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FORCED VIBRATION OF A NONLINEAR STATIONARY SYSTEM WITH A FRICTION CLUTCH, STRUCTURAL FRICTION BEING TAKEN INTO CONSIDERATION

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The considerations are concerned with a nonlinear discrete stationary system containing a multiple-disc friction clutch, structural friction being taken into consideration. The system vibrates under harmonic excitation. Nonuniform distribution of pressure between the friction discs is taken into account. The influence of the dynamic system parameters upon the amplitude, vibration frequency and the phase shift angle is analysed. The equation of motion of the system is solved by the Van der Pol method.

1. Introduction

Advanced methods of design of friction clutches require a thorough analysis of systems incorporating such clutches, the principal problem being that of making use of the elastic and damping properties of the clutch as a natural means of energy dissipation, therefore also a means of vibration damping. By judicious selection of geometrical parameters and loads acting on the system the resonance amplitude can be considerably reduced. The establishment of relevant relations requires a mathematical model representing in a correct manner the real system. The studies which have hitherto been made by the present author in the domain of mechanical systems with multiple-disc friction clutches, structural friction being taken into consideration, are based on the assumption of uniform pressure distribution between the surfaces of the cooperating plates. Stationary and non-stationary operation conditions are considered, with vibration exciting forces of deterministic or random character. In real mechanical systems the distribution of friction forces and temperatures is not uniform, the study of the problem of non-uniform pressure distribution thus appearing to be justified. The causes of non-uniform pressure distribution are diversified. Let us mention, for instance, incorrect assemblage of the clutch itself or the power transmission system, excessive clearances in the bearings, thermal overload etc. The assumptions concerning the properties of the material are those of the classical theory of elasticity.

2. Determination of the total increase in angular displacement $\varphi_{n+1}(M)$ in any stage of motion of the mechanical system

We assume that the pressure distribution between cooperating surfaces of a clutch is parabolic. Such an assumption approaches the real pressure distribution and varies in the radial direction; it is based on the results of extensive experimental research (Figs.1, 2) as well as the results of measurement of the friction forces acting on the cooperating objects. Our point

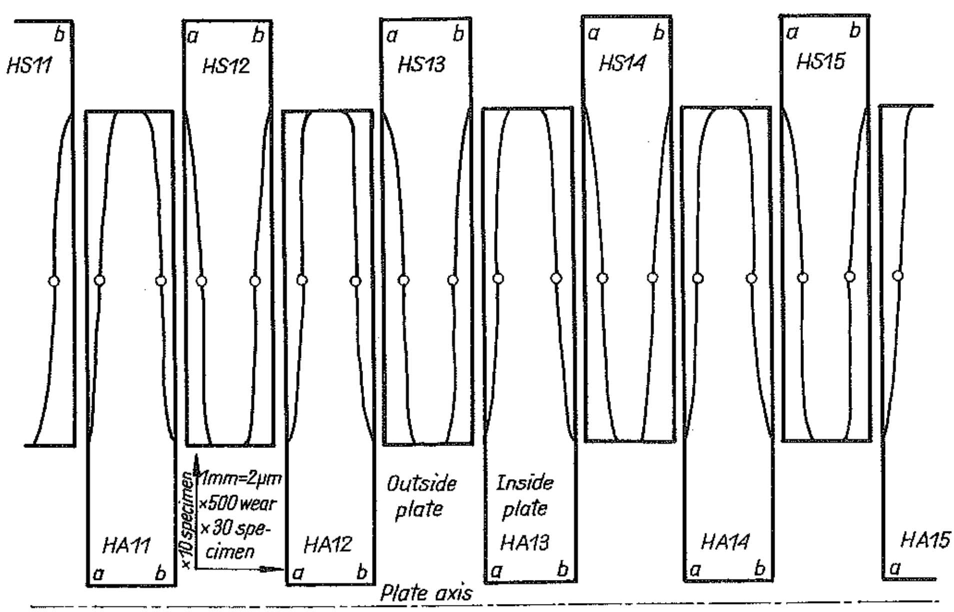

FIG. 1. The wear process of clutch plates after 25000 cycles. Type B.

of departure will be the formulae for circumferential displacements $V_1(r)$, $V_2(r)$, $V_3(r)$ of a clutch plate as functions of the external load and the

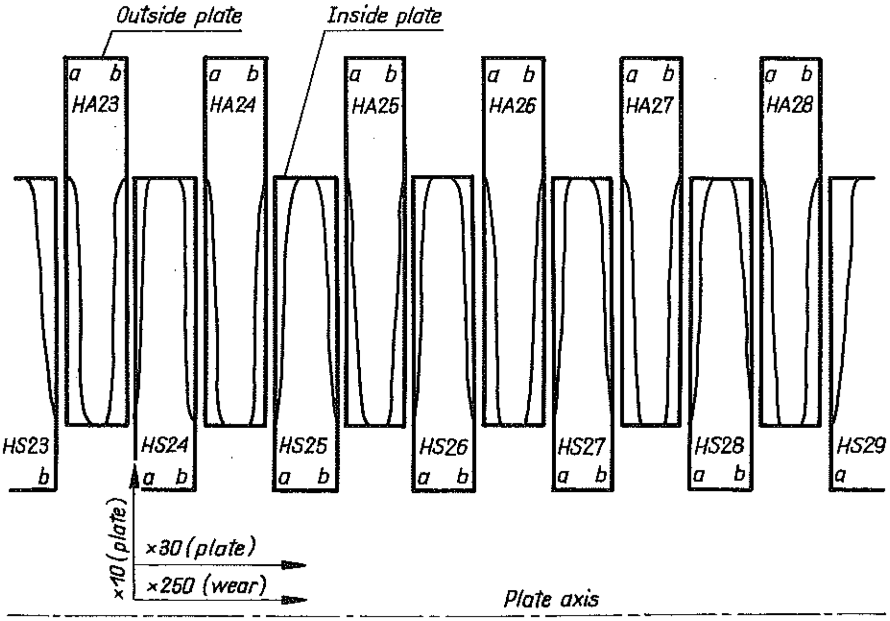

FIG. 2. The wear process of clutch plates after 25000 cycles. Type C.

