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FREE VIBRATIONS OF THIN, ELASTIC, ORTHOGONALLY STIFFENED SHELLS OF REVOLUTION WITH STIFFENERS TREATED AS DISCRETE ELEMENTS

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This paper presents the method and numerical examples of the calculation of free-vibration frequencies and modes of thin, elastic, orthogonally stiffened shells of revolution. The variational formulation has been employed. The integro-differential Hamilton functional is brought to the algebraic form by separation of the variables, then expanded into trigonometric series in the circumferential direction and discretized by the finite difference scheme in the meridional direction. The three calculation models are compared: a shell with stringers modelled by a smooth orthotropic shell, a shell with stringers treated as discrete elements without couplings between harmonics and with the couplings. It is shown that taking couplings into account may influence the frequencies and modes calculated. The results obtained are compared with experimental ones published in the literature.

1. Introduction

Shells of revolution occur in many recent engineering structures. Due to the strength functional and technological conditions, axially symmetric shell structures and elements of the shell are often stiffened with a discrete system of meridional and/or circumferential ribs.

A number of papers and specialistic computer programs have been devoted to the dynamical analysis of a smooth shell or of a shell with circular rings discussed in [5]. In the majority of papers concerning the dynamical analysis of meridionally stiffened shells, the stiffeners are taken into account indirectly, via smearing out and via replacing the meridionally stiffened structure with an appropriate smooth orthotropic shell. In these programs, the problem is transformed to the problem of the estimation of the stiffness and inertia parameters of the smooth, equivalent shell [2, 3, 8]. These approximations are sufficient if the meridional stiffeners are dense and equally spaced along the shell circumference. However, in the case of a small number of meridional stringers, a substitute smooth orthotropic continuum cannot represent the dynamic behaviour of the real stiffened structure with sufficient accuracy. The frequencies calculated are most often inaccurate and the corresponding modes may significantly differ from the eigen-modes of the real stiffened structure. One cannot find basing on those results, even in the reasonable approximation, the state of dynamic stresses of a vibrating structure, which may be necessary for the examination of dynamic response of the shell structure to the external excitation.

Most of the works which present the analysis of meridionally stiffened shells of revolution with stringers treated as discrete elements refer to shells with zero Gaussian curvature, mainly to cylindrical shells. A wide review of the literature concerning the problem is presented in [1, 4, 14].

In this paper the free-vibration analysis of orthogonally-stiffened shells of revolution with arbitrary geometry of the shell meridian is presented. The paper is based on the method brought out in [5] where the free-vibration analysis of the segmented shells of revolution reinforced with circular rings are described. Thus in this paper the effect of meridional stiffeners is mainly considered. As in [5], the problem is formulated in the general form as a minimization of a functional arising from Hamilton's principle. The strain and kinetic energy of elements (shell and stiffeners) are calculated with the use of the linear theory of shells [13] and the theory of weakly curved rods [12]. The problem formulated in this way is solved numerically by bringing it to the generalized eigenproblem. The numerical examples presented here show the influence of the way in which the meridional stiffeners are taken into account on the eigen frequencies and the modes. It is shown that the meridional stringers generate couplings between the circumferential numbers of waves. The coupling effect, when taken into account, may change significantly the numerically calculated values of frequencies and modes of free vibration, and then the results agree with the experimental ones described in the literature [6, 11].

2. Formulation of the problem

Similarly as in [5], the problem considered here has been formulated as a variational one [2], i.e.,

$$\delta H = \delta \int_{t_0}^{t_1} (T-E)\,dt = 0, \tag{2.1}$$

where E denotes the elastic energy of all the elements of the system and T denotes their kinetic energy. The components of the energies may be presented in the form

$$E_q = \frac{1}{2}\int_{\alpha_q} \mathbf{n}_q^T \boldsymbol{\varepsilon}_q \, d\alpha_q, \qquad T_q = \frac{\omega^2}{2}\int_{\alpha_q} \mathbf{u}_q^T \bar{\mathbf{M}}_q \mathbf{u}_q \, d\alpha_q, \qquad q = s, c, l. \tag{2.2}$$

The subscript q stands for the letters: s — for shell segments, c — for circular rings, l — for meridional stringers. Moreover,

$$\alpha_q = \begin{cases} \text{— the areas of the middle shell surface for } q = s, \\ \text{— the length of the ring axis for } q = c, \\ \text{— the length of the stringer axis for } q = l. \end{cases}$$

The magnitudes $\mathbf{n}_q$ are the generalized force vectors and $\boldsymbol{\varepsilon}_q$ are the corresponding strain vectors. The quantities $\mathbf{u}_q$ are generalized displacement vectors and $\bar{\mathbf{M}}_q$ are the corresponding matrices of inertia coefficients of the particular elements of the system.

The constitutive and kinematical relations can be written in the form

$$\text{(2.3)} \qquad \mathbf{n}_q = \mathbf{C}_q \boldsymbol{\varepsilon}_q, \quad \boldsymbol{\varepsilon}_q = \bar{\mathbf{A}}_q \mathbf{u}_q, \quad q = s, c, l,$$

where $\mathbf{C}_q$ are the elasticity matrices, $\bar{\mathbf{A}}_q$ are the differential operators. The form of the relations (2.3) results from the hypothesis assumed for the model of the shell segments and stiffeners. The hypotheses assumed here are the Sander's variant of the linear theory of shells [13] and the theory of weakly curved beams [12] for stiffeners, as in [5]. For the shell segments ($q = s$) and the circular rings ($q = c$), the vector appearing in the relations (2.3) are given in [5] Chapter 3, and for meridional stringers the vectors $\mathbf{n}_l$, $\boldsymbol{\varepsilon}_l$, $\mathbf{u}_l$ have the components:

$\mathbf{n}_l = \{N_l \; M_{zl} \; M_{yl} \; M_l\}^T$ generalized force vector (Fig. 1),
$\boldsymbol{\varepsilon}_l = \{\varepsilon_l \; \varkappa_{yl} \; \varkappa_{zl} \; \varkappa_l\}^T$ strain vector,
$\mathbf{u}_l = \{u_l \; v_l \; w_l \; v_{yl} \; v_{zl}\}$ generalized displacement vector (Fig. 1).

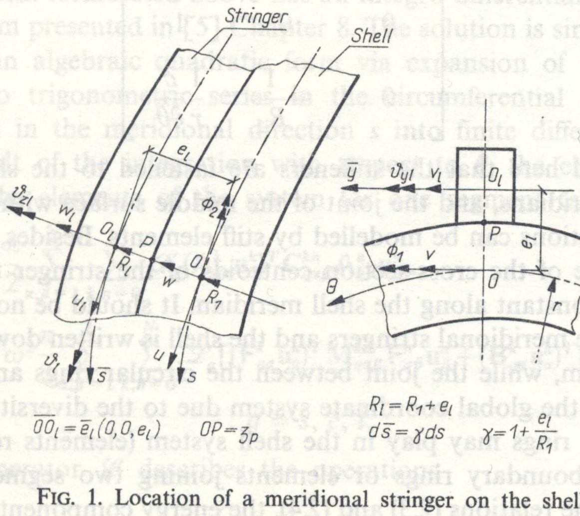


FIG. 1. Location of a meridional stringer on the shell.

