) c(\alpha, \omega_0, T) D_{11}(\infty) + c^2(\alpha, \omega_0, T) c_2(\infty)\Big\}.$$

$$(3.7)\quad \varkappa_3 = \nu E[F^3]\omega_0^{-6}\Big\{T - 3\frac{\alpha}{\sqrt{1-\alpha^2}} s_1(T) - 3c_1(T) + 3\frac{\alpha^2}{1-\alpha^2} s_2(T) + \frac{6\alpha}{\sqrt{1-\alpha^2}} D_{11}(T) + 3c_2(T) - \frac{\alpha^3}{(1-\alpha^2)^{3/2}} s_3(T) - \frac{3\alpha^2}{1-\alpha^2} D_{21}(T) - 3\frac{\alpha}{\sqrt{1-\alpha^2}} D_{12}(T) - c_3(T) + s^3(\alpha, \omega_0, T) s_3(\infty) + 3s^2(\alpha, \omega_0, T) c(\alpha, \omega_0, T) D_{21}(\infty) + 3s(\alpha, \omega_0, T) c^2(\alpha, \omega_0, T) D_{12}(\infty) + c^3(\alpha, \omega_0, T) c_3(\infty)\Big\}.$$

$$(3.8)\quad \varkappa_4 = \nu E[F^4]\omega_0^{-8}\Big\{T - 4\frac{\alpha}{\sqrt{1-\alpha^2}} s_1(T) - 4c_1(T) + 6\frac{\alpha^2}{1-\alpha^2} s_2(T) + 12\frac{\alpha}{\sqrt{1-\alpha^2}} D_{11}(\mathrm{T}) + 6c_2(T) - 4\frac{\alpha^3}{(1-\alpha^2)^{3/2}} s_3(T) - \frac{12\alpha^2}{1-\alpha^2} D_{21}(T) - \frac{12\alpha}{\sqrt{1-\alpha^2}} D_{12}(T) - 4c_3(T) + \frac{\alpha^4}{(1-\alpha^2)^2} s_4(T) + \frac{4\alpha^3}{(1-\alpha^2)^{3/2}} D_{31}(T) + \frac{6\alpha^2}{1-\alpha^2} D_{22}(T) + \frac{4\alpha}{\sqrt{1-\alpha^2}} D_{13}(T) + c_4(T) + s^4(\alpha, \omega_0, T) s_4(\infty) + 4s^3(\alpha, \omega_0, T) D_{31}(\infty) + 6s^2(\alpha, \omega_0, T) c^2(\alpha, \omega_0, T) D_{22}(\infty) + 4s(\alpha, \omega_0, T) c^3(\alpha, \omega_0, T) D_{13}(\infty) + c^4(\alpha, \omega_0, T) c_4(\infty)\Big\},$$

where

$$s_n(T) = \int_0^T e^{-\alpha\omega_0 n u} \sin^n \zeta u \, du,$$

$$c_n(T) = \int_0^T e^{-\alpha\omega_0 n u} \cos^n \zeta u \, du,$$

(3.9)
$$D_{mn}(T) = \int_0^T e^{-\alpha\omega_0 (m+n) u} \sin^m \zeta u \cos^n \zeta u \, du,$$

$$s_n(\infty) = \int_T^\infty e^{-\alpha\omega_0 n u} \sin^n \zeta u \, du,$$

$$c_n(\infty) = \int_T^\infty e^{-\alpha\omega_0 n u} \cos^n \zeta u \, du,$$

$$D_{mn}(\infty) = \int_T^\infty e^{-\alpha\omega_0 (m+n) u} \sin^m \zeta u \cos^n \zeta u \, du.$$

The coefficients of skewness λ_3 and of excess λ_4 are expressed, respectively, as

(3.10)
$$\lambda_3 = \frac{\varkappa_3}{\sigma_q^3} = \frac{16\alpha\sqrt{\alpha}}{3(1+8\alpha^2)} \sqrt{\frac{\omega_0}{\nu}} \frac{E[F^3]}{\{E[F^2]\}^{3/2}} \frac{\tilde{\varkappa}_3}{\tilde{\sigma}_q^3},$$

(3.11)
$$\lambda_4 = \frac{\varkappa_4}{\sigma_q} = \frac{3\alpha}{2(1+3\alpha^2)} \frac{\omega_0}{\nu} \frac{E[F^4]}{\{E[F^2]\}^2} \frac{\tilde{\varkappa}_4}{\tilde{\sigma}_q^4}.$$

It is interesting to note that in view of Eq. (3.1) the coefficients $\lambda_j = \varkappa_j/\varkappa_2^{j/2}$ appear to be of the order $(\omega_0/\nu)^{j/2-1}$ (cf. [15]).

It can be proved that as $T \to 0$ in such a way that the following products are kept constant: $T^2 E[F^2] = \text{const}$, $T^3 E[F^3] = \text{const}$. $T^4 [F^4] = \text{const}$, the expressions for σ_q^2, $\varkappa_3$ and $\varkappa_4$ approach the respective solutions for Dirac delta impulses and so do the coefficients λ_3 and λ_4. Hence it follows that $\lim_{T\to 0} \tilde{\varkappa}_3/\tilde{\sigma}_q^3 = 1$ and $\lim_{T\to 0} \tilde{\varkappa}_4/\tilde{\sigma}_q^4 = 1$ and these are the normalized coefficients of skewness and excess, respectively.

The coefficients of skewness $\tilde{\varkappa}_3/\tilde{\sigma}_q^3$ and of excess $\tilde{\varkappa}_4/\tilde{\sigma}_q^4$, plotted against the pulse duration, are shown in Figs. 1 and 2, respectively. Both coefficients are always positive and reveal very pronounced maxima at the values of pulse duration equal to the multiple natural period, i.e., for $\omega_0 T = n \cdot 2\pi$, $n = 1, 2, 3, \ldots$. The minima of these curves are flat and occur at $\omega_0 T = (2n+1)\pi$, $n = 1, 2, 3, \ldots$. The heights of maxima decrease and the heights of minima increase as the pulse duration T increases.