geometrical parameters of the system. These formulae are valid in the case of parabolic pressure distribution and have been derived in [6].

a. In the first stage of the loading process the load increases from 0 to αM $(0 < \alpha \leq 1)$

$$V_1(r) = \frac{fr^2}{24k_2h}\Big\{(P_3 - P_4)\left[(1+h)^{\frac{4}{3}} - 1\right] + 2\left[(1+h)^{\frac{-2}{3}} - 1\right] \times \left(t_1 - \frac{3\alpha M \cdot h}{\Pi f r^3}\right) + 8l_1\left[(1+h)^{\frac{1}{3}} - 1\right]\Big\}, \tag{2.1}$$

where

$$h = \nu_1(\alpha M - M_2), \qquad \nu_1 = \frac{3}{\Pi f (P_3 + P_4) r^3}, \tag{2.2}$$

$$k_1 = G \cdot h_1, \qquad k_2 = G \cdot h_2, \tag{2.3}$$

$$M_2 = \frac{\alpha M \cdot k_1}{(k_1 + k_2)}, \qquad t_1 = P_4 - P_3(1 + 2h), \tag{2.4}$$

$$l_1 = P_4 - P_3(1 + h), \tag{2.5}$$

and h_1 is the internal plate thickness, h_2 – external plate thickness, k_1 – internal plate stiffness, k_2 – external plate stiffness, α – a dimensionless parameter taking values from 0 to 1, G – shear modulus, P_3 – maximum unit pressure on the internal side of the clutch plate, P_4 – minimum unit pressure on the external side of the clutch plate, f – friction coefficient, r – internal radius of the clutch plate, M – maximum value of the periodic torque acting on the mechanical system.

b. In the second stage of the process loading (unloading) we have $1 \geq \alpha \geq r_1$

$$V_2(r) = \frac{fr}{24k_2}\left\{\frac{(P_3 - P_4)}{r^3 M_3}\left[r^4(1+M_3)^{\frac{4}{3}} - \rho_2^4\right] + 2\left[\frac{1}{r^2(1+M_3)^{\frac{2}{3}}} - \frac{1}{\rho_2^2}\right] \times \left(\frac{r^3 g_1}{M_3} - \frac{3M}{\Pi f}\right) + \frac{8S_1}{M_3}\left[r(1+M_3)^{\frac{1}{3}} - \rho_2\right] + \left(\frac{1}{r^2} - \frac{1}{\rho_2^2}\right) \times \left(\frac{6\alpha M}{\Pi f} + \frac{2r^3 t_1}{h}\right) + \frac{1}{h}\left[\frac{(P_3 - P_4)}{r^3}(r^4 - \rho_2^4) + 8l_1(r - \rho_2)\right]\right\}, \tag{2.6}$$

where

$$M_3 = \nu_1(M - M_{2A}), \qquad M_{2A} = \frac{Mk_2}{k_1 + k_2}, \tag{2.7}$$

$$g_1 = P_4 - P_3(1 + 2M_3), \qquad S_1 = P_4 - P_3(1 + M_3), \tag{2.8}$$

$$\rho_2 = \left[\frac{1}{(P_4 - P_3)}\left\{\left(\frac{M_3^2}{2}\left[P_4^2 + P_3^2 - \frac{h}{M_3}(P_4^2 - P_3^2)\right]\right)^{\frac{1}{2}} + S_1\right\}\right]^{\frac{1}{3}}, \quad P_3 \neq P_4. \tag{2.9}$$

c. In the stage of repeated loading ($r_1 \leq \alpha \leq 1$) we have

$$V_3(r) = \frac{fr}{24k_2}\left[\frac{(P_3 - P_4)}{r^3 M_3}\left[r^4(1+M_3)^{\frac{4}{3}} - \rho_{2b}^4\right] + 2\left[\frac{1}{r^2(1+M_3)^{\frac{2}{3}}} - \frac{1}{\rho_{2b}^2}\right] \times \left(\frac{r^3 g_1}{M_3} - \frac{3M}{\Pi f}\right) + \frac{8S_1}{M_3}\left[r(1+M_3)^{\frac{1}{3}} - \rho_{2b}\right] + \left(\frac{1}{\rho_3^2} - \frac{1}{\rho_{2b}^2}\right) \times \left(\frac{6r_1 M}{\Pi f} + \frac{2t_1 r^3}{h}\right) + \frac{1}{h}\left[\frac{(P_3 - P_4)}{r^3}(\rho_3^4 - \rho_{2b}^4) + 8l_1(\rho_3 - \rho_{2b})\right] + \frac{1}{h}\left\{\frac{(P_4 - P_3)}{r^3}(r^4 - \rho_3^4) + \left(\frac{1}{r^2} - \frac{1}{\rho_3^2}\right)\left[2r^3(P_4 - P_3) - 4l_1 r^3 + \frac{6\alpha M h}{\Pi f}\right] + 8l_1(\rho_3 - r)\right\}\right], \tag{2.10}$$

where

$$(2.11)\quad \rho_{2b} = r\left[\frac{1}{(P_4 - P_3)}\left\{\left(\frac{M_3^2}{2}\left[P_3^2 + P_4^2 - \frac{n_1}{M_3}(P_4^2 - P_3^2)\right]\right)^{\frac{1}{2}} + S_1\right\}\right]^{\frac{1}{3}},$$

$$(2.12)\quad n_1 = \nu_1(r_1 M - M_{2b}), \qquad M_{2b} = \frac{r_1 M k_2}{k_1 + k_2},$$

$$(2.13)\quad \rho_3 = r\left[\frac{1}{(P_4 - P_3)}\left\{\left(S_1^2 - \frac{(P_3^2 - P_4^2)}{2}\left[M_3(h - M_3) - \frac{4S_1\rho_{2b}^3}{r^3(P_3 + P_4)}\right] + \frac{\rho_{2b}^6(P_3 - P_4)^2}{r^6}\right)^{\frac{1}{2}} + S_1\right\}\right].$$

The symbols $V_1(r)$, $V_2(r)$ and $V_3(r)$ denote the circumferential displacements measured from the internal side of the clutch plate at a radius r in three consecutive stages of the loading process.