Employing geometrical relations, which result from the hypothesis assumed for the model of the system, and the compatibility conditions for the strains of the stiffeners and the shell, all displacement parameters may be expressed through the three components of the displacement of the shell middle surface. These relations may be written in the form

$$\left.\begin{array}{l}\{\phi_1\ \phi_2\}^T\\ \{\vartheta_{xc}\ \vartheta_{zc}\}^T\\ \{\vartheta_{yl}\ \vartheta_{zl}\}\end{array}\right\} = \mathbf{B}_q\mathbf{u}, \quad q = s, c, l,$$

$$(2.4) \qquad \{u_q\, v_q\, w_q\, \vartheta_q\}^T = \mathbf{F}_q\mathbf{u}, \quad q = c, l.$$

$$\mathbf{u} = \{u\, v\, w\}^T.$$

The differential operators $\mathbf{B}_q(q = s, c, l)$ contain relations between the inter-dependent components of the vectors $\mathbf{u}_q$ and the vector $\mathbf{u}$ due to the Kirchhoff–Love hypothesis. The operators $\mathbf{F}_q(q = c, l)$ represent the transformation of the independent displacement components of the cross-section centroids of the stiffeners to the middle shell surface displacements u, v, w. The operator $\mathbf{F}_c$ for the circular rings is given in [5] Chapter 6, and for the meridional stringers $\mathbf{F}_l$ is calculated at $\theta = \theta_l$, i.e., where the stringers are fastened to the shell, and has the following form:

$$\mathbf{F}_l = \begin{bmatrix} 1+\dfrac{e_l}{R_1} & 0 & -e_l\dfrac{\partial}{\partial s} \\ 0 & 1+\dfrac{e_l}{R_1} & -e_l\dfrac{\partial}{\partial\theta} \\ 0 & 0 & 1 \\ 0 & -\dfrac{1}{R_2} & \dfrac{1}{r}\dfrac{\partial}{\partial\theta} \end{bmatrix}.$$

It is assumed here that the stiffeners are fastened to the shell along the parallels and meridians, and the joint of the middle surface with the centroids of their cross-sections can be modelled by stiff elements. Besides, it is assumed that the distance of the cross-section centroids of the stringer to the middle shell surface is constant along the shell meridian. It should be noticed that the joint between the meridional stringers and the shell is written down in the local coordinate system, while the joint between the circular rings and the shell is written down in the global coordinate system due to the diversity of functions that the circular rings may play in the shell system (elements reinforcing the shell segments, boundary rings or elements joining two segments).

Employing the relations (2.3) and (2.4), the energy components (2.2) may be expressed only in terms of the displacements of the shell middle surface u, v, w:

$$E_q = \frac{1}{2}\mathscr{L}_q(\mathbf{A}_q\mathbf{u})^T\mathbf{C}_q\mathbf{A}_q\mathbf{u}, \quad (q = s, c, l),$$

$$(2.5)$$

$$T_q = \frac{\omega^2}{2}\mathscr{L}_q[(\mathbf{F}_q\mathbf{u})^T\mathbf{M}_q\mathbf{F}_q\mathbf{u}+(\mathbf{B}_q\mathbf{u})^T\mathbf{I}_q\mathbf{B}_q\mathbf{u}],$$

where

$$\bar{\mathscr{L}}_q = \begin{cases} \int\limits_s\int\limits_\theta (\ ) r d\theta ds & \text{for} \quad q = s, \\ \int\limits_\theta (\ ) R d\theta & \text{for} \quad q = c \quad \text{and} \quad s = s_c, \\ \int\limits_s (\ ) \gamma ds & \text{for} \quad q = l \quad \text{and} \quad \theta = \theta_l. \end{cases}$$

The quantities $\mathbf{C}_q$ are the elasticity matrices and $\mathbf{M}_q$, $\mathbf{I}_q$ are proper inertia matrices. The entries in the matrices $\mathbf{M}_q$, $\mathbf{I}_q$ for $q = s, c$ are given in [5] Chapter 7, and for $q = l$ are as follows:

(2.6) $$\mathbf{M}_l = \varrho_l \operatorname{diag} \lceil A_l, A_l, A_l, I_l \rfloor, \quad \mathbf{I}_l = \varrho_l \operatorname{diag} \lceil I_{yl}, I_{zl} \rfloor,$$

where ϱ_l is the density of the stringer and A_l, I_l, I_{yl}, I_{zl} are geometrical parameters of the cross-section of the stringer.

The kinetic energies $(2.5)_2$ are composed of two parts. The first part is connected with the translation inertias of each element of the system and the twist of stiffeners and the second with the rotational inertias of these elements. Of course, the operator $\mathbf{F}_q$ for the shell segment ($q = s$) is the unit matrix.

3. Solution of the problem

The problem formulated above has an integro-differential form, identically as the problem presented in [5] Chapter 8. The solution is similar too, i.e., it is reduced to an algebraic quadratic form via expansion of the displacement vector $\mathbf{u}$ into trigonometric series in the circumferential direction θ, and discretization in the meridional direction s into finite differences.

As a result of the integration with respect to θ, the elastic and kinetic energies of the elements of the system for one segment take the form

(3.1)
$$E_q = \frac{\pi}{2} \sum_{\alpha,\beta=1}^{2} \sum_{k,n=0}^{N} \mathscr{L}_q (\mathbf{A}_{q\alpha}^k \mathbf{u}_\alpha^k)^T \mathbf{C}_{q\alpha\beta}^{kn} \mathbf{A}_{q\beta}^n \mathbf{u}_\beta^n,$$
$$T_q = \omega^2 \frac{\pi}{2} \sum_{\alpha,\beta=1}^{2} \sum_{k,n=0}^{N} \mathscr{L}_q [(\mathbf{F}_{q\alpha}^k \mathbf{u}_\alpha^k)^T \mathbf{M}_{q\alpha\beta}^{kn} \mathbf{F}_{q\beta}^n \mathbf{u}_\beta^n + (\mathbf{B}_{q\alpha}^k \mathbf{u}_\alpha^k)^T \mathbf{I}_{q\alpha\beta}^{kn} \mathbf{B}_{q\beta}^n \mathbf{u}_\beta^n],$$
$$q = s, c, l,$$

where the operator $\mathscr{L}$ describes the operations

(3.2) $$\mathscr{L}_q = \begin{cases} \int\limits_s (\ ) ds & \text{for } q = s, l, \\ \sum\limits_{c=1}^{C} (\ ) & \text{for } q = c \end{cases}$$

and C is the number of rings on the shell segment.