An important information is that the skewness of the probability density of response to square pulses is always greater than in the case of Dirac delta

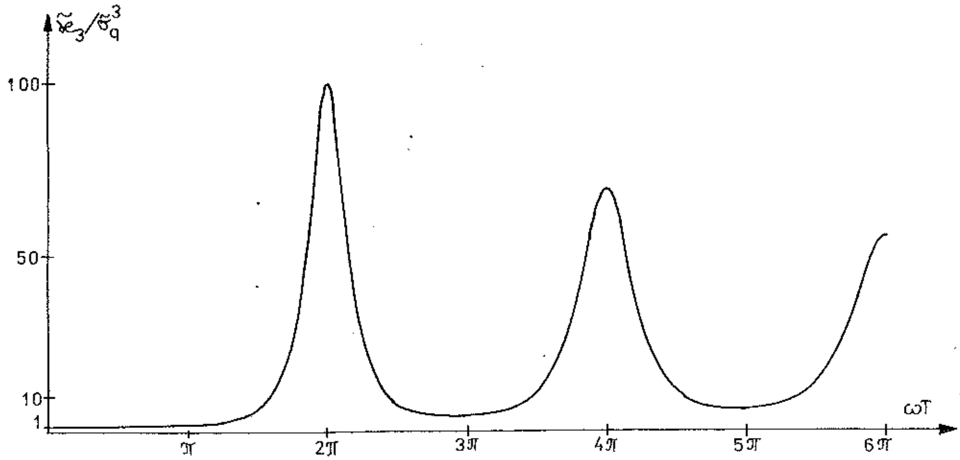

FIG. 1.

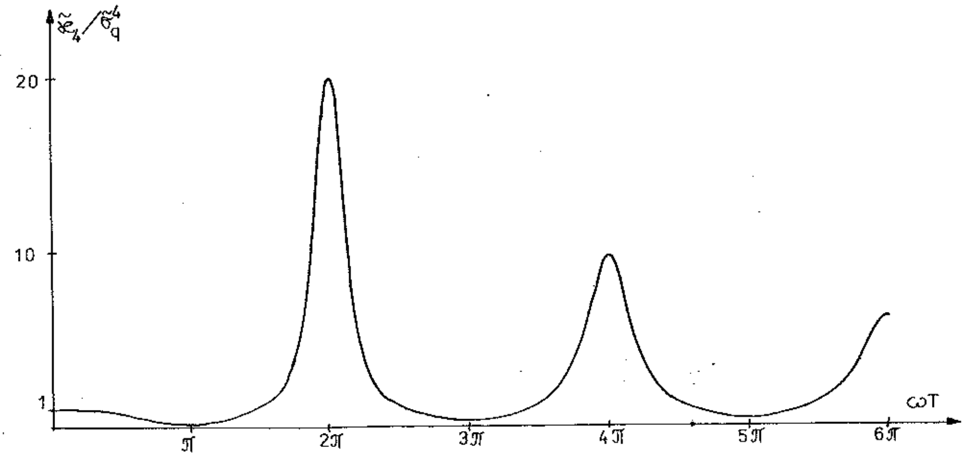

FIG. 2.

impulses. On the other hand, the behaviour of the excess coefficient is different; for example, at the values of $\omega_0 T$ in the neighbourhood of $(2n+1)\pi$, $n = 1, 2, 3, \ldots$, the excess is much smaller than in the case of Dirac delta impulses.

The absolute maxima of both coefficients occur when the pulse duration equals the natural period of the system, i.e., $\omega_0 T = 2\pi$. This means that the departure of the probability density from the Gaussian behaviour is then expected to be the largest.

4. Analysis of the approximate stationary probability density function

Let us confine our attention to the probability density $f(\xi)$ of the steady-state response to the stationary train of square pulses.

Expansion (2.13) of the probability density may be viewed in two ways. Collecting the terms of expansion according to the orders of Hermite polynomials (or, equivalently, according to the order of derivatives of the corresponding differential series), one obtains the Gram–Charlier series.

On the other hand, since the coefficients λ_j are of the order $(\omega_0/\nu)^{j/2-1}$, the expansion (2.13) is the one in powers of $(\omega_0/\nu)^{1/2}$. Grouping the terms according to these powers yields the Edgeworth series

(4.1) $$f(\xi) = \frac{1}{\sqrt{2\pi}} \exp(-\xi^2/2) \left\{ 1 + \sum_{j=1}^{\infty} \left(\frac{\omega_0}{\nu} \right)^{j/2} c_j(\xi) \right\}.$$

where the term $c_j(\xi)$ includes all possible products of the coefficients λ_j for which the product is of the order $(\omega_0/\nu)^{j/2}$.

Two questions arise in a natural way. The first one concerns the convergence of the power series (4.1) and hence the validity of such an expansion. The approximation accuracy attained by using a truncated series, consisting preferably of the first few terms of expansion only, is a second question.

Cramer [24] stated the following condition for the validity of the Edgeworth expansion for the density function $f(x)$: the expansion is convergent to $f(x)$ in every point where $f(x)$ is continuous if $f(x)$ is of finite variation in $(-\infty, \infty)$ and the integral $\int_{-\infty}^{\infty} \exp(x^2/4) f(x)\, dx$ is convergent, i.e., the tails of the density function approach zero faster than $\exp(-x^2/4)$. Unfortunately, this condition is of little practical value because the rate of approaching zero is for $f(x)$ unknown.

Some insight into the question of validity of the Edgeworth expansion is found in the references [25] and [26]. The paper by Wallace [25] deals with the distribution function of the standardized sum of n independent, identically distributed random variables. The asymptotic expansion of the distribution function is obtained, which is a formal Edgeworth expansion in powers of $n^{-1/2}$ (cf. the powers of $(\omega_0/\nu)^{1/2}$ in the case of the expansion (4.1)). Moreover, the rate of convergence to normality is estimated to depend on $n^{-1/2}$. The bounds for the error of approximation by using the truncated Edgeworth series are given, which depend on the unknown distribution function entering through the absolute moments or through the characteristic function, but their numerical usefulness is rather poor. It appears that this error is of the same order of magnitude as the first neglected term. An important information is that the asymptotic property of expansion is a property of finite partial sums (as the number of components $n \to \infty$). From the explanations given in the reference [25] concerning the convergence of the infinite series (dependence on the value of n), it follows that the convergence of the infinite series (4.1) is assured if the ratio ω_0/ν is sufficiently small (i.e., ν is large), but for an arbitrary ω_0/ν the series may or may not be convergent. Usually for large ω_0/ν (small ν) only the addition of the first few terms improves the approximation.