The total increase in angular displacement $\varphi_{n+1}(M)$ in the $(n+1)$-st loading cycle is determined for the mechanical system considered by adding the increases in angular displacement $\Delta\varphi_1(M)$ of the clutch shaft and the relative displacement $\Delta\varphi_{n+1}(M)$ of clutch plates, that is

$$(2.14)\quad \varphi_{n+1}(M) = \Delta\varphi_1(M) + \Delta\varphi_{n+1}(M),$$

where

$$(2.15)\quad \Delta\varphi_1(M) = \frac{(M_{n+1} - M_n)l}{GJ_0},$$

$$(2.16)\quad J_0 = \frac{\Pi d^4}{32},$$

and J_0 is the geometrical moment of inertia of the clutch shaft, d – diameter of the clutch shaft, l – length of the clutch shaft, $M_{n+1} - M_n$ – increase in load between the $(n+1)$-st and the n-th stage of motion of the system.

By proceeding in the same manner as was done in [7] it is impossible to obtain a generalized formula expressing the displacement in any stage of loading. It could be done in the case of uniform pressure distribution but now it becomes impossible in view of the complex form of the formulae (2.1), (2.6), (2.10) for the individual loading stages under non-uniform pressure distribution. A possbility of overcoming these difficulties occurs, however, if the function involving the cubic roots appearing in Eqs. (2.1), (2.6), (2.10) is expanded in power series. As a result of the theoretical and numerical analysis, it has been found that the values of the roots and the first two terms of the series are almost identical. In view of the properties of those series and complexity of the solutions only the first two terms of each series will be retained for subsequent considerations. After transformation the formulae for the angular deformations $\varphi_1(r)$, $\varphi_2(r)$ and $\varphi_3(r)$ take the form

$$\varphi_1(r) = \frac{uM^2}{6\Pi k_2 r^2}, \tag{2.17}$$

$$\varphi_2(r) = \frac{uM^2}{6\Pi k_2 r^2} - \frac{c_1(1-\alpha)^2 M^2}{6\Pi k_2 r^2} = \varphi_1 - \frac{c_1(M_{n+1}-M_n)^2}{6\Pi k_2 r^2}, \tag{2.18}$$

$$\varphi_3(r) = \varphi_2 + \frac{c_1}{6\Pi k_2 r^2}(M_{n+2}-M_{n+1})^2, \tag{2.19}$$

where $M_n = \alpha M$ is the variable value of the torque, $M_{n+1} = M$ – maximum torque, $M_{n+2} = r_1 M$ – minimum torque in the second stage of motion

$$c_1 = \frac{b_1 - uP_3}{P_4 - P_3}, \qquad P_4 \neq P_3, \tag{2.20}$$

$$u = \nu_1\left(1 - \frac{k_2}{k_1+k_2}\right), \tag{2.21}$$

$$b_1 = \frac{u}{\sqrt{2}}\left[P_3^2 + P_4^2 + r_1(P_3^2 - P_4^2)\right]^{\frac{1}{2}}, \tag{2.22}$$

$$\varphi_1 = \frac{V_1(r)}{r}, \qquad \varphi_2 = \frac{V_2(r)}{r}, \qquad \varphi_3(r) = \frac{V_3(r)}{r}. \tag{2.23}$$

The above equations (2.17), (2.18) and (2.19) may be generalized for any stage of motion. As regards the entire mechanical system, the formula (2.14) takes the form

$$\varphi_{n+1}(M) = \varphi_n(M_n) + \frac{(M_{n+1}-M_n)l}{GJ_0} + \frac{c_1}{6\Pi k_2 r^2}(M_{n+1}-M_n)^2 \operatorname{sign}\frac{dM}{dt}. \tag{2.24}$$

3. Determination of the torque as a function of the twist angle of the mechanical system

The motion begins at the moment of the load M_n being applied. The relevant angle of twist is $\varphi_n(M)$. As a result of the angular displacement from the initial position to the new position φ_n some elastic strain and friction forces will appear in a number of plates. These forces will perform some work, which has been determined in [6]. As a result of unloading, the plate will be displaced under the action of the elasticity forces from the position φ_n to a new, extreme position φ_{n+1}. From this new position the motion will be resumed and repeated as long as the elasticity forces will be able to overcome the friction forces. Because it is the inverse relation $\varphi_{n+1} = \psi(M)$ which is known, not the direct relation $M = f_g(\varphi_{n+1})$ the formula (2.24) is transformed to the following equation describing the variation of the torque as a function of time

$$\frac{c_1}{6\Pi k_2 r^2}\text{sign}\frac{dM}{dt}(M_{n+1}-M_n)^2 + \frac{l}{GJ_0}(M_{n+1}-M_n) - (\varphi_{n+1}-\varphi_n) = 0. \tag{3.1}$$

One of its roots, namely

$$(M_{n+1}-M_n) = k\left[\frac{2}{\eta}\text{sign}\frac{d\varphi}{dt}\left\{-1+\left[1+\eta(\varphi_{n+1}-\varphi_n)\text{sign}\frac{d\varphi}{dt}\right]^{\frac{1}{2}}\right\}\right], \tag{3.2}$$

where

$$k = \frac{GJ_0}{l}, \tag{3.3}$$

$$\eta = \frac{2c_1G^2J_0^2}{3\Pi k_2 r^2 l^2} \tag{3.4}$$

has a physical sense.