The operators $\mathbf{A}^k_{q\alpha}$, $\mathbf{F}^k_{q\alpha}$, $\mathbf{B}^k_{q\alpha}$ resulted from the differentiation of the operators $\mathbf{A}_q$, $\mathbf{F}_q$, $\mathbf{B}_q$ with respect to θ. The matrices of stiffness $\mathbf{C}^{kn}_{q\alpha\beta}$ and inertia $\mathbf{M}^{kn}_{q\alpha\beta}$, $\mathbf{I}^{kn}_{q\alpha\beta}$ coefficients for $q = s, c$ are given in [5] Chapter 8, and for meridional stringers ($q = l$) they have the form

$$\mathbf{C}^{kn}_{l\alpha\beta} = \frac{\gamma}{\pi}\sum_{l=1}^{L}(\mathbf{T}^{lk}_{\alpha})^T \mathbf{C}_l \mathbf{T}^{ln}_{\beta},$$

(3.3)

$$\mathbf{M}^{kn}_{l\alpha\beta} = \frac{\gamma}{\pi}\sum_{l=1}^{L}(\bar{\mathbf{T}}^{lk}_{\alpha})^T \mathbf{M}_l \bar{\mathbf{T}}^{ln}_{\beta}, \quad \mathbf{I}^{kn}_{l\alpha\beta} = \frac{\gamma}{\pi}\sum_{l=1}^{L}(\bar{\bar{\mathbf{T}}}^{lk}_{\alpha})\mathbf{I}_l \bar{\bar{\mathbf{T}}}^{ln}_{\beta},$$

where L is the number of meridional stringers on the shell segment. $\mathbf{T}^{lk}_{\alpha}$, $\bar{\mathbf{T}}^{lk}_{\alpha}$, $\bar{\bar{\mathbf{T}}}^{lk}_{\alpha}$ are diagonal matrices with trigonometric functions of the type $\sin k\theta_l$, $\cos k\theta_l$. The subscripts α, β assume the values 1,2 and are connected with skew-symmetric ($\alpha, \beta = 1$) and symmetric ($\alpha, \beta = 2$) components of the displacement vector $\mathbf{u}$, and the subscripts k, n, are connected with the numbers of circumferential waves.

Employing the finite difference method along the shell meridian, the integro-differential problem is converted into the algebraic one [5]. During the process the differential operators $\mathbf{A}^k_{q\alpha}$, $\mathbf{F}^k_{q\alpha}$, $\mathbf{B}^k_{q\alpha}$ are replaced by suitable matrices and the integrals — by sums. Eventually the algebraic form of the functional H which describes free vibrations of orthogonally stiffened shells of revolution may be written in the form

$$\bar{H} = \sum_{\alpha,\beta=1}^{2}\sum_{k,n=0}^{N}\sum_{p=1}^{P}\sum_{i=1}^{m_p}(\mathbf{u}^{pk}_i)^T[\mathbf{K}^{\alpha\beta kn}_{pi} - \omega^2 \mathbf{M}^{\alpha\beta kn}_{pi}]\mathbf{u}^{pn}_{\beta i}, \tag{3.4}$$

where P is the number of shell segments, m_p — the number of subsegments in the segment p. Local stiffness matrices $\mathbf{K}^{\alpha\beta kn}_{pi}$ and inertia matrices $\mathbf{M}^{\alpha\beta kn}_{pi}$ are calculated as a sum of the respective local matrices for the shell and stiffeners, what can be written symbolically in the form

$$\mathbf{K}^{\alpha\beta kn}_{pi} = \sum_{q=s,c,l}\mathbf{K}^{\alpha\beta kn}_{pqi}, \quad \mathbf{M}^{\alpha\beta kn}_{pi} = \sum_{q=s,c,l}\mathbf{M}^{\alpha\beta kn}_{pqi},$$

where

$$\mathbf{K}^{\alpha\beta kn}_{pqi} = \delta(\mathbf{A}^{\alpha k}_{pqi})^T \mathbf{C}^{\alpha\beta kn}_{pqi}\mathbf{A}^{\beta n}_{pqi},$$

$$\mathbf{M}^{\alpha\beta kn}_{pqi} = \delta[(\mathbf{F}^{\alpha k}_{pqi})^T \tilde{\mathbf{M}}^{\alpha\beta kn}_{pqi}\mathbf{F}^{\beta n}_{pqi} + (\mathbf{B}^{\alpha k}_{pqi})^T \mathbf{I}^{\alpha\beta kn}_{pqi}\mathbf{B}^{\beta n}_{pqi}],$$

and δ denotes Δ_i for $q = s, l$ or $\sum_{c=1}^{C}\delta_{ci}$ for $q = c$ (δ_{ci} denotes the place of fastening of the ring to the shell segment).

Summation with respect to the indices appearing in Eq. (3.4) is carried out via introduction of the global vectors and matrices [5]. The stationarity condition for the quadratic form (3.4) leads to the generalized eigen-problem, similarly as it did for the shell reinforced only with circular rings [5], that is

$$\mathbf{K}\mathbf{q} = \omega^2 \mathbf{M}\mathbf{q}. \tag{3.5}$$

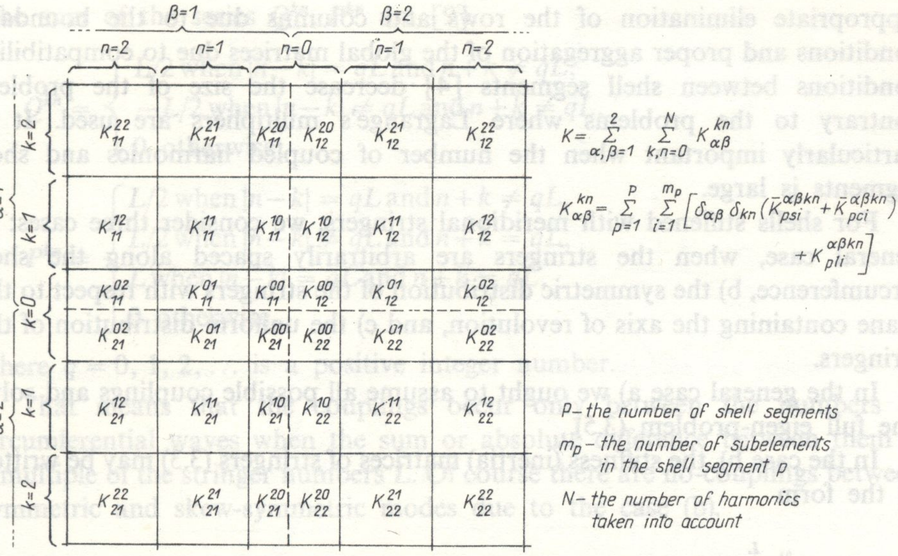


FIG. 2. The structure of the global matrix **K**.

The important difference between these two problems lies in the structure of the global matrices **K**, **M** and the global vector **q**. For the shell stiffened with meridional stringers, the global vector **q** has components which are symmetric and skew-symmetric global vectors of the individual harmonics n

$$\mathbf{q} = \{\mathbf{u}_1^N, \mathbf{u}_1^{(N-1)}, \ldots, \mathbf{u}_1^k, \ldots, u_1^0, \mathbf{u}_2^0, \ldots, \mathbf{u}_2^n, \mathbf{u}_2^{(N-1)}, \mathbf{u}_2^N\}^T,$$

and the structure of the global matrices **K**, **M** is shown in Fig. 2. These matrices have block structure. It is visible that the couplings between individual numbers of circumferential waves and between symmetric and skew-symmetric mode shapes are induced only by the term $\mathbf{K}_{pli}^{\alpha\beta kn}$ which refers to the meridional stringers. So, in the case of smooth shells or shells reinforced only with the circular rings, the problem is uncoupled to $N+1$ eigen-problems as it was shown in [5]. All matrix blocks lying outside the diagonal are zero.

For shells stiffened with meridional stringers, we need, in general, to examine the couplings between the symmetric and skew-symmetric components of vibrations as well as between the individual harmonics.

4. THE GLOBAL MATRICES — DISCUSSION

The total size of the eigen-problem (3.5) depends essentially on the number N of the terms in the trigonometric development series. But it depends on the number of the shell segments P and the finite difference mesh, too. Important here is the way in which the constraints are introduced into the functional H.

Appropriate elimination of the rows and columns due to the boundary conditions and proper aggregation of the global matrices due to compatibility conditions between shell segments [4] decrease the size of the problem contrary to the problems where Lagrange's multipliers are used. It is particularly important when the number of coupled harmonics and shell segments is large.