In the reference [26] more precise formulation of the assumptions under which the formal Edgeworth expansion is valid is given. The estimates for the rate of convergence to normality and the error due to using the partial sum of expansion are given not for the distribution function, but for non-zero probabilities defined on Borel sets.

The most important thing in practical applications is to know whether the truncated series (preferably at a low level) may provide an adequate approximation to the actual probability density. For example, satisfactory results have been obtained in approximating the experimental histograms by the first few terms only of the Edgeworth series [18]. When it is known that the truncated series is a good approximation, then the question of convergence is of minor importance.

There exist also some results concerning the total deviation from Gaussianity of the Poisson driven shot noise processes. For example, in the reference [27] the upper bound for the maximum of the difference of the shot noise distribution function $F(x, t)$ and normal distribution function $G(x, t)$ is evaluated as

$$|F(x, t) - G(x, t)| < \frac{4}{3}\left[2\pi \frac{I_3^2(t)}{I_2^3(t)}\right]^{1/2}, \tag{4.2}$$

where $I_3(t)$ is the third absolute moment

$$I_3(t) = \int_{-\infty}^{\infty} |h(t, \tau)|^3 \nu(\tau)\, d\tau, \tag{4.3}$$

where $h(t, \tau)$ is the pertinent filtered pulse shape function, and $I_2(t)$ is the variance.

Under the assumption that the pulse shape function $h(t, \tau)$ of the shot noise process is bounded by a constant M, i.e., $|h(t, \tau)| \leqslant M$, the bound given by the inequality (4.3) is estimated from above and is shown to depend on the inverse standard deviation. This means that as the variance tends to infinity (e.g. when $\nu \to \infty$ cf. [2]), the shot noise approaches the Gaussian process.

On the other hand, the obvious inequality

$$\varkappa_3 = \int_{-\infty}^{\infty} h^3(t, \tau)\nu(\tau)\, d\tau < \int_{-\infty}^{\infty} |h(t, \tau)|^3 \nu(\tau)\, d\tau, \tag{4.4}$$

which holds for the oscillatory function $h(t, \tau)$ (e.g., the response of a vibratory system to an impulse or general pulse) may be used. Then the bound given by the inequality (4.2) is estimated from below by the skewness coefficient $\lambda_3 = \varkappa_3/(\varkappa_2)^{3/2}$. Thus the skewness coefficient behaviour provides information about the departure of the shot noise process from normality. Let us explain it in more detail. In the case of a small skewness coefficient, two situations are possible: the upper bound of the departure from normality may be low and then only small departure can be expected, or this bound may be high – then both large and small deviations are likely to occur. On the other hand, when the skewness coefficient is large, the upper bound has to be accordingly high, hence the first situation described above is impossible; the deviation cannot be bounded at a low level.

Altogether it may be concluded that the successive approximations including the first few terms of the Edgeworth expansion are certainly improvements with respect to the Gaussian approximation (Gaussian term of the expansion) and provide the information about the tendency of the departure from normality.

In view of all the above observations and remarks, let us truncate the series (2.13) in such a way as to retain the terms of the order $(\omega_0/\nu)^{1/2}$ and $(\omega_0/\nu)^1$. The result is

$$f(\xi) = \frac{1}{\sqrt{2\pi}} \exp(-\xi^2/2) \left\{ 1 + \left(\frac{\omega_0}{\nu} \right)^{1/2} c_1(\xi) + \frac{\omega_0}{\nu} c_2(\xi) \right\}, \tag{4.5}$$

where

$$c_1(\xi) = a_3 H_3(\xi), \tag{4.6}$$

$$c_2(\xi) = a_4 H_4(\xi) + a_6 H_6(\xi), \tag{4.7}$$

$$a_3 = \frac{8\alpha\sqrt{\alpha}}{9(1+8\alpha^2)} \frac{E[F^3]}{\{E[F^2]\}^{3/2}} \frac{\tilde{\varkappa}_3}{\tilde{\sigma}_q^3}, \tag{4.8}$$

$$a_4 = \frac{\alpha}{16(1+3\alpha^2)} \frac{E[F^4]}{\{E[F^2]\}^2} \frac{\tilde{\varkappa}_4}{\tilde{\sigma}_q^4}, \tag{4.9}$$

$$a_6 = \frac{32}{81} \frac{\alpha^3}{(1+8\alpha^2)^2} \frac{\{E[F^3]\}^2}{\{E[F^2]\}^3} \frac{\tilde{\varkappa}_3^2}{\tilde{\sigma}_q^6}, \tag{4.10}$$

$$H_3(\xi) = \xi^3 - 3\xi, \tag{4.11}$$

$$H_4(\xi) = \xi^4 - 6\xi^2 + 3, \tag{4.12}$$

$$H_6(\xi) = \xi^6 - 15\xi^4 + 45\xi^2 - 15. \tag{4.13}$$

It is seen that the contribution of the non-Gaussian terms (the departure of the probability density from the Gaussian distribution) increases as the damping ratio α increases and as the average rate ν of pulse occurrence decreases; moreover, it depends on the distribution of the random amplitudes and, of course, on the pulse duration T.