On substituting Eqs. (2.20), (2.21) and (2.22) into Eqs. (3.4), we find

$$\eta = \frac{Gd^8h_1\left\{[P_3^2+P_4^2+r_1(P_3^2-P_4^2)]^{\frac{1}{2}} - \sqrt{2}P_3\right\}}{512\sqrt{2}h_2l^2fr^5(h_1+h_2)(P_4^2-P_3^2)}, \tag{3.5}$$

where η is a dimensionless parameter.

New initial conditions must be taken into consideration in Eq. (3.2) for each stage of the motion. This inconvenience can be avoided by displacing the system of coordinates of the structural hysteresis loop of the function $M = f_z(\varphi_{1,2,3})$ by performing a transformation as suggested in [5]. Finally we obtain

$$M(\varphi, \dot{\varphi}) = \frac{k}{\eta}\operatorname{sign}\frac{d\varphi}{dt}\left\{-1 + 2\left[1 + \eta(\varphi + \varphi_s)\operatorname{sign}\frac{d\varphi}{dt}\right]^{\frac{1}{2}} - \left(1 + 2\eta\varphi_s\operatorname{sign}\frac{d\varphi}{dt}\right)^{\frac{1}{2}}\right\}, \tag{3.6}$$

where

$$\varphi_s = \frac{\varphi_{n+1} - \varphi_n}{2}, \tag{3.7}$$

and φ_{n+1} is the maximum value of the relative angle of twist of the clutch plates, that is the internal (active) and external (passive) plate at the end of the $(n+1)$-st half-period of vibration, φ_n – maximum value of the relative angle of twist of clutch plates at the end of the n-th half-period of vibration, φ – current value of the relative angle of twist of the clutch plates during one half-period of vibration.

4. Equation of torsional vibrations of the mechanical system

Let us consider a two-mass model describing a mechanical system composed of an engine (s), a friction clutch (S_p) and a working machine (MR) as represented in Fig. 3. The assumptions concerning the properties of the material will be those of the classical theory of elasticity.

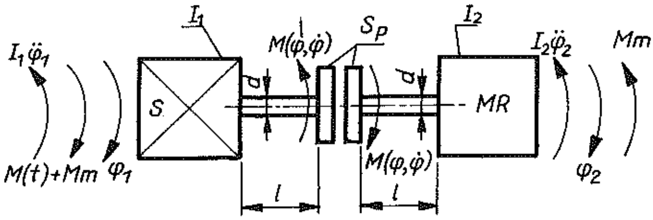

FIG. 3. The model of the mechanical system under consideration.

The equations of a system with two degrees of freedom as represented in Fig. 3 have the form

$$\begin{aligned} J_1\ddot{\varphi}_1 + M(\varphi, \dot{\varphi}) - M_m &= M(t), \\ J_2\ddot{\varphi}_2 - M(\varphi, \dot{\varphi}) + M_m &= 0, \end{aligned} \tag{4.1}$$

where J_1 is the reduced moment of inertia of the moving masses of the engine and the active part of the clutch, J_2 – reduced moment of inertia of the moving masses of the working machine and the passive part of the clutch, φ_1 – angle of twist of the active plate of the clutch, φ_2 – angle of twist of the passive plate of the clutch, M_m – nominal loading torque, M(t) – harmonic excitation and $M(\varphi, \dot{\varphi})$ – the torque transmitted by the clutch, the influence of frictional (forces which depend on the sign of velocity and displacement).

On introducing, as a coordinate, the relative displacement angle of the clutch plates

$$\varphi = \varphi_1 - \varphi_2, \tag{4.2}$$

and denoting

$$J_z = \frac{J_1 J_2}{J_1 + J_2}, \tag{4.3}$$

we obtain the equation

$$\ddot{\varphi} + \frac{M(\varphi, \dot{\varphi})}{J_z} = \frac{M(t)}{J_1} + \frac{M_m}{J_z}. \tag{4.4}$$

It is assumed that the variable engine torque is described by a constant average value M_m and a discrete torque $M(t)$ in the form of harmonic excitation

$$M(t) = M_0 \cos \omega t, \tag{4.5}$$

M_0 – excitation amplitude, ω – angular velocity of the excitation torque and t – time.

The expected form of solution of the nonlinear differential equation (4.4) is

$$\varphi = A \cos z, \tag{4.6}$$

where

$$z = \omega t + \varphi_0, \tag{4.7}$$

φ_0 is the phase shift angle between the excitation torque amplitude and the displacement, A – the displacement amplitude in the form of the relative twist angle of the clutch plates.

The Van der Pol method presented in [1] yields

$$\dot{A}\cos z - A\dot{\varphi}_0 \sin z = 0. \tag{4.8}$$

On differentiating the expected solution (4.6) twice and substituting into Eq. (4.4), we find

$$- \dot{A}\omega J_z \sin z - A\omega^2 J_z \cos z - A\omega\dot{\varphi}_0 J_z \cos z + M(\varphi, \dot{\varphi}) = \beta_1 M_0 \cos(z - \varphi_0) + M_m, \tag{4.9}$$

where

$$\beta_1 = \frac{J_2}{J_1}. \tag{4.10}$$

On multiplying Eq. (4.8) by $\omega \cos z$, Eq. (4.9) – by $\sin z$ and subtracting, we obtain

$$- \dot{A}\omega J_z - A\omega^2 J_z \sin z \cos z + M(\varphi, \dot{\varphi}) \sin z = \beta_1 M_0 \sin z \cos(z - \varphi_0) + M_m \sin z. \tag{4.11}$$

Since A and φ_0 are slowly varying parameters in Eq. (4.4), Eq. (4.11) takes, after integrating over the interval $(0,\ 2\pi)$, the form

$$- 2\dot{A}\omega J_z + \frac{1}{\Pi}\int_0^{2\pi} M(\varphi, \dot{\varphi}) \sin z dz = \beta_1 M_0 \sin \varphi_0. \tag{4.12}$$

On multipyling Eq. (4.8) by $\omega \sin z$, Eq. (4.9) – by $\cos z$, adding and integrating over the interval $(0,\ 2\pi)$, we obtain

$$- 2J_z A\omega\dot{\varphi}_0 - J_z A\omega^2 + \frac{1}{\Pi}\int_0^{2\pi} M(\varphi, \dot{\varphi}) \cos z dz = \beta_1 M_0 \cos \varphi_0. \tag{4.13}$$

In the steady state we have

$$\dot{A} = \dot{\varphi}_0 = 0\,,$$

therefore Eqs. (4.12) and (4.13) are reduced to the form

$$\sin \varphi_0 = \frac{1}{\Pi \beta_1 M_0}\int_0^{2\pi} M(\varphi, \dot{\varphi}) \sin z dz, \tag{4.14}$$

(4.15)
$$J_z\omega^2 + \frac{\beta_1 M_0}{A}\cos\varphi_0 = \frac{1}{\Pi A}\int\limits_0^{2\pi} M(\varphi,\dot\varphi)\cos z dz.$$

If the solution of Eq. (4.4) is analysed by variational means (the Ritz method) we obtain a result which is identical with Eqs. (4.14) and (4.15).

As regards integration of Eq. (4.4) it is necessary to match the solutions, as a consequence of a discontinuity of $M(\varphi,\dot\varphi)$ for $\dot\varphi = 0$. To avoid this necessity, we confine our consideration to a single half-period (the motion between two stops).

Thus the integration interval, from 0 to 2Π, of the rigid-hand terms of the above equations is divided into two sub-intervals, from 0 to Π for negative sign $d\varphi/dt$ and from Π to 2Π for positive sign $d\varphi/dt$. This is, for instance, the procedure adopted by the authors of [1, 2] and [5].

By denoting

(4.16)
$$A = \varphi_s \operatorname{sign}\frac{d\varphi}{dt}$$

and making use of Eq. (4.6) the formula (3.6) can be reduced to the form

(4.17)
$$M(\varphi,\dot\varphi) = \begin{cases} \frac{k}{\eta}\left\{(1+2\eta A)^{\frac{1}{2}} + 1 - 2\left[1+\eta A(1-\cos z)\right]^{\frac{1}{2}}\right\} & \text{for } \operatorname{sign}\frac{d\varphi}{dt} < 0, \\ \frac{k}{\eta}\left\{2\left[1+\eta A(1+\cos z)\right]^{\frac{1}{2}} - 1 - (1+2\eta A)^{\frac{1}{2}}\right\} & \text{for } \operatorname{sign}\frac{d\varphi}{dt} > 0. \end{cases}$$

Evaluation of the integral in Eq. (4.15) by analytical methods being impossible, the terms of Eq. (4.17) involving square roots were expanded in power series. As a result of the numerical analysis it was found that the value of the root and the sum of the first five terms of the series approach each other very closely. The remainder of the convergent binomial series, beginning from the sixth term, tends rapidly to zero with increasing number of the rejected terms of the series. In view of the properties of that series the first five terms are assumed for further considerations.

After some manipulations we obtain

(4.18)
$$M(\varphi,\dot\varphi) = kA\left\{\cos z + \frac{1}{2}\eta A\left[\frac{1}{2}(1-\cos z)^2 - 1\right] + \frac{1}{2}\eta^2 A^2 \times\left[1 - \frac{1}{4}(1-\cos z)^3\right] + \frac{5}{8}\eta^3 A^3\left[\frac{1}{8}(1-\cos z)^4 - 1\right]\right\} \quad \text{for } \operatorname{sign}\frac{d\varphi}{dt} < 0,$$

(4.19)
$$M(\varphi,\dot\varphi) = kA\left\{\cos z + \frac{1}{2}\eta A\left[1 - \frac{1}{2}(1+\cos z)^2\right] + \frac{1}{2}\eta^2 A^2 \times\left[\frac{1}{4}(1+\cos z)^3 - 1\right] + \frac{5}{8}\eta^3 A^3\left[1 - \frac{1}{8}(1+\cos z)^4\right]\right\} \quad \text{for } \operatorname{sign}\frac{d\varphi}{dt} > 0,$$

On substituting these expressions into Eqs. (4.14) and (5.15) and integrating, we find

$$\sin\varphi_0 = \frac{1}{\Pi\beta_1 M_0}\left(\int_0^{\pi} M(\varphi,\dot\varphi)\sin z dz\Big|_{\mathrm{sign}\frac{d\varphi}{dt}<0} + \int_{\pi}^{2\pi} M(\varphi,\dot\varphi)\sin z dz\Big|_{\mathrm{sign}\frac{d\varphi}{dt}>0}\right) = \frac{2k\eta A^2}{3\Pi M_0\beta_1}\left[\frac{3}{2}\eta A\left(1-\frac{3}{2}\eta A\right)-1\right], \tag{4.20}$$

$$J_z\omega^2 + \frac{\beta_1 M_0}{A}\cos\varphi_0 = \frac{1}{\Pi A}\left(\int_0^{\pi} M(\varphi,\dot\varphi)\cos z dz\Big|_{\mathrm{sign}\frac{d\varphi}{dt}<0} + \int_{\pi}^{2\pi} M(\varphi,\dot\varphi)\cos z dz\Big|_{\mathrm{sign}\frac{d\varphi}{dt}>0}\right) = k\left\{1-\frac{\eta A}{2}\left[1-\frac{5}{16}\eta A\left(3-\frac{7}{2}\eta A\right)\right]\right\}. \tag{4.21}$$