For shells stiffened with meridional stringers, we consider three cases: a) general case, when the stringers are arbitrarily spaced along the shell circumference, b) the symmetric distribution of the stringers with respect to the plane containing the axis of revolution, and c) the uniform distribution of the stringers.

In the general case a) we ought to assume all possible couplings and solve one full eigen-problem (3.5).

In the case b), the stiffness (inertia) matrices of stringers (3.3) may be written in the form

$$\mathbf{C}^{kn}_{l\alpha\beta} = \frac{\gamma}{\pi}\sum_{l=1}^{L} \mathbf{T}(\theta)^{lk}_{\alpha}\mathbf{C}_l\mathbf{T}(\theta)^{ln}_{\beta} =$$

$$= \frac{\gamma}{\pi}\Big[\sum_{l=1}^{L/2} \mathbf{T}(\theta_l)^{lk}_{\alpha}\mathbf{C}_l\mathbf{T}(\theta_l)^{ln}_{\beta} + \sum_{l=1}^{L/2} \mathbf{T}(-\theta_l)^{lk}_{\alpha}\mathbf{C}_l\mathbf{T}(-\theta_l)^{ln}_{\beta}\Big].$$

Since $\mathbf{T}^{lk}_{\alpha}$ are diagonal matrices of the type

$$\mathbf{T}^{lk}_1 = \operatorname{diag}\lceil \sin k\theta_l, \sin k\theta_l, \cos k\theta_l, \cos k\theta_l \rfloor,$$

$$\mathbf{T}^{lk}_2 = \operatorname{diag}\lceil \cos k\theta_l, \cos k\theta_l, \sin k\theta_l, \sin k\theta_l \rfloor,$$

and the matrices of the stiffness coefficients $\mathbf{C}_l$ (inertia coefficients $\mathbf{M}_l$, $\mathbf{I}_l$) are diagonal, too, due to the assumed hypothesis, the stiffness and inertia matrices (3.3) differ from zero only when $\alpha = \beta$. That results in uncoupling of the symmetric and skew-symmetric modes.

For the uniform distribution of the stringers (c), the places of fastening of the stringers to the shell are described by $\theta_l = 2\pi l/L$. Then each element of the matrices (3.3) has one of the factors

$$(4.1)\quad \sum_{l=1}^{L} \sin k\frac{2\pi l}{L}\sin n\frac{2\pi l}{L} = Q^{kn}, \quad \sum_{l=1}^{L} \cos k\frac{2\pi l}{L}\cos n\frac{2\pi l}{L} = P^{kn} \quad \text{for } \alpha = \beta,$$

and

$$\sum_{l=1}^{L} \sin k\frac{2\pi l}{L}\cos n\frac{2\pi l}{L} = 0 \quad \text{for } \alpha \neq \beta.$$

The sum of the series Q^{kn}, P^{kn} are [9]

$$Q^{kn} = \begin{cases} L/2 \text{ when } |n-k| = qL \text{ and } n+k \neq qL, \\ -L/2 \text{ when } |n-k| \neq qL \text{ and } n+k \neq qL, \\ 0 \text{ otherwise,} \end{cases}$$

$$P^{kn} = \begin{cases} L/2 \text{ when } |n-k| = qL \text{ and } n+k \neq qL, \\ L/2 \text{ when } |n-k| \neq qL \text{ and } n+k = qL, \\ L \text{ when } |n-k| = qL \text{ and } n+k = qL, \\ 0 \text{ otherwise,} \end{cases}$$

where $q = 0, 1, 2, \ldots$ is a positive integer number.

That means that the couplings occur only between the numbers of circumferential waves when the sum or absolute difference between them is a multiple of the stringer numbers L. Of course there are no-couplings between symmetric and skew-symmetric modes due to the case (b).

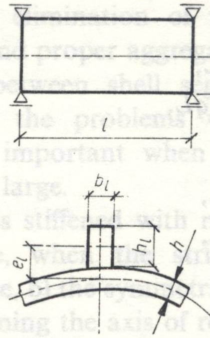
5. Numerical examples

The numerical examples enclosed here are treated as the verification of the method of analysis presented above. A comparison between the three calculation models of shells with meridional stiffeners is also shown (Example 5.2). It is shown that each of the models gives quantitatively different results and that couplings between harmonics have an important influence on the frequencies and modes calculated when the shell is stiffened only with a few meridional stringers.

5.1. *The cylindrical shell with 60 stringers*

The influence of the stiffeners on the frequencies and mode shapes of free vibration of non-stiffened shell has been examined. A simply supported cylindrical nonstiffened shell (N), a shell with 60 external stringers (Z) and internal stringers (W) has been calculated. The geometry and physical data of the shell and meridional stringers are given in Fig. 3. During the calculation, the meridional of the shell was divided into 15 subsegments.

In this case, the couplings between harmonics are negligible due to the large number of the stringers. The frequencies calculated for the number $n = 0, 1, \ldots, 18$ of circumferential waves, and results given by Egle and Sewall [7] are shown in Fig. 4. Comparison of the stiffened (Z), (W) and non-stiffened shells leads to the conclusion that the stringers may have an important influence on the frequencies of free vibration (Figs. 4, 5, 6). In the example



	Example 5.1	Example 5.2
$l=$	$0.6096\,m$	$0.9868\,m$
$r=$	$0.2422\,m$	$0.1945\,m$
$h=$	$6.452\cdot10^{-4}m$	$4.638\cdot10^{-4}m$
$e_l=$	$\pm3.655\cdot10^{-3}m$	$-5.287\cdot10^{-3}m$
$b_l=$	$2.840\cdot10^{-3}m$	$1.0389\cdot10^{-3}m$
$h_l=$	$2.840\cdot10^{-3}m$	$1.0112\cdot10^{-2}m$
$E=$	$6.89\cdot10^{10}Nm^{-2}$	$2.0\cdot10^{11}Nm^{-2}$
$\nu=$	0.315	0.3
$\rho=$	$2.714\cdot10^{3}Ns^{2}m^{-4}$	$7.844\cdot10^{4}Ns^{2}m^{-4}$
$L=$	60	4

FIG. 3. Geometry and material constants of the cylindrical shell and stringers (Examples 5.1 and 5.2).

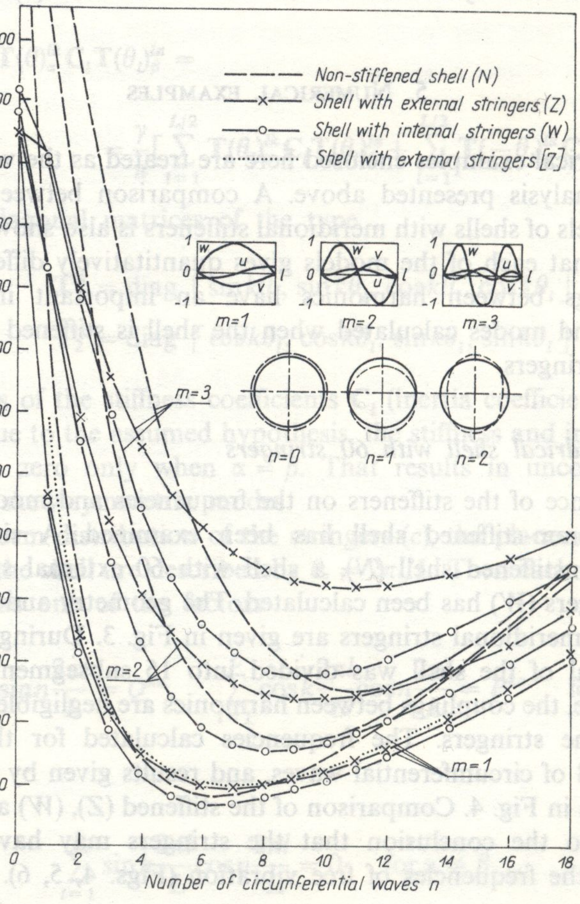


FIG. 4. Eigen-frequencies ω [Hz] of the cylindrical shells (N), (Z), (W) versus n.

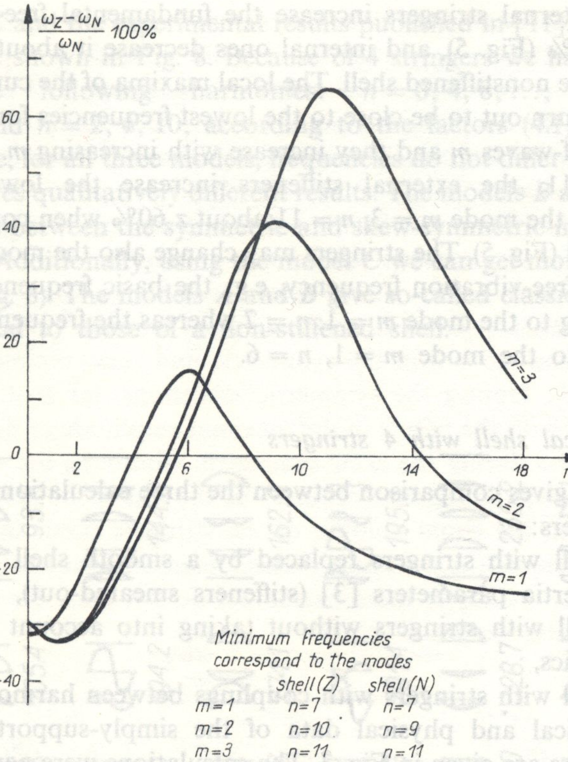


FIG. 5. Relative differences of eigen-frequencies of the shell (Z) with respect to the shell (N).

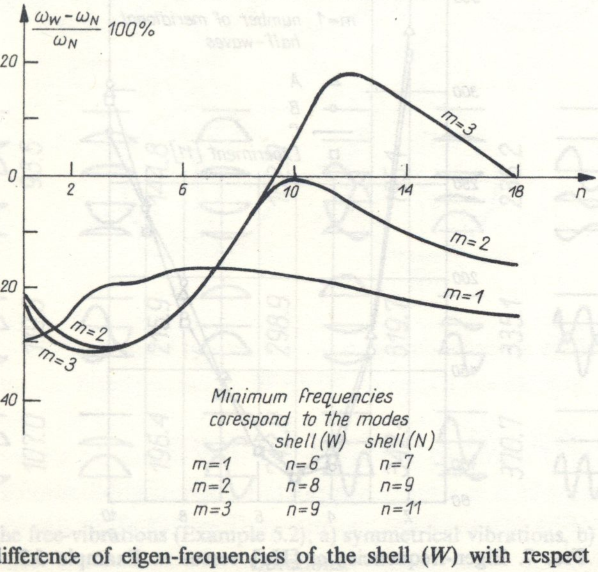


FIG. 6. Relative difference of eigen-frequencies of the shell (W) with respect to the shell (N).

discussed, the external stringers increase the fundamental free-vibration frequency about 10% (Fig. 5), and internal ones decrease it about 20% (Fig. 6) with respect to the nonstiffened shell. The local maxima of the curvature shown in Figs. 5 and 6 turn out to be close to the lowest frequencies for each number of meridional half-waves m and they increase with increasing m. For example, for $m = 3$, $n = 11$, the external stiffeners increase the lowest frequency corresponding to the mode $m = 3$, $n = 11$ about z 60% when compared to the non-stiffened shell (Fig. 5). The stringers may change also the mode corresponding to the basic free-vibration frequency, e.g., the basic frequency of the shell (N) corresponding to the mode $m = 1$, $n = 7$ whereas the frequency of the shell W corresponds to the mode $m = 1$, $n = 6$.

5.2. *The cylindrical shell with 4 stringers*

This example gives comparison between the three calculation models of the shell with stringers:

A – the shell with stringers replaced by a smooth shell with modified stiffeners and inertia parameters [3] (stiffeners smeared-out),

B – the shell with stringers without taking into account the couplings between harmonics,

C – the shell with stringers with couplings between harmonics included.

The geometrical and physical data of the simply-supported shell with 4 internal stringers are given in Fig. 3. The calculations were performed for 10 subsegments in the finite difference mesh.

Figure 7 shows some of the frequencies calculated for each calculation

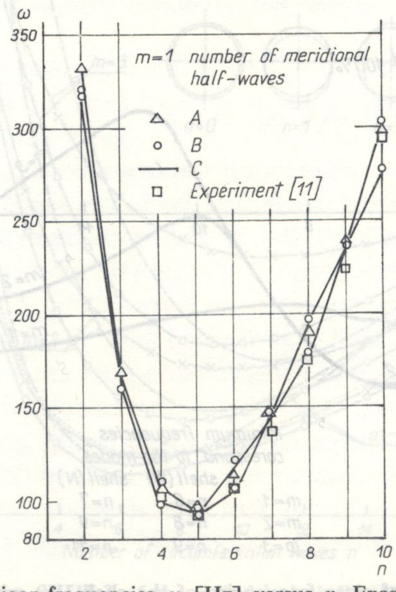


FIG. 7. Eigen-frequencies ω [Hz] versus n. Example 5.2.

model A, B, C and the experimental results published in [11]. The corresponding modes are shown in Fig. 8. Because of 4 stringers we have the couplings between the following harmonics: $n = 0, 4, 8, \ldots,$ and $n = 1, 3, 5, 7, 9, \ldots,$ and $n = 2, 6, 10,$ according to the factors (4.1).

In this case, for all three models, frequencies do not differ much but each of the models gives qualitatively different results. The models B and C allow to see the differences between the symmetric and skew-symmetric modes contrary to the model A. Additionally, using the model C we can get more exact modes of vibrations (Fig. 8). The models A and B give so-called classical modes, that is modes identical to those of a non-stiffened shell.

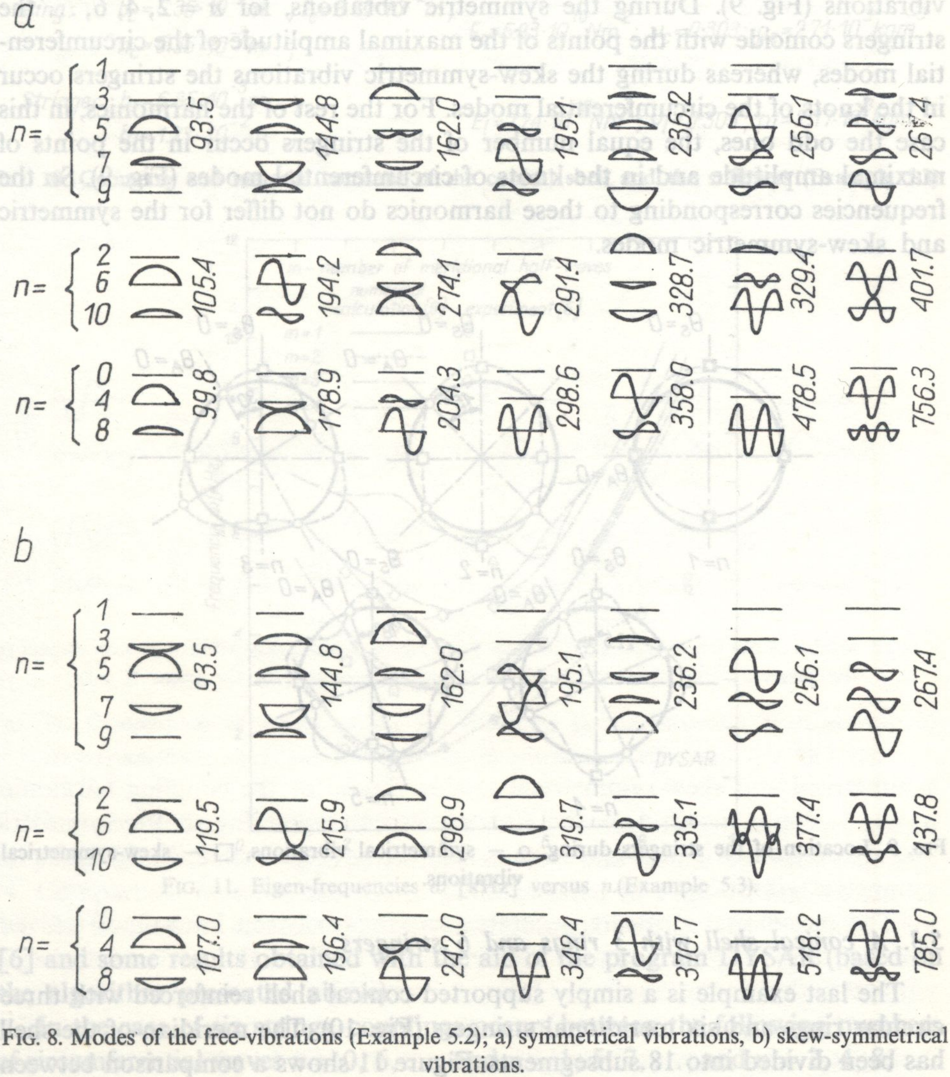


FIG. 8. Modes of the free-vibrations (Example 5.2); a) symmetrical vibrations, b) skew-symmetrical vibrations.

It should be noticed here that when couplings between harmonics are considered, it is often not possible to find the unique relation between the frequencies and the circumferential number n, contrary to the shell without stringers or when the couplings are not taken into account. But for the first few free-vibration frequencies, usually one of the modes dominates (Fig. 8), thus we can plot ω versus n.

In this case the differences between the frequencies corresponding to the symmetric and skew-symmetric modes occur only for the even number of harmonics $n = 2, 4, 6, 8, 10, \ldots$, i.e., when the value $2n/L$ (L – the number of stringers) is an integer. It arises from the different location of the stringers with respect to the plane $x,\theta = 0$ in the symmetric and skew-symmetric modes of vibrations (Fig. 9). During the symmetric vibrations, for $n = 2, 4, 6, \ldots$ the stringers coincide with the points of the maximal amplitude of the circumferential modes, whereas during the skew-symmetric vibrations the stringers occur in the knots of the circumferential modes. For the rest of the harmonics, in this case the odd ones, the equal number of the stringers occur in the points of maximal amplitude and in the knots of circumferential modes (Fig. 9). So the frequencies corresponding to these harmonics do not differ for the symmetric and skew-symmetric modes.

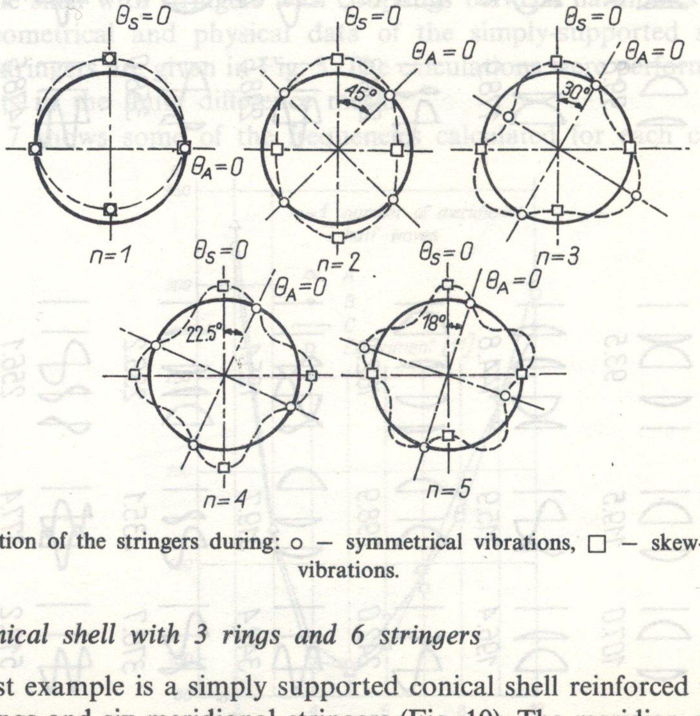


FIG. 9. Location of the stringers during: ○ – symmetrical vibrations, □ – skew-symmetrical vibrations.

5.3. A conical shell with 3 rings and 6 stringers

The last example is a simply supported conical shell reinforced with three circular rings and six meridional stringers (Fig. 10). The meridian of the shell has been divided into 18 subsegments. Figure 11 shows a comparison between the experimental and numerical results published by CRENWELGE and MUSTER

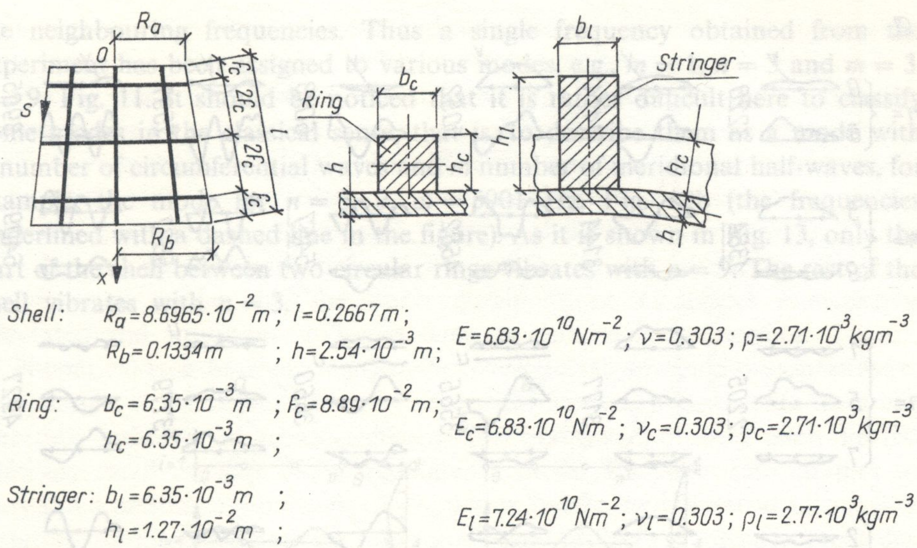


FIG. 10. Geometry and material constant of the conical shell and the stiffeners (Example 5.3)

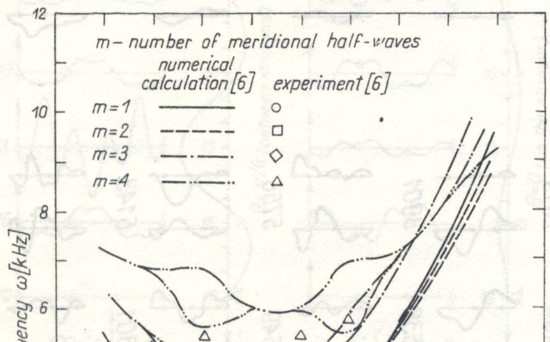


FIG. 11. Eigen-frequencies ω [kHz] versus n.(Example 5.3).

[6] and some results obtained with the aid of the program DYSAR (based on the algorithm presented above).

In the case of six strings, couplings occurs between the following numbers of circumferential waves $n = 0, 6, \ldots$, and $n = 1, 5, 7, \ldots$, and $n = 2, 4, 8, \ldots$. Figure 12 shows the modes and the corresponding frequencies obtained for the

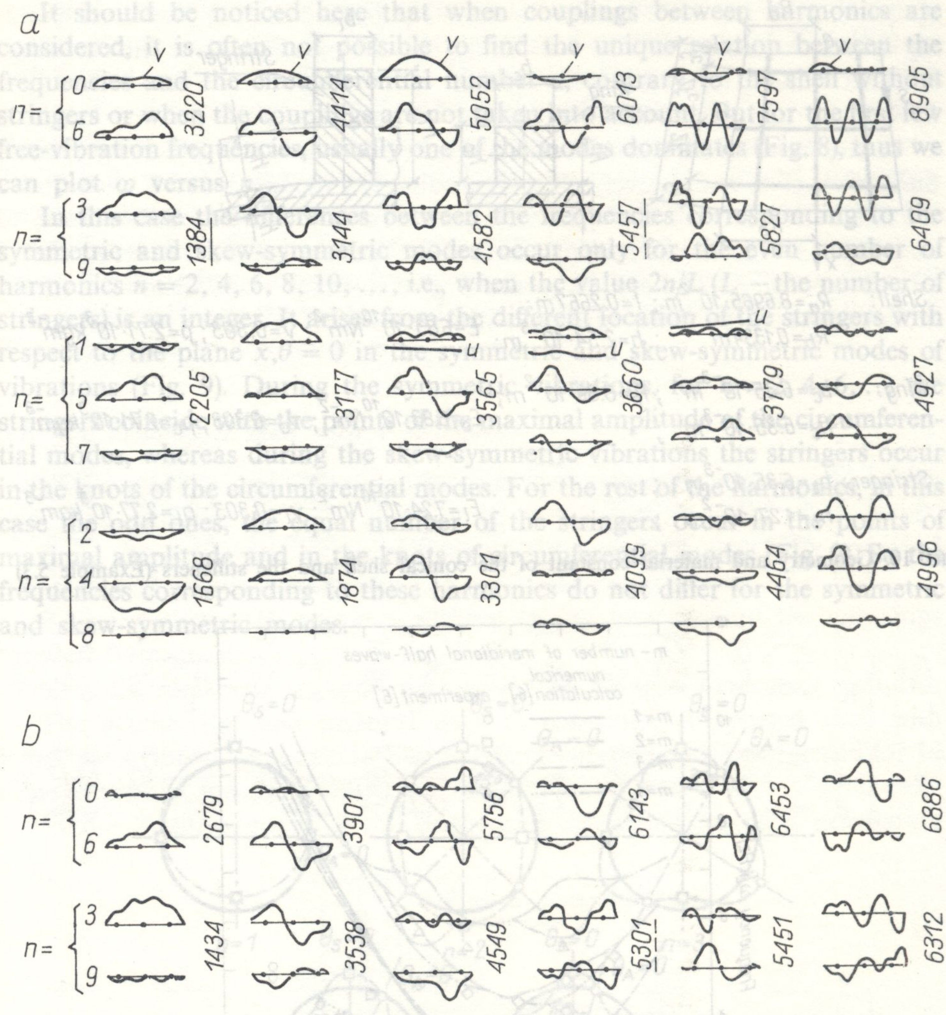


FIG. 12. Modes of the free-vibrations (Example 5.3) a) skew-symmetrical vibrations with respect to the plane $\theta_A = 0$, b) symmetrical vibrations with respect to the plane $\theta_s = 0$.

symmetrical and skew-symmetrical modes. Since, for the coupling harmonics $n = 1, 5, 7$ and $n = 2, 4, 8$, the frequencies corresponding to the symmetric and skew-symmetric modes are equal, they are not repeated in the figure for the symmetric vibrations.

The meridional stringers generate couplings between harmonics and the circular rings may generate modes in which each of the parts of the shell vibrates with different numbers of circumferential waves n, as it has been shown in [5, 10]. In [6] it was mentioned that the experiment has given complex modes and it has been explained by aggregation of vibrations corresponding to the neighbouring frequencies. Thus a single frequency obtained from the experiment has been assigned to various modes, e.g., $m = 1$, $n = 3$ and $m = 3$, $n = 9$, Fig. 11. It should be noticed that it is rather difficult here to classify some modes in the classical sense, that is, to describe them as a mode with n number of circumferential waves and m number of meridional half-waves, for example, the mode for $n = 3$, 9, $\omega = 5901$ Hz, Fig. 12b (the frequencies underlined with a dashed line in the figure). As it is shown in Fig. 13, only the part of the shell between two circular rings vibrates with $n = 9$. The rest of the shell vibrates with $n = 3$.

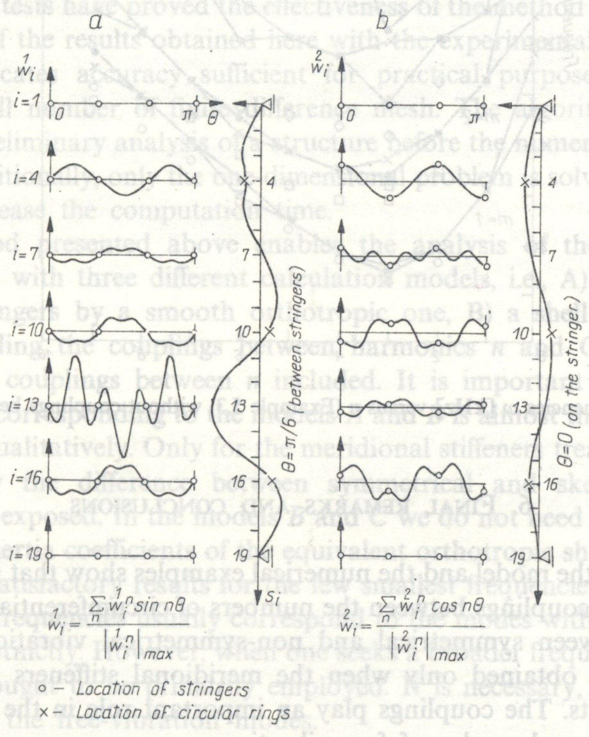


FIG. 13. Composition of modes of the free-vibrations in the successive cross-sections; a) skew-symmetrical vibrations $\overset{1}{\omega} = 5457$ Hz, b) symmetrical vibrations $\overset{2}{\omega} = 5301$ Hz.

For the stiffened shell described above, calculations, without taking couplings into account, were performed as well. The results are shown in Fig. 14. Comparison between the results demonstrated in Figs. 12 and 14 shows that the frequencies calculated without including couplings (Fig. 14) differ only slightly from the ones with the couplings included (Fig. 12) when the frequencies correspond to the modes with one harmonic dominating distinctly. In the other cases, when the couplings are stronger, both results may differ more than 20%. Therefore, including couplings into calculation is necessary.

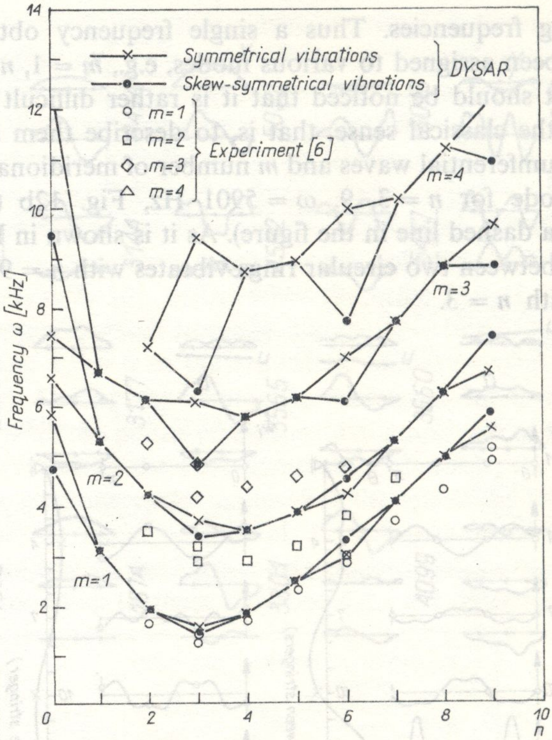


FIG. 14. Eigen-frequencies ω [kHz] versus n (Example 5.3) without couplings between n included.

6. Final remarks and conclusions

Analysis of the model and the numerical examples show that the meridional stringers cause couplings between the numbers of circumferential waves n, and differences between symmetrical and non-symmetrical vibrations. But these results may be obtained only when the meridional stiffeners are treated as discrete elements. The couplings play an important role in the calculation of the frequencies and modes of free vibrations.

In general, the problem of free-vibration analysis of orthogonally stiffened shells of revolution is reduced to a single generalized eigen-problem of comparatively large size (Fig. 2), contrary to the free-vibration analysis of shells without meridional stringers [5]. For the symmetric distribution of the stringers with respect to the plane containing the axis of revolution, the symmetric and skew-symmetric modes become uncoupled. Thus the problem is reduced to two generalized eigen-problems half the size of the original one. For uniform distribution of the stringers along the shell circumference, couplings occur only when the absolute sum or difference of the numbers of circumferential waves n, k is a multiple of the stringer number L, i.e., $|n \pm k| = qL$ (q — integer number). Thus the problem is reduced to several eigen-problems of smaller size.

This shortens the time of numerical calculations (the time of calculation increases progressively with the size of the eigenproblem).

For a shell with meridional stringers the values of the frequencies corresponding to symmetrical modes may differ from the ones corresponding to skew-symmetrical ones. This phenomenon is not observed for smooth shells or for shells with circumferential rings because of the axial symmetry of the systems [5]. Similarly as for smooth shells or shells with circumferential rings [5], due to the concentration of frequencies in the spectrum, the real free-vibration modes may be linear combinations of modes numerically calculated and connected with the same or proximate frequencies.

Numerical tests have proved the effectiveness of the method presented here. Comparison of the results obtained here with the experimental data from the literature indicates accuracy sufficient for practical purposes even at the relatively small number of finite difference mesh. The algorithm enables to carry out a preliminary analysis of a structure before the numerical calculation is started. Additionally, only the one-dimensional problem is solved numerically. All these decrease the computation time.

The method presented above enables the analysis of the orthogonally stiffened shells with three different calculation models, i.e., A) a replacement shell with stringers by a smooth orthotropic one, B) a shell with stringers without including the couplings between harmonics *n* and C) a shell with stringers with couplings between *n* included. It is important that the computation time corresponding to the models *A* and *B* is almost the same, but the results differ qualitatively. Only for the meridional stiffeners treated as discrete elements, may the difference between symmetrical and skew-symmetrical frequencies be exposed. In the models *B* and *C* we do not need to estimate the stiffness and inertia coefficients of the equivalent orthotropic shell. Each of the models gives satisfactory results for the few smallest frequencies of the system because these frequencies usually correspond to the modes with one harmonic dominating distinctly. However, when one seeks a broader frequency spectrum, the model *C* ought to be primarily employed. It is necessary, too, when one wants to find the free-vibration modes.

The method developed here can be easily extended to other dynamical problems of stiffened shells of revolution, such as taking into account the distribution of initial stresses or linear stability of shells. It can also be used as an important part of the complete dynamical analysis of shells under given, time-varying loads.

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Streszczenie

DRGANIA SWOBODNE CIENKICH, SPRĘŻYSTYCH I ORTOGONALNIE USZTYWNIONYCH POWŁOK OBROTOWYCH Z ŻEBRAMI TRAKTOWANYMI JAKO ELEMENTY DYSKRETNE

Przedstawiono metodę rozwiązania i przykłady liczbowe dotyczące wyznaczania częstości i postaci drgań własnych cienkich powłok obrotowych z ortogonalnym usztywnieniem. Całkowo-różniczkowy funkcjonał Hamiltona sprowadzono do postaci algebraicznej metodą rozdziału zmiennych, a następnie rozwinięto w szereg trygonometryczny w kierunku obwodowym oraz zdyskretyzowano metodą różnic skończonych w kierunku południkowym. Porównano ze sobą trzy modele obliczeniowe: powłoka z żebrami zastąpiona ortotropową powłoką jednorodną, powłoka z żebrami traktowanymi jako elementy dyskretne bez sprzężenia postaci drgań harmonicznych oraz ze sprzężeniami. Wykazano, że uwzględnienie sprzężeń może wpływać na wyznaczone częstości i postacie drgań swobodnych powłoki. Wyniki porównano z danymi doświadczalnymi znanymi z literatury.

Р е з ю м е

СВОБОДНЫЕ КОЛЕБАНИЯ ТОНКИХ, УПРУГИХ И С ОРТОГОНАЛЬНО ПРИДАННОЙ ЖЕСТКОСТЬЮ ВРАЩАТЕЛЬНЫХ ОБОЛОЧЕК С РЕБРАМИ ТРАКТОВАННЫМИ КАК ДИСКРЕТНЫЕ ЭЛЕМЕНТЫ

Представлен метод решения и числовые примеры, касающиеся определения частоты и вида собственных колебаний тонких вращательных оболочек с ортогонально приданной жесткостью. Интегро-дифференциальный функционал Гамильтона сведен к алгеобраическому виду методом разделения переменных, а затем он разложен в тригонометрический ряд в периметрическом направлении и дискретизирован методом конечных разностей в меридиональном направлении. Сравнены с собой три рачетные модели: оболочка с ребрами, заменена ортотропной однородной оболочкой, оболочка с ребрами, трактованными как дискретные элементы без сопряжения вида гармонических колебаний, а также с сопряжениями. Показано, что учет сопряжений может влиять на определенные частоты и виды свободных колебаний оболочки. Результаты сравнены с экспериментальными данными известными из литературы.

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