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ON IRREDUCIBLE NUMBER OF INVARIANTS AND GENERATORS IN THE CONSTITUTIVE RELATIONSHIPS(1)

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The Pipkin-Rivlin method for determining the generators of a polynomial representation of a symmetric isotropic second-order tensor-valued function is modified. Generators of an anisotropic and orthotropic symmetric second-order tensor-valued function are thus shown as dependent on a finite number of symmetric second-order tensors. The obtained results coincide with those arrived at by Boehler. Next, irreducible invariants of anisotropic scalar functions, depending on a single symmetric second-order tensor are found. Types of anisotropy are considered in which the material symmetry group is described by means of vectors and symmetric second-order tensors. The anisotropic scalar functions derived can be used to construct the constitutive equations for nonlinear elasticity of Green's material as well as potentials and yield conditions in plasticity. As an example, the equations are derived for a material reinforced with two orthogonal families of bars.

1. Introduction

The theory of tensor functions together with the theorems on their representations have been recognized to be an efficient mathematical tool for the formulation of constitutive relationships to describe an arbitrary behaviour of a continuum. This approach is found to ensure both the desirable analytical clarity and the required generality of the equations in question. It also allows to account in a straightforward manner for the invariance requirements of the principle of isotropy of space and the material symmetries as well as additional internal constraints such as inextensibility in certain directions, plane states and so on. Construction of a representation of a constitutive relationship consists in the determination of the types and the number of tensor generators as well as of the invariants of the polynomial or functional basis. The polynomial basis is termed irreducible if none of its elements can be expressed in terms of a polynomial depending on the remaining invariants. Pipkin and Wineman [8, 19] proved that the assumption of a polynomial type of a representation is not essential. Thus the

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necessity arises to determine, for given arguments, an irreducible functional basis whose set of invariants is in turn irreducible, provided any other invariant of these arguments can be expressed as an arbitrary scalar function of the invariants of the basis. The generator is said to be irreducible if it cannot be expressed as a linear combination of other generators, the coefficients being: polynomial functions of polynomial basis invariants in the case of polynomial representation, or arbitrary functions of the functional basis in the general case.

In the paper two types of nonpolynomial representations of anisotropic tensor functions will be considered: a scalar function and a symmetric second-order tensor-valued function. The three-dimensional Euclidean space is assumed as a convenient frame of reference. Among many types of material anisotropy there are such whose material symmetry group is characterized by unit vectors and symmetric second-order tensors. Classification of those groups, together with the corresponding sets of parametric tensors is given by I-SHIH LIU [6]. Complete list of parametric tensors for all classes of crystals is shown by SEDOV and LOKHIN [7, 11].

Let g denote a group of material symmetry

$$g = \{\mathbf{Q} \in G : \mathbf{Q}\mathbf{e}_p = \mathbf{e}_p, \mathbf{Q}\mathbf{M}_r\mathbf{Q}^T = \mathbf{M}_r\}, \qquad p = 1,\ldots,n, \qquad r = 1,\ldots,m, \tag{1.1}$$

where $\mathbf{Q}$ is an orthogonal tensor, $\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$, $\mathbf{I}$ stands for unit tensor, $\mathbf{e}_p$ are unit vectors, $\mathbf{M}_r$ denote symmetric second-order tensors, G is subgroup of a full orthogonal group O of all rotations and reflections.

Representations of scalar function

$$t = f(\mathbf{A}_k) \tag{1.2}$$

and a symmetric second-order tensor-valued function

$$\mathbf{T} = \mathbf{F}(\mathbf{A}_k), \tag{1.3}$$

where $\mathbf{A}_k$, $k = 1,2,\ldots,a$ denote symmetric second-order tensors, are sought such that, for $\forall\mathbf{Q} \in g$, the following relations be satisfied:

$$t = f(\mathbf{Q}\mathbf{A}_k\mathbf{Q}^T), \tag{1.4}$$

$$\mathbf{Q}\mathbf{T}\mathbf{Q}^T = \mathbf{F}(\mathbf{Q}\mathbf{A}_k\mathbf{Q}^T). \tag{1.5}$$