On introducing the notations: $a = A/\varphi_{st}$ – dimensionless vibration amplitude, $\varphi_s = M_0/k$ – static displacement in the form of relative angular displacement of clutch plates, $\gamma = \omega/\omega_0$ – dimension frequency, $\psi = \eta(M_0/k)$ – dimensionless parameter, we have the following relations for the tangent of the phase displacement angle φ_0 and the dimensionless frequency γ as functions of the external load, the geometrical parameters of the system, the friction forces and the dimensionless amplitude a:

$$\mathrm{tg}\varphi_0 = \frac{\frac{2\psi a}{3\Pi}\left[\frac{3}{2}\psi a\left(1-\frac{3}{2}\psi a\right)-1\right]}{1-\frac{\psi a}{2}\left[1-\frac{5}{16}\psi a\left(3-\frac{7}{2}\psi a\right)\right]-\gamma^2}, \tag{4.22}$$

$$\gamma = \left[1-\frac{\psi a}{2}\left[1-\frac{5}{16}\psi a\left(3-\frac{7}{2}\psi a\right)\right] \mp\frac{\beta_1}{a}\left\{1-\frac{4\psi^2 a^4}{9\Pi^2\beta_1^2}\left[\frac{3}{2}\psi a\left(1-\frac{3}{2}\psi a\right)-1\right]^2\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}}. \tag{4.23}$$

Numerical computation was performed with the following data $h_1 = 0.00125\,\mathrm{m}$, $h_2 = 0.00103\,\mathrm{m}$, $r = 0.0585\,\mathrm{m}$, $l = 0.15\,\mathrm{m}$, $d = 0.035\,\mathrm{m}$.

The results of computation are shown in Figs.4 to 10.

5. Conclusions

On the basis of the results of numerical analysis it has been found that all the resonance curves start from the dimensional resonance amplitude, somewhat below unity, and tend asymptotically to zero in the postresonance range. They tend also to a more smooth form in that range. In all the cases the dimensionless resonance amplitudes take, under constant pressure distribution, the values which are higher than those for variable pressure distribution, the difference ranging from a few percent to a dozen percent or so. We can also observe a phenomenon of displacement of the resonance in the direction to the left in the case of variable pressure distribution as compared to that of uniform pressure distribution, other parameters of the system remaining unchanged. The above inferences are confirmed by diagrams, some examples of which are shown in Figs.4 and 5. In the case of resonance curves moving farther to the left, the damping in the system is stronger, which is seen from the decreasing resonance amplitudes. From the diagrams in Fig. 5 it is seen that an increasing excitation amplitude M_0 results in a smaller dimensionless resonance amplitude. This fact is caused by the increase in the load M_0 being accompanied by an increase in area of the zone of slip which means increased energy dissipation, which results, in turn, in more intensive vibration damping in the power transmission system. The damping effect is the best for an appropriate value of the product $q_2 = pf$, because the zone of relative slip between the clutch plates is the largest. This inference is confirmed by the diagrams of Fig. 4 showing clearly the influence of product on the resonance amplitude, with the same geometrical parameters and the same load, but a variable coefficient of friction f. The influence of the dimensionless parameter $\psi = g(M_0,\ d,\ h_1,\ h_2,\ l,\ r,\ P_3,\ P_4,\ f)$ on the dimensionless resonance amplitude can be easily analysed by the designer. The nonlinearity of the system is observed at any frequency and amplitude of vibration. For an excitation frequency ω approaching the natural frequency ω_0, the dimensionless amplitudes a reach very high values. From Figs.4 and 5 it is seen that the most dangerous frequency interval is

$$0.73 < \gamma < 1.15$$

If γ approaches zero, the dimensionless amplitude of vibration tends to a value between 0.85 and 0.9. This means that slowly acting excitation torque produces an angular displacement of the system approaching its static value.

The dependence of the phase shift angle φ_0 on the dimensionless fre-

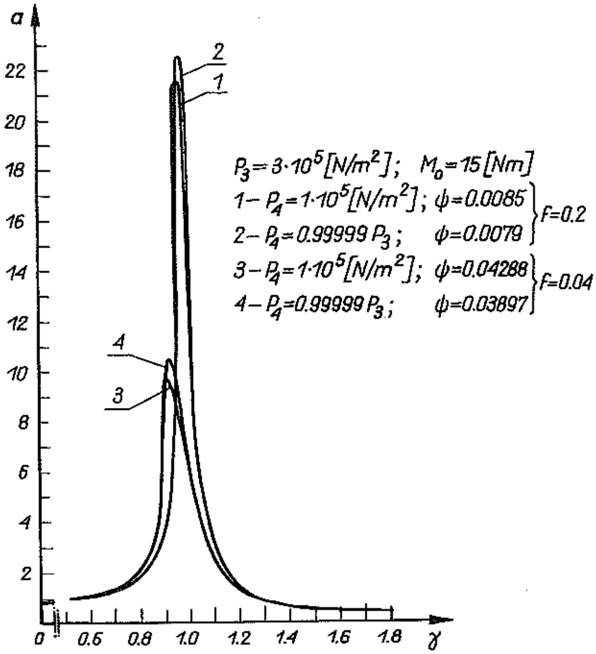

FIG. 4. Resonance curves for uniform and non-uniform pressure distribution.

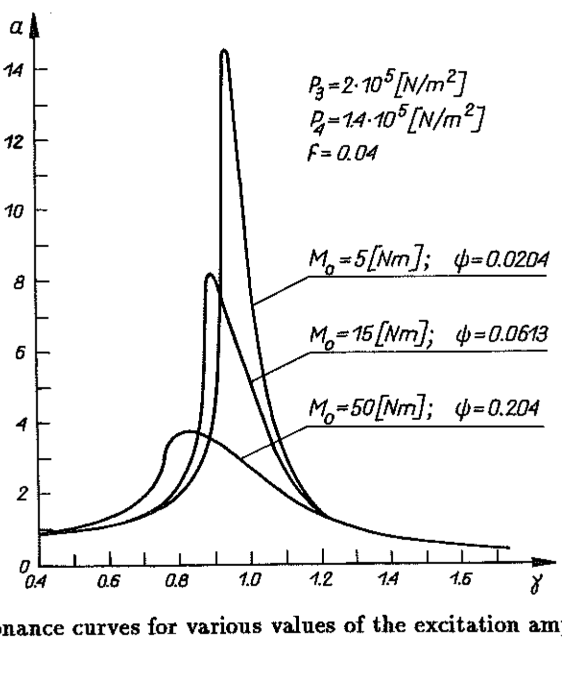

FIG. 5. Resonance curves for various values of the excitation amplitude M_0.

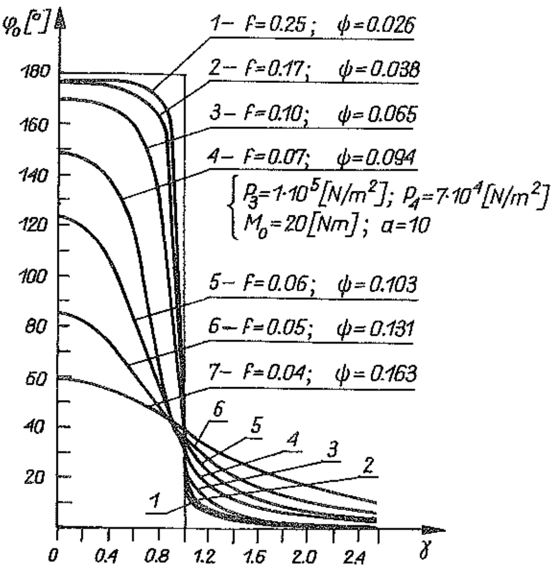

FIG. 6. Diagrams of phase displacement angle φ_0 as a function of the dimensionless frequency γ, for various friction coefficients f.

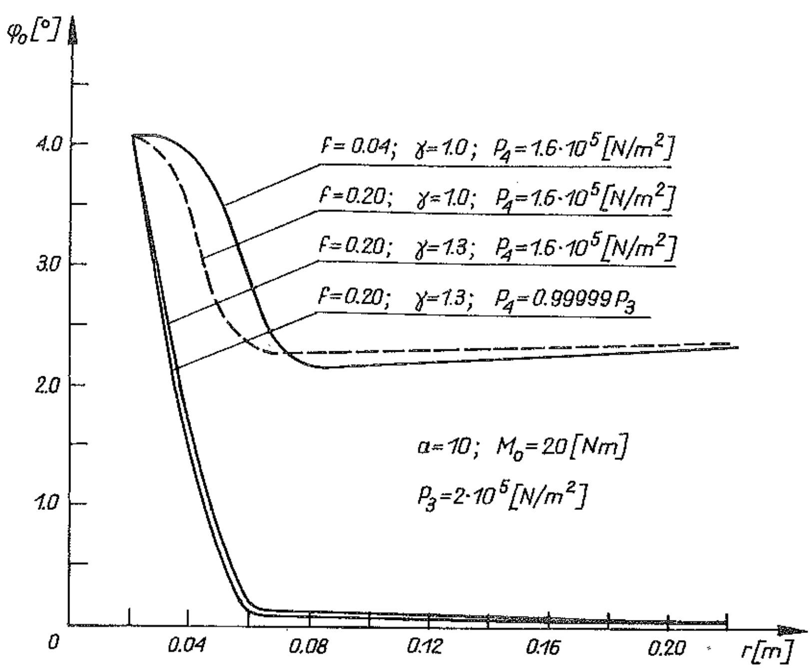

FIG. 7. Diagrams of phase displacement φ_0 as a function of the internal radius r of a clutch plate for $\gamma = 1$ and $\gamma = 1.3$.

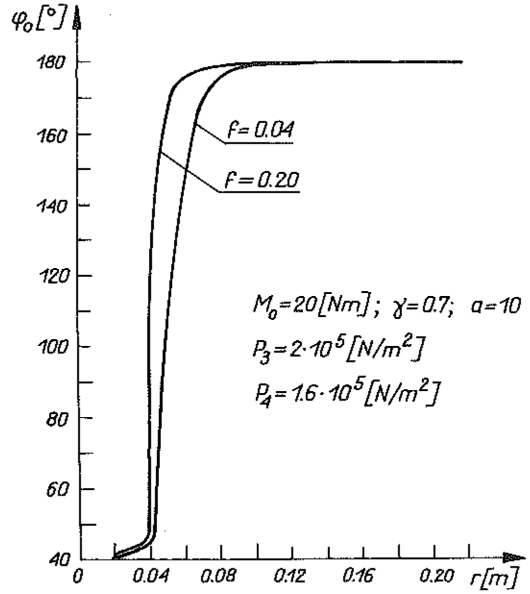

FIG. 8. Diagram of phase angle φ_0 as a function of the internal radius r of a clutch plate for $\gamma = 0.7$.

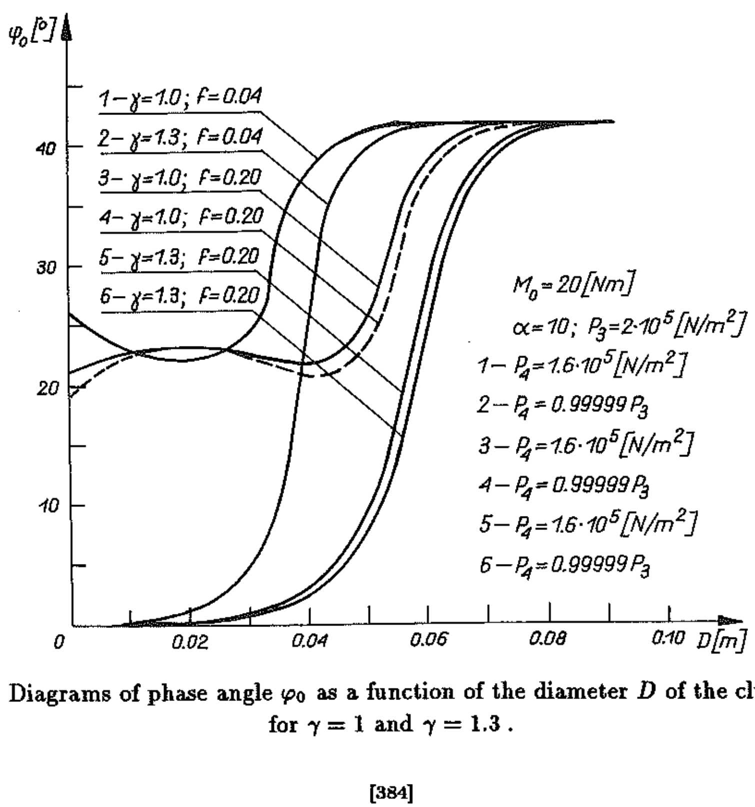

FIG. 9. Diagrams of phase angle φ_0 as a function of the diameter D of the clutch shaft for $\gamma = 1$ and $\gamma = 1.3$.

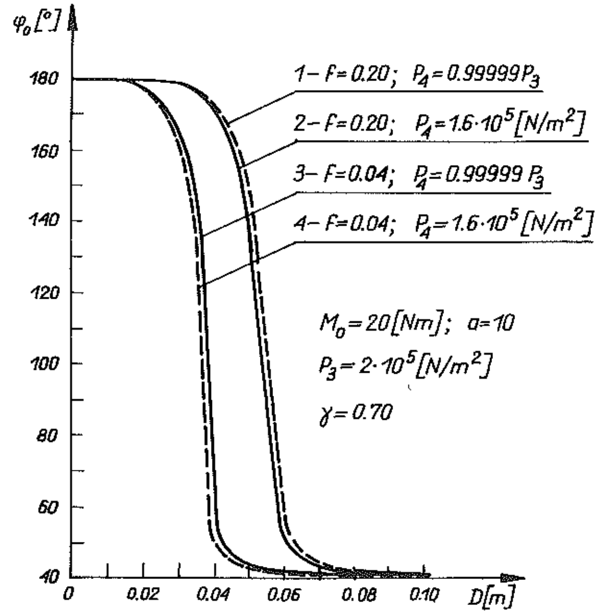

FIG. 10. Diagrams of phase angle φ_0 as a function of the diameter D of the clutch shaft for $\gamma = 0.7$.

quency γ for various values of the friction coefficient f is represented by the diagrams in Fig. 6. For a high value of the friction coefficient f the angle φ_0 varies rapidly in the neighbourhood of $\gamma = 1$. At decreasing values of γ the angle φ_0 becomes large, therefore there appears a phase shift between the forced vibration and the excitation torque.

For $\gamma > 1$ the angle φ_0 decreases and tends to zero, which shows that the forced vibration is the same phase as the excitation torque. With decreasing $\gamma(\gamma < 1)$ the value of the angle φ_0 increases which means that the forced vibration is shifted in phase with respect to the excitation torque. Then the vibration damping in the system increases. The diagrams of $\varphi_0 = g(r)$ shown in Figs.7 and 8 differ form those presented in Figs. 9 and 10 both in the pre-resonance and post-resonance ranges. In the post-resonance range $(\gamma > 1)$ the curves $\varphi_0 = g(d)$ begin at $\varphi_0 = 0$ and tend to $\varphi_0 \approx 41°$. Before resonance $(\gamma = 0.7)$ the value of the phase shift angle φ_0 decreases rapidly depending on the diameter of the clutch shaft (Fig. 10). These diagrams show clearly that the angle φ_0 is influenced by the friction coefficient f. The phase shift angle is a measure of vibration damping in the system. If it increases, the energy dissipation increases also, as well as the vibrations occuring in the mechanical system. The effects of structural friction can be made use of to improve the design methods of dynamic systems.

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Streszczenie

DRGANIA WYMUSZONE NIELINIOWEGO UKŁADU STACJONARNEGO ZE SPRZĘGŁEM CIERNYM PRZY UWZGLĘDNIENIU TARCIA KONSTRUKCYJNEGO

Rozważania przeprowadzono dla nieliniowego układu dyskretnego stacjonarnego zawierającego sprzęgło cierne wielopłytowe przy uwzględnieniu tarcia konstrukcyjnego. Drgania występują pod wpływem wymuszenia harmonicznego. Zagadnienie rozpatrywane jest przy założeniu nierównomiernego rozkładu nacisków występujących pomiędzy współpracującymi powierzchniami tarcz ciernych. Zbadano wpływ parametrów układu dynamicznego na charakterystykę amplitudowo-częstotliwościową i na kąt przesunięcia fazowego. Równanie ruchu badanego układu rozwiązano metodą Van der Pola.

Р е з ю м е

ВОЭМУШЕННЫЕ КОЛЕБАНИЯ НЕЛИНЕЙНОЙ СТАЦИОНАРНОЙ СИСТЕМЫ С ФРИКЦИОННОЙ МУФТОЙ ПРИ УЧТЕНИИ КОНСТРУКЦИОННОГО ТРЕНИЯ

Рассуждения провидено для нелинейного дискретного стационарного система содержающего многопластинковой фрикционной муфтой с учтением конструкционного трения. Колебания выступают под влиянием гармонического возмущениая. Проблема рассматриваются на основании неравномиерного расположения выступляющих между срабатывающими поверхностями фрикционных щитов. Исследовано влияние параметров динамической системы на амплитудночастотной характеристику и на угол фазного сдвига. Уравнение движения исследователий системы решено методом Ван дер Пола.

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