Let us first discuss the effect of the pulse magnitudes (amplitudes) distribution on the probability density $f(\xi)$. In the case of normally distributed pulses magnitudes, denoting $\frac{\sigma_F}{E[F]} = \varphi$, one obtains

$$\frac{E[F^3]}{\{E[F^2]\}^{3/2}} = \frac{1+3\varphi^3}{(1+\varphi^2)^{3/2}} = \varphi_{III}. \tag{4.14}$$

$$\frac{E[F^4]}{\{E[F^2]\}^2} = \frac{3\varphi^4+6\varphi^2+1}{(1+\varphi^2)^2} = \varphi_{IV}. \tag{4.15}$$

It appears that the coefficient φ_{III} attains maximum when $\varphi = 1$, and then $\varphi_{III} = \sqrt{2}$, $\varphi_{IV} = 2.5$. The other extreme case takes place when φ_{IV} attains maximum, i.e. when $\varphi \to \infty$. Then $E[F] = 0$ and also $E[F^3] = 0$, which implies that $\varphi_{III} = 0$, $\varphi_{IV} = 3$. This means that in the case of zero-mean normally distributed pulse magnitudes the response probability density curve is symmetric.

If the pulse magnitudes are Rayleigh distributed, then it can be shown that $\varphi_{III} = \frac{3}{4}\sqrt{\pi} \cong 1.329$ and $\varphi_{IV} = 2$. In the case of uniformly distributed positive pulse magnitudes, with probability density function, $g_F(\eta) = \frac{1}{a}$, $\eta \in (0, a)$, the coefficients are $\varphi_{IIII} = \frac{3\sqrt{3}}{4} \cong 1.299$, $\varphi_{IV} = 1.8$. Of course if the pulse magnitudes F are deterministic, both coefficients are equal to unity: $\varphi_{III} = \varphi_{IV} = 1$.

Hence, except for the case of zero-mean normally distributed pulse magnitudes, the coefficients φ_{III} and φ_{IV} expressed in terms of third- and fourth-order moments, respectively, take the values which are not very much different for different probability distributions assumed. In the following analysis the pulse magnitudes are assumed to be Gaussian distributed, with $\varphi_{III} = \sqrt{2}$ and $\varphi_{IV} = 2.5$.

The results of the analysis of the approximate probability density are shown in Fig. 3 through 10.

In the case of very short pulses ($\omega_0 T < 1/4$), Fig 3., the probability density is practically the same as for Dirac delta impulses; even for a large value $\omega_0/\nu = 100$ the effect of the skewness term of expansion (4.5) is very weak (dotted line), but taking into account the excess term (dashed line) reveals the large departure of the probability density from the Gaussian distribution (represented by a solid line).

When the pulse duration is equal to half of the natural period ($\omega_0 T = \pi$),

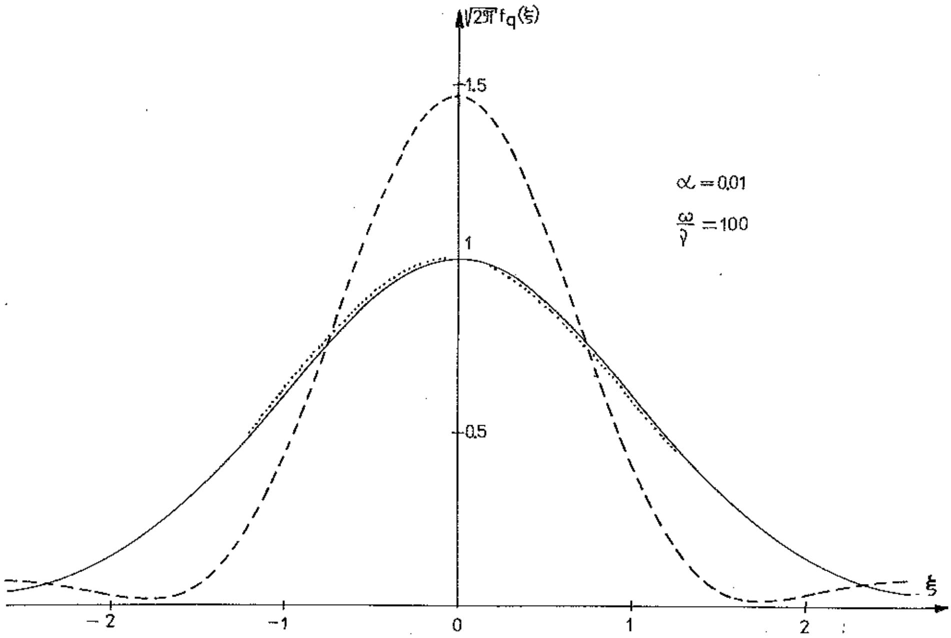

FIG. 3.

Fig 4, also the contribution of the excess term is very low as it might be expected from the behaviour of $\tilde{\varkappa}_4/\tilde{\sigma}_q^4$ shown in Fig. 2. Consequently, even for a large value $\omega_0/\nu = 50$ the probability density is merely slightly non-Gaussian.

In the subsequent analyses the skewness λ_3, the excess λ_4 and the λ_3^2 terms are taken into account. The evolution of the probability density curve with increasing pulse duration can be observed in each of the Figs. 5, 6 and 7, for each of the values of the ratio $\omega_0/\nu = 1, 2$, and 5, respectively. Indeed, the probability density curves reveal positive skewness, i.e., the probability of occurrence of large positive values of the response is greater than that of large negative values, whereas the probability of occurrence of small positive values is less than that of small negative values. The approximate probability density curves attain negative values in some small regions in the vicinity of $\xi = -2$ (i.e., for $\omega_0 T = \pi$ and $\omega_0/\nu = 1$ – Fig. 5, for $\omega_0 T = \frac{15}{8}\pi$; 2π and $\frac{\omega_0}{\nu} = 2$ – Fig 6 and for $\omega_0 T = \frac{15}{8}\pi$; 2π and $\omega_0/\nu = 5$ – Fig. 7), which should be regarded as a result of the insufficient number of terms of expansion. This also means that departure from the Gaussian behaviour is large in these cases.

The evolution of the probability density curve for $\omega_0 T = 2\pi$ and with an increasing ratio ω_0/ν is shown in Fig. 8.

As the pulse duration increases (up to $\omega_0 T = 2\pi$) and as the ratio ω_0/ν increases (the average rate decreases), the positive skewness of the probability

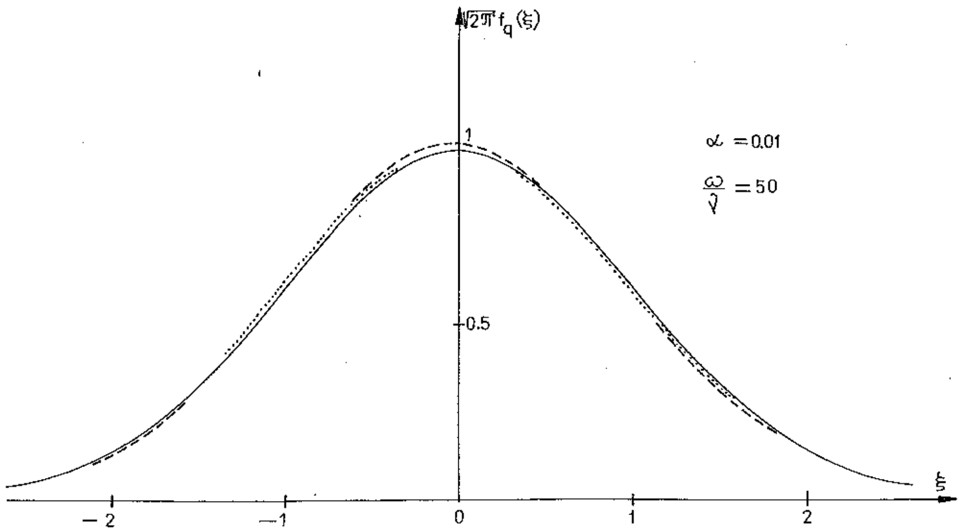
$\sqrt{2\pi}\, f_q(\xi)$
1
0.5
$\alpha = 0.01$
$\frac{\omega}{\gamma} = 50$
ξ
−2
−1
0
1
2

FIG. 4.

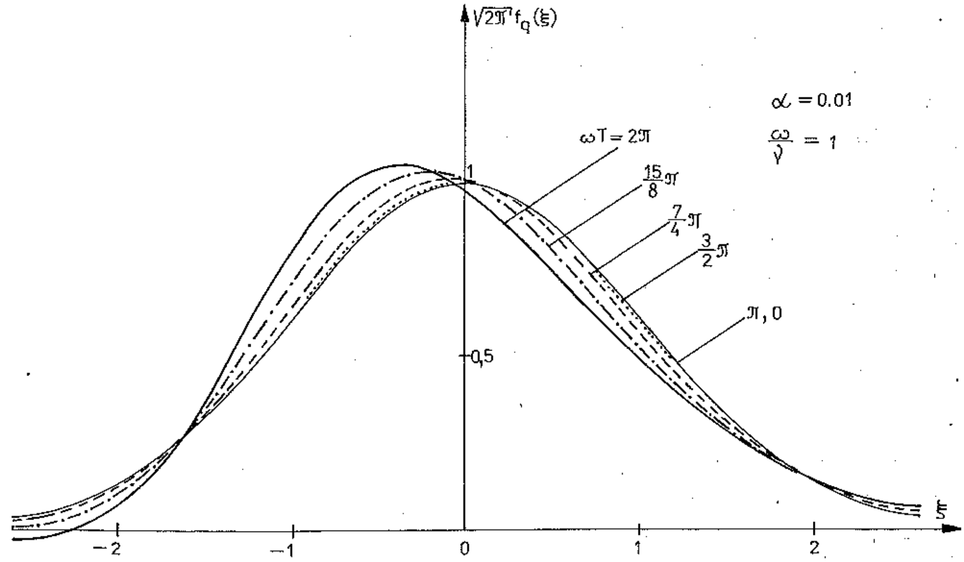
$\sqrt{2\pi}\, f_q(\xi)$
$\alpha = 0.01$
$\frac{\omega}{\gamma} = 1$
$\omega T = 2\pi$
$\frac{15}{8}\pi$
$\frac{7}{4}\pi$
$\frac{3}{2}\pi$
$\pi, 0$
1
0,5
ξ
−2
−1
0
1
2

FIG. 5.

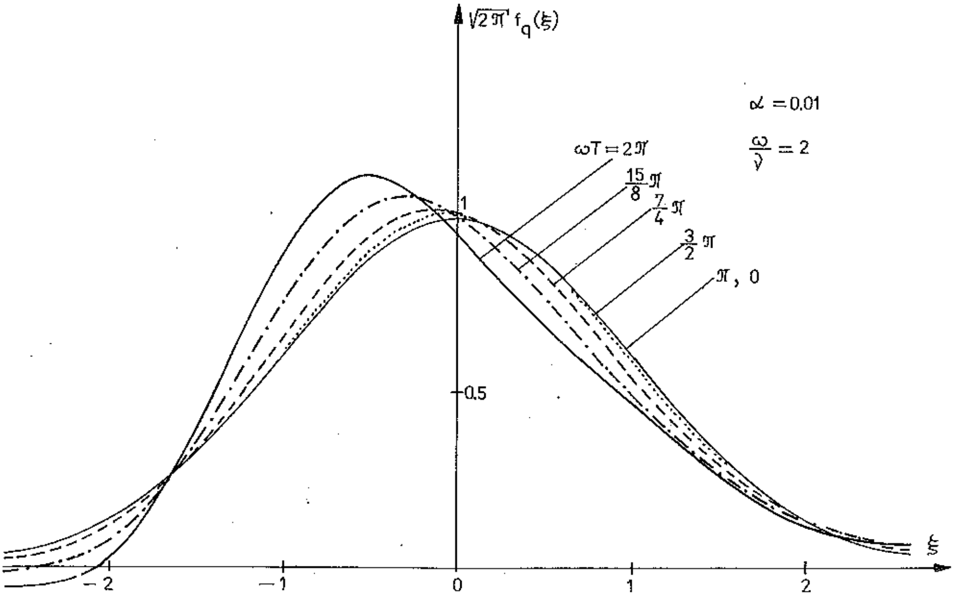
$\sqrt{2\pi}\, f_q(\xi)$
$\alpha = 0.01$
$\frac{\omega}{\nu} = 2$
$\omega T = 2\pi$
$\frac{15}{8}\pi$
$\frac{7}{4}\pi$
$\frac{3}{2}\pi$
π, 0
1
0.5
−2
−1
0
1
2
ξ

FIG. 6.

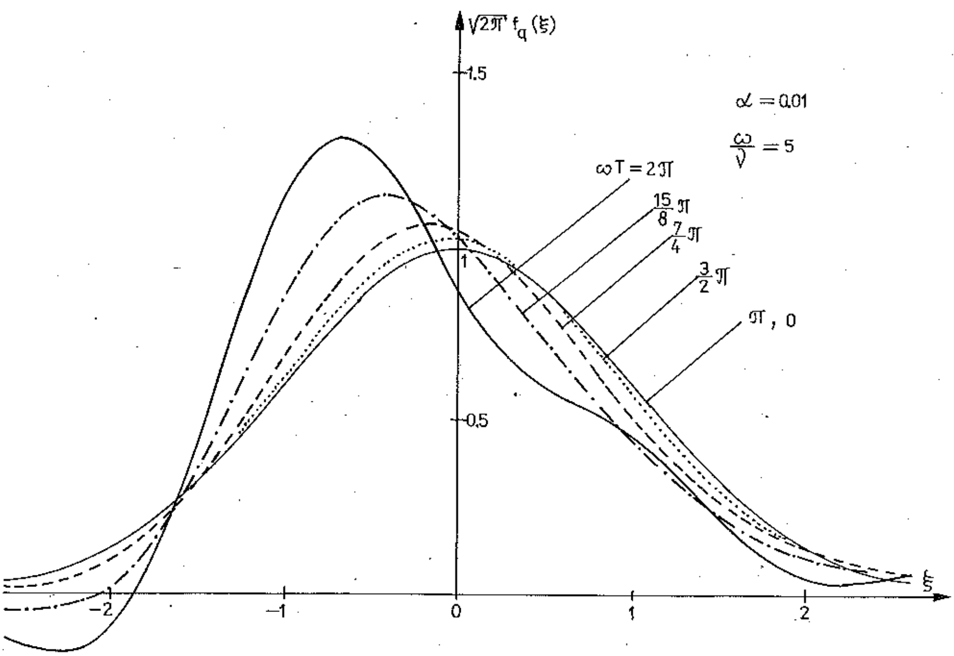
$\sqrt{2\pi}\, f_q(\xi)$
1.5
$\alpha = 0.01$
$\frac{\omega}{\nu} = 5$
$\omega T = 2\pi$
$\frac{15}{8}\pi$
$\frac{7}{4}\pi$
$\frac{3}{2}\pi$
π, 0
1
0.5
−2
−1
0
1
2
ξ

FIG. 7.

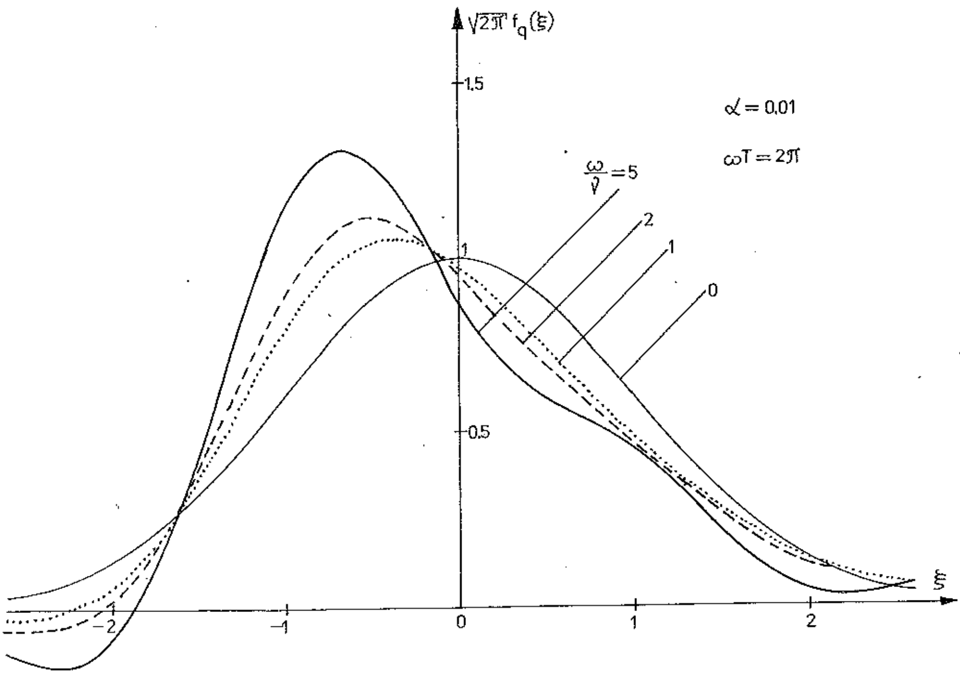

FIG. 8.

density curve increases, what corresponds to an increase of the probability of large positive and small negative values of the response and to the decrease of the probability of small positive and large negative values.

The contribution of consecutive terms of the expansion (4.5) in the case of large departure from normality ($\omega_0 T = 2\pi$) is shown in Fig. 9 for $\omega_0/\nu = 2$ and in Fig. 10 for $\omega_0/\nu = 5$.

Positive skewness of the probability density function can be explained by the quasi-static effect of the general pulses, which is the most clear when the pulse duration equals the natural period of the system $T = 2\pi/\omega_0$. Then the value of the response $z_1(T)$ decisive for the induced free vibration is very small, hence the free vibration has the nature of small amplitude oscillations about the equilibrium level; the response is essentially quasi-static. However, the probability density considered herein is that of the standarized variable, hence these small oscillations should be regarded as negative values with respect to the mean level. Therefore the small negative values of the response are more frequent (more probable) than small positive ones and large positive values are more frequent (more probable) than large negative ones.

The departure of the probability density from the Gaussian distribution may be explained as follows. The response of the system at the time t is, of course, due to all the pulses which occurred before the instant t and, as the

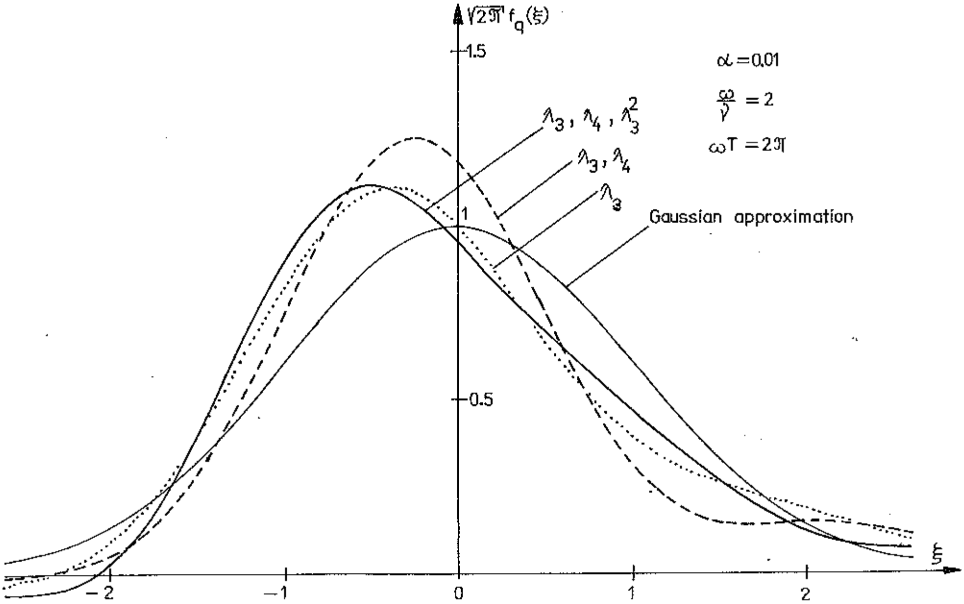
$\sqrt{2\pi}\, f_q(\xi)$
1.5
1
0.5
−2
−1
0
1
2
ξ
$\alpha = 0.01$
$\frac{\omega}{\gamma} = 2$
$\omega T = 2\pi$
$\lambda_3, \lambda_4, \lambda_3^2$
λ_3, λ_4
λ_3
Gaussian approximation

FIG. 9.

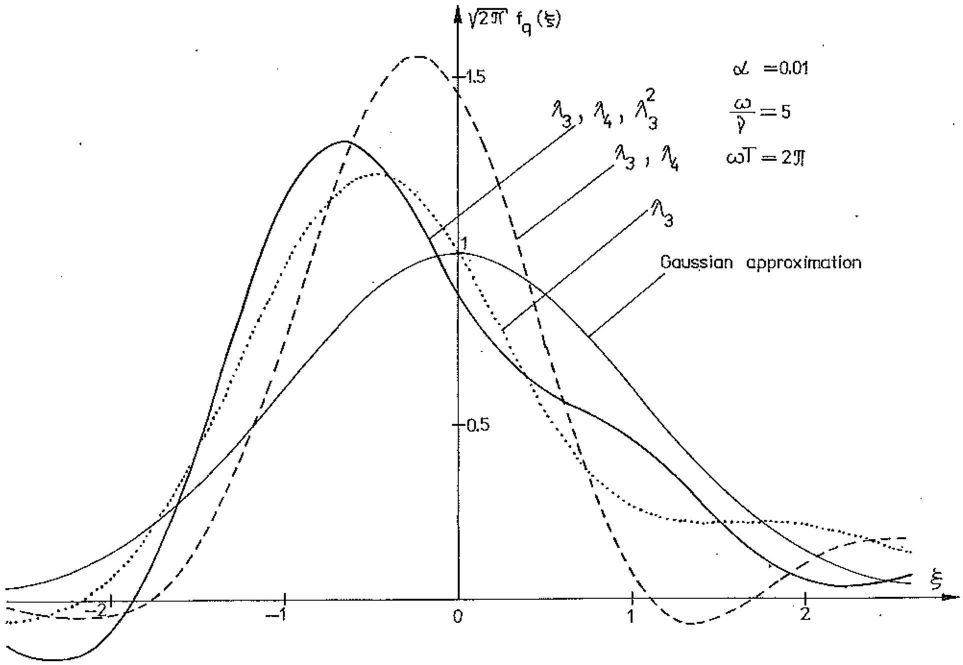
$\sqrt{2\pi}\, f_q(\xi)$
1.5
1
0.5
−2
−1
0
1
2
ξ
$\alpha = 0.01$
$\frac{\omega}{\gamma} = 5$
$\omega T = 2\pi$
$\lambda_3, \lambda_4, \lambda_3^2$
λ_3, λ_4
λ_3
Gaussian approximation

FIG. 10.

number of pulses tends to infinity, the response probability density approaches the Gaussian one (cf. e.g. [2]). However, the physical interpretation of the splitting of the integral (2.6) is such that the first term accounts for the free vibration due to the pulses which originated before the instant $t-T$ (cf. [21, 22]). When the pulse duration is equal to the natural period of the structure, the induced free vibration is not important (see the explanation above) and the response is essentially quasi-static. Consequently, the pulses which occurred in the time interval $(t-T, t)$ are decisive for the response at the time t. However, from the Poisson distribution law it follows that for finite ν only very small probabilities correspond to the occurrences of large numbers of pulses in such a short time interval. The response is then effectively due to the relatively small number of pulses and hence it is far from being Gaussian.

The effect of damping can be explained in a similar way. As the damping ratio α increases, the free vibration induced by each pulse is more strongly damped out and hence only the pulses which occurred in a certain limited time interval prior to the instant t are practically decisive for the response at that instant.

5. Discussion and conclusions

The verification of the accuracy of the approximation used should be made by comparing the results presented with those obtained by other, highly reliable methods, e.g., digital simulation for a sufficiently large number of sample functions. Unfortunately no studies are available dealing with the problem of random general pulses which would make possible a direct comparison of results.

However, as far as the departure of the response from Gaussianity is concerned, some extra insight into the question may be gained by comparing the findings of the present paper with others, also dealing with the problem of Poisson driven responses.

In the reference [19] Roberts developed the saddle point approximation technique for evaluating the probability density of the response to Poisson distributed pulses. While the Edgeworth series may be regarded as the result of inverting the characteristic function with the help of expanding the cumulant generating function about the origin, this alternative technique is based on the expansion of the cumulant generating function about the saddle point. The advantage of saddle point approximation over the Edgeworth series approach is that the resulting expansion can yield accurate approximations to the probability of exceeding extreme amplitudes, whereas the approximations obtained from the Edgeworth series assume negative values in some regions at the tails of the probability density curve. Though the numerical analysis based on the saddle point approximation is performed in the reference [19] for Dirac delta impulses only, the findings of a qualitative nature may be helpful.

The results obtained from saddle point approximation and from digital simulation for Dirac delta impulses with Gaussian magnitudes and for $\omega_0/\nu = 5$, $\alpha = 0.2$ as well as for $\omega_0/\nu = 44.2$, $\alpha = 0.0226$ show that the departure of the density curve from the Gaussian behaviour is large in these cases. It is worthwhile noting that JANSEEN and LAMBERT [14] have proposed the parameter $\gamma = \left(2\alpha \frac{\omega_0}{\nu}\right)^{-1}$ for the characterization of the departure from normality for the response to Poisson distributed impulses and adopted the condition $\gamma < 1$ as the requirement for a highly non-Gaussian response. This means that the response is highly non-Gaussian if the damping ratio is sufficiently large and/or the train is sparse enough (sufficiently large ratio ω_0/ν); the combination $\alpha\omega_0/\nu$ being decisive for the departure from normality. In fact, in both cases mentioned above of the reference [19], it is $\alpha\omega_0/\nu = 1$.

If the damping ratio equals $\alpha = 0.01$, as in the present paper, the requirement for a highly non-Gaussian response to Dirac delta impulses is $\omega_0/\nu > 50$. As a matter of fact, the approximate probability density curve for Dirac delta impulses with $\omega_0/\nu = 100$, as it is seen in Fig. 3, is highly non-Gausian.

However, in the case of general pulses, owing to the behaviour of the skewness and excess coefficients, the highly non-Gaussian response should be expected even for much lover values of ω_0/ν. This conjecture is certainly supported by the occurrence of negative values of the approximate probability density for $\omega_0/\nu = 1, 2, 5$ and for $\omega_0 T = 2\pi$ (see e.g., Fig. 8), which can be interpreted as the effect of the insufficient number of terms of expansion in the case of highly non-Gaussian density function being difficult to approximate.

The findings of the present paper concerning the influence of the ratio ω_0/ν and damping ratio α on departure from normality are in accordance with the observation due to JANSSEN and LAMBERT [14].

The numerical results reported herein have been obtained for the particular case of square pulses. However, the qualitative features of the response probability density behaviour which have been revealed will be similar for other types of general pulses.

It may be concluded that in spite of the fact that the approximate probability density is obtained for the first few terms of expansion only, it reveals the behaviour which is supported by the results of other studies on Poisson driven responses of linear systems. Although the Edgeworth series truncated at a low level may yield very reliable results if the departure from Gaussianity is not large (e.g. the train is dense enough), the approximations constructed by including the consecutive non-Gaussian terms of the expansion are useful also in other situations since they are improvements with respect to the Gaussian approximation. This way it provides information about the probability density of the response to Poisson distributed general pulses, in particular about the tendency of the departure from Gaussianity.

The approach presented may be generalized to the case of an acutal continuous structure. If the problem is described by the set of equations governing the modal responses, then the cumulants of the structural response are expressed in terms of the cross-cumulants of the modal responses. The generalization of the pertinent formulae to the case of cross-cumulants is straightforward.

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STRESZCZENIE

O ROZKŁADZIE PRAWDOPODOBIEŃSTWA ODPOWIEDZI OSCYLATORA LINIOWEGO NA DZIAŁANIE LOSOWEJ SERII IMPULSÓW ROZŁOŻONYCH W CZASIE

Rozważane są drgania oscylatora liniowego poddanego poissonowskiej serii impulsów rozłożonych w czasie. Pełne rozwinięcie w szereg jednowymiarowej funkcji gęstości prawdopodobieństwa odpowiedzi układu jest przedstawione w jawnej postaci. Zanalizowano współczynniki skośności i spłaszczenia rozkładu wyznaczone dla ustalonego stanu drgań pod wpływem stacjonarnej serii impulsów prostokątnych. W celu przybliżonego zbadania gęstości prawdopodobieństwa odpowiedzi dokonano obcięcia omawianego szeregu. Zbadano wpływ czasu trwania impulsu oraz średniego natężenia pojawiania się impulsów na gęstość prawdopodobieństwa odpowiedzi. Wyjaśniono zjawisko dodatniej skośności oraz odchylenia rozkładu prawdopodobieństwa od rozkładu normalnego.

РЕЗЮМЕ

О РАСПРЕДЕЛЕНИИ ВЕРОЯТНОСТИ ОТВЕТА ЛИНЕЙНОГО ОСЦИЛЛЯТОРА НА ДЕЙСТВИЕ СЛУЧАЙНОЙ СЕРИИ ИМПУЛЬСОВ, РАСПРЕДЕЛЕННЫХ ВО ВРЕМЕНИИ

Рассматриваются колебания линейного осциллятора, подверженного пуассоновской серии импульсов, распределенных во времени. Полное разложение в ряд одномерной функции плотности вероятности ответа системы представлено в явном виде. Проанализированы коэффициенты асимметрии и сплющенности распределения, определенные для установленных колебаний под действием стацинонарной серии плямоугольных импульсов. С целью приближенного исследования плотности вероятности ответа было произведено усечение рассматриваемого ряда. Исследовалось влияние продолжительности импульса,

а также средней интенсивности появления импульсов на плотность вероятности ответа. Было выяснено явление положительной асимметрии, а также уклона распределения вероятности от нормального распределения.

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