Such a dependence of the functions (1.2), (1.3) on the parametric tensors is aimed at that the relationships (1.4), (1.5) hold true for $\forall\ \mathbf{Q} \in O$. Then the functions sought are isotropic functions with respect to arguments $\mathbf{A}_k$ and parametric tensors as well. Isotropy of thus constructed functions within the group g can be found in [6]. For the general case of arbitrary functional relationships RYCHLEWSKI [10] proved that it is always possible to select such parameters that the constitutive relation be a universal rule conforming to the principle of isotropy of the physical space. In our case it is sufficient that such $\mathbf{e}_p$ and $\mathbf{M}_r$ exist, defining g, which lead to the following relationships being satisfied:

$$(1.6)\qquad \forall \mathbf{Q} \in O, \qquad t = f(\mathbf{Q}\mathbf{A}_k\mathbf{Q}^T, \mathbf{Q}\mathbf{e}_p, \mathbf{Q}\mathbf{M}_r\mathbf{Q}^T),$$

$$(1.7)\qquad \mathbf{Q}\mathbf{T}\mathbf{Q}^T = \mathbf{F}(\mathbf{Q}\mathbf{A}_k\mathbf{Q}^T, \mathbf{Q}\mathbf{e}_p, \mathbf{Q}\mathbf{M}_r\mathbf{Q}^T)$$

and

$$\forall \mathbf{Q} \in G, \qquad \mathbf{Q}\mathbf{e}_p = \mathbf{e}_p, \qquad \mathbf{Q}\mathbf{M}_r\mathbf{Q}^T = \mathbf{M}_r.$$

Determination of the representation of functions (1.2) and (1.3) consists in finding the representation of a suitable isotropic scalar function and an isotropic symmetric second-order tensor-valued function depending on the arguments $\mathbf{A}_k$, $\mathbf{e}_p$, $\mathbf{M}_r$. Since the parameters $\mathbf{e}_p$, $\mathbf{M}_r$ are not variables, the determined invariants of the functional basis and the sets of generators are, according to the theorems given by WANG [16, 17, 18], SMITH [12, 13] and BOEHLER [1], generally reducible. That is why a direct analysis of invariants and a representation of generators is always necessary, possibly in a chosen, most simple, Cartesian frame of reference oriented with the body considered. It is perfectly reasonable since the physical space is a Euclidean one for which all the considerations can be made in the Cartesian coordinates. Reduction of invariants and, in particular, generators, in the case of nonpolynomial anisotropic tensor functions is usually very cumbersome and not infrequently too complicated to be performed.

2. GENERATORS FOR ANISOTROPIC AND ORTHOTROPIC SYMMETRIC SECOND-ORDER TENSOR-VALUED FUNCTION

Irreducible nonpolynomial representations of the functions (1.2) and (1.3) for the cases of general anisotropy, orthotropy and one of the types of transverse isotropy are given by BOEHLER in [3, 4, 5]. In what follows a different, simpler way will be shown to obtain the generators of the function (1.3) for the cases of anisotropy and orthotropy. The obtained coincidence of results is by no means incidental; the proposed method can be employed in all those anisotropies whose material symmetry groups are finite ones.

Following BOEHLER [3, 5], two corollaries can be formulated as below:

2.1. COROLLARY 1

In the three-dimensional space a complete and irreducible representation of an arbitrary anisotropic scalar function (1.4) has the form

$$(2.1)\qquad t = \varphi(\mathrm{tr}\mathbf{M}_{11}\mathbf{A}_k, \mathrm{tr}\mathbf{M}_{22}\mathbf{A}_k, \mathrm{tr}\mathbf{M}_{33}\mathbf{A}_k, \mathrm{tr}\mathbf{M}_{12}\mathbf{A}_k, \mathrm{tr}\mathbf{M}_{13}\mathbf{A}_k, \mathrm{tr}\mathbf{M}_{23}\mathbf{A}_k) = \varphi(l_{\mathbf{A}_k}),$$

where $\mathbf{M}_{ij}=\mathbf{e}_i\otimes\mathbf{e}_j$, $i,j = 1,2,3$, $\mathrm{tr}\mathbf{M}_{ij}\mathbf{A}_k = \mathrm{tr}\mathbf{M}_{ji}\mathbf{A}_k$. For example, the representations of tensors $\mathbf{M}_{11}$ and $\mathbf{M}_{12}$ are as follows

$$\mathbf{M}_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{M}_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

2.2. Corollary 2

In the three-dimensional space a complete and irreducible representation of an arbitrary orthotropic scalar function (1.4) has the form

$$t = \psi(l_{\mathbf{A}_k,\mathbf{M}_{ii}}), \qquad i = 1,2,3, \qquad \mathbf{M}_{ii} = \mathbf{e}_i \otimes \mathbf{e}_i \quad \text{(no summation)}, \tag{2.2}$$

where $l_{\mathbf{A}_k,\mathbf{M}_{ii}}$ denote all the invariants combined from one, two or three variable arguments $\mathbf{A}_k$ as shown in Table 1.

Table 1. Functional basis for orthotropic scalar function.

Variables	Invariants
1	2
$\mathbf{A}$	$\mathrm{tr}\mathbf{M}_{11}\mathbf{A}$, $\mathrm{tr}\mathbf{M}_{11}\mathbf{A}^2$, $\mathrm{tr}\mathbf{A}^3$, $\mathrm{tr}\mathbf{M}_{22}\mathbf{A}$, $\mathrm{tr}\mathbf{M}_{22}\mathbf{A}^2$, $\mathrm{tr}\mathbf{M}_{33}\mathbf{A}$, $\mathrm{tr}\mathbf{M}_{33}\mathbf{A}^2$,
$\mathbf{A}_1$,$\mathbf{A}_2$	$\mathrm{tr}\mathbf{M}_{11}\mathbf{A}_1\mathbf{A}_2$, $\mathrm{tr}\mathbf{A}_1^2\mathbf{A}_2$, $\mathrm{tr}\mathbf{A}_1\mathbf{A}_2^2$, $\mathrm{tr}\mathbf{M}_{22}\mathbf{A}_1\mathbf{A}_2$, $\mathrm{tr}\mathbf{M}_{33}\mathbf{A}_1\mathbf{A}_2$,
$\mathbf{A}_1$,$\mathbf{A}_2$,$\mathbf{A}_3$	$\mathrm{tr}\mathbf{A}_1\mathbf{A}_2\,\mathbf{A}_3$

In order to establish an anisotropic nonpolynomial representation of the second-order tensor-valued function (1.3), an auxiliary anisotropic scalar function α is first assumed,

$$\alpha = \mathrm{tr}\,\mathbf{CT}, \tag{2.3}$$

which remains linear with respect to the symmetric second-order tensor $\mathbf{C}$. $\mathbf{T}$ is a function sought. Assume that

$$\alpha = \varphi_h(l_{\mathbf{A}_k})\, l^h_{\mathbf{C}}, \qquad h = 1,\ldots,6, \tag{2.4}$$

where the dummy, or summation, index $h = 1,\ldots,6$, $l^h_{\mathbf{C}}$ stand for linear invariants of the tensor $\mathbf{C}$ determined in accordance with Corollary 1. The function $\mathbf{T}$ to be found will be calculated from the formula

$$\mathbf{T} = \frac{\partial\alpha}{\partial\mathbf{C}} = \varphi_h(l_{\mathbf{A}_k})\,\frac{\partial l^h_{\mathbf{C}}}{\partial\mathbf{C}} = \varphi_h\mathbf{G}^h. \tag{2.5}$$

We finally obtain

$$T = \varphi_1 \mathrm{M}_{11} + \varphi_2 \mathrm{M}_{22} + \varphi_3 \mathrm{M}_{33} + \varphi_4(\mathrm{M}_{12} + \mathrm{M}_{21}) + \varphi_5(\mathrm{M}_{13} + \mathrm{M}_{31}) + \varphi_6(\mathrm{M}_{23} + \mathrm{M}_{32}). \tag{2.6}$$

Remember that $\varphi_h = \varphi_h(l_{\mathbf{A}_k})$.

Similarly, to determine a nonpolynomial representation of an orthotropic symmetric second-order tensor-valued function **T**, see Eq. (1.3), an auxiliary orthotropic scalar function β is again assumed,

$$\beta = \mathrm{tr}\,\mathbf{CT}\,, \tag{2.7}$$

which stays linear with respect to **C** as before. Its form is the following:

$$\beta = \psi_g(l_{\mathbf{A}_k,\mathbf{M}_{ii}})\, l^g_{\mathbf{C},\mathbf{M}_{ii}}, \tag{2.8}$$

where the summation index $g = 1,\ldots,s$ and $l^g_{\mathbf{C},\mathbf{M}_{ii}}$ are linear invariants of the tensor **C** determined according to COROLLARY 2. These invariants are given in Table 2.

Table 2. Linear invariants $l^g_{\mathbf{C},\mathbf{M}_{ii}}$.

Variables	Invariants
$\mathbf{C}$	$\mathrm{tr}\mathbf{M}_{11}\mathbf{C}$, $\mathrm{tr}\mathbf{M}_{22}\mathbf{C}$, $\mathrm{tr}\mathbf{M}_{33}\mathbf{C}$
$\mathbf{C}$, $\mathbf{A}$	$\mathrm{tr}\mathbf{M}_{11}\mathbf{CA}$, $\mathrm{tr}\mathbf{M}_{22}\mathbf{CA}$, $\mathrm{tr}\mathbf{M}_{33}\mathbf{CA}$, $\mathrm{tr}\mathbf{CA}^2$
$\mathbf{C}$, $\mathbf{A}_1$, $\mathbf{A}_2$	$\mathrm{tr}\mathbf{CA}_1\mathbf{A}_2$

The orthotropic function **T** can be found from the formula

$$\mathbf{T} = \frac{\partial \beta}{\partial \mathbf{C}} = \psi_g(l_{\mathbf{A}_k,\mathbf{M}_{ii}})\,\frac{\partial l^g_{\mathbf{C},\mathbf{M}_{ii}}}{\partial \mathbf{C}} = \psi_g \mathbf{G}^g. \tag{2.9}$$

Simple calculations lead to the set of generators $\mathbf{G}^g$ which are shown in Table 3.

The above generators appear to coincide with those given by BOEHLER in [3, 5].

It is worth noting that the application of the method put forward above, which can be looked upon as a certain generalization of the Pipkin - Rivlin method [9] valid for polynomial representations of isotropic symmetric second-order tensor-valued functions, leads for transverse isotropy to the results different from those given in [3, 5].

Table 3. Set of generators.

Variables	Generators
–	M_{11}, M_{22}, M_{33}
A	$M_{11}A + AM_{11}, A^2$ $M_{22}A + AM_{22}$ $M_{33}A + AM_{33}$
A_1, A_2	$A_1A_2 + A_2A_1$

3. Functional basis for certain anisotropic scalar functions

We shall now deal with an irreducible basis of invariants for a scalar function (1.6). The considerations will be confined to one argument A only. Theorems of Wang [17] and Boehler [5] enable the invariants to be formed for particular material symmetry groups and their number to be reduced. Relevant results are shown in Table 4.

In the first column the names of crystallographic systems or type of anisotropy are given whereas the second column provides the names of geometric symmetry classes within each system together with suitable parametric tensors (after [6]). The third column supplies the minimal numbers of irreducible invariants. Their determination consists in the reduction of invariants obtained from appropriate theorems for isotropic functions [17, 5]. Let us demonstrate that, taking for example sphenoidal class of monoclinic system (Table 4, case 4b), there are only 8 invariants, out of 34, that constitute an irreducible set.

P r o o f. In the Cartesian frame of reference the following relationships are known to hold good:

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = 1, \quad \mathbf{e}_1 \cdot \mathbf{M}_{22}\mathbf{e}_1 = \mathbf{e}_1 \cdot \mathbf{M}_{22}^2\mathbf{e}_1 = 0, \quad \mathrm{tr}\mathbf{N}_1^2 = -2,$$

$$\mathrm{tr}\mathbf{M}_{22} = \mathrm{tr}\mathbf{M}_{22}^2 = \mathrm{tr}\mathbf{M}_{22}^3 = 1, \quad \mathrm{tr}\mathbf{M}_{22}\mathbf{N}_1^2 = \mathrm{tr}\mathbf{M}_{22}^2\mathbf{N}_1^2 = -1,$$

$$\mathrm{tr}\mathbf{M}_{22}^2\mathbf{N}_1^2\mathbf{M}_{22}\mathbf{N}_1 = 0, \tag{3.1}$$

$$\mathbf{e}_1 \cdot \mathbf{N}_1^2\mathbf{e}_1 = \mathbf{M}_{22}\mathbf{e}_1 \cdot \mathbf{N}_1\mathbf{e}_1 = \mathbf{M}_{22}^2\mathbf{e}_1 \cdot \mathbf{N}_1\mathbf{e}_1 = \mathbf{M}_{22}^2\mathbf{N}_1\mathbf{e}_1 \cdot \mathbf{N}_1^2\mathbf{e}_1 = 0,$$

$$\mathbf{A}\mathbf{e}_1 \cdot \mathbf{M}_{22}\mathbf{e}_1 = \mathbf{A}\mathbf{e}_1 \cdot \mathbf{N}_1\mathbf{e}_1 = \mathbf{A}^2\mathbf{e}_1 \cdot \mathbf{N}_1\mathbf{e}_1 = \mathbf{A}^2\mathbf{N}_1\mathbf{e}_1 \cdot \mathbf{N}_1\mathbf{e}_1 = 0.$$

This means that the above invariants are reducible and, in addition, since $\mathbf{M}_{22}^2 = \mathbf{M}_{22}$, the invariants $\mathrm{tr}\mathbf{A}\mathbf{M}_{22}^2, \mathrm{tr}\mathbf{A}^2\mathbf{M}_{22}^2, \mathrm{tr}\mathbf{A}\mathbf{M}_{22}^2\mathbf{N}_1$ are also redundant.

The irreducible set of invariants is:

Table 4. Sets of irreducible invariants.

Type of anisotropy, crystallo-graphic system	Geometric symmetry class, parametric tensors	Irreducible invariants
1	2	3
1) Transverse isotropy	a) $\mathbf{e}_1$, $\mathbf{N}_1 = \mathbf{e}_2 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_2$	$\mathrm{tr}\mathbf{A}$, $\mathrm{tr}\mathbf{A}^2$, $\mathrm{tr}\mathbf{A}^3$, $\mathbf{e}_1 \cdot \mathbf{A}\mathbf{e}_1$, $\mathbf{e}_1 \cdot \mathbf{A}^2\mathbf{e}_1$, $\mathrm{tr}\mathbf{A}^2\mathbf{N}_1^2\mathbf{A}\mathbf{N}_1$
	b) $\mathbf{e}_1$	$\mathrm{tr}\mathbf{A}$, $\mathrm{tr}\mathbf{A}^2$, $\mathrm{tr}\mathbf{A}^3$, $\mathbf{e}_1 \cdot \mathbf{A}\mathbf{e}_1$, $\mathbf{e}_1 \cdot \mathbf{A}^2\mathbf{e}_1$
	c) $\mathbf{N}_1$	$\mathrm{tr}\mathbf{A}$, $\mathrm{tr}\mathbf{A}^2$, $\mathrm{tr}\mathbf{A}^3$, $\mathrm{tr}\mathbf{A}\mathbf{N}_1^2$, $\mathrm{tr}\mathbf{A}^2\mathbf{N}_1^2$, $\mathrm{tr}\mathbf{A}^2\mathbf{N}_1^2\mathbf{A}\mathbf{N}_1$
	d) $\mathbf{M}_{11} = \mathbf{e}_1 \otimes \mathbf{e}_1$	[3] $\mathrm{tr}\mathbf{A}$, $\mathrm{tr}\mathbf{A}^2$, $\mathrm{tr}\mathbf{A}^3$, $\mathrm{tr}\mathbf{A}\mathbf{M}_{11}$, $\mathrm{tr}\mathbf{A}^2\mathbf{M}_{11}$
2) orthotropy	$\mathbf{M}_{22} = \mathbf{e}_2 \otimes \mathbf{e}_2$, $\mathbf{M}_{33} = \mathbf{e}_3 \otimes \mathbf{e}_3$	$\mathrm{tr}\mathbf{A}$, $\mathrm{tr}\mathbf{A}^2$, $\mathrm{tr}\mathbf{A}^3$, $\mathrm{tr}\mathbf{A}\mathbf{M}_{22}$, $\mathrm{tr}\mathbf{A}^2\mathbf{M}_{22}$, $\mathrm{tr}\mathbf{A}\mathbf{M}_{33}$, $\mathrm{tr}\mathbf{A}^2\mathbf{M}_{33}$
3) triclinic system	a) predial class, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$	COROLLARY 1 [3, 4]
	b) pinacoidal class, $\mathbf{N}_1$, $\mathbf{N}_2 = \mathbf{e}_1 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_1$	$\mathrm{tr}\mathbf{A}$, $\mathrm{tr}\mathbf{A}\mathbf{N}_1^2$, $\mathrm{tr}\mathbf{A}\mathbf{N}_2^2$, $\mathrm{tr}\mathbf{A}\mathbf{N}_1\mathbf{N}_2$, $\mathrm{tr}\mathbf{A}\mathbf{N}_1^2\mathbf{N}_2$, $\mathrm{tr}\mathbf{A}\mathbf{N}_1\mathbf{N}_2^2$
4) monoclinic system	a) domatic class, $\mathbf{e}_2$, $\mathbf{e}_3$	$\mathrm{tr}\mathbf{A}$, $\mathbf{e}_2 \cdot \mathbf{A}\mathbf{e}_2$, $\mathbf{e}_2 \cdot \mathbf{A}^2\mathbf{e}_2$, $\mathbf{e}_3 \cdot \mathbf{A}\mathbf{e}_3$, $\mathbf{e}_3 \cdot \mathbf{A}^2\mathbf{e}_3$, $\mathbf{e}_2 \cdot \mathbf{A}\mathbf{e}_3$, $\mathbf{e}_2 \cdot \mathbf{A}^2\mathbf{e}_3$
	b) sphenoidal class, $\mathbf{e}_1$, $\mathbf{M}_{22}$, $\mathbf{N}_1$	$\mathrm{tr}\mathbf{A}^3$, $\mathbf{e}_1 \cdot \mathbf{A}\mathbf{e}_1$, $\mathbf{e}_1 \cdot \mathbf{A}^2\mathbf{e}_1$, $\mathrm{tr}\mathbf{A}\mathbf{M}_{22}$, $\mathrm{tr}\mathbf{A}^2\mathbf{M}_{22}$, $\mathrm{tr}\mathbf{A}^2\mathbf{N}_1^2\mathbf{A}\mathbf{N}_1$, $\mathrm{tr}\mathbf{A}\mathbf{M}_{22}\mathbf{N}_1$, $\mathrm{tr}\mathbf{A}^2\mathbf{M}_{22}\mathbf{N}_1$
	c) prismatic class, $\mathbf{M}_{22}$, $\mathbf{N}_1$	$\mathrm{tr}\mathbf{A}$, $\mathrm{tr}\mathbf{A}^2$, $\mathrm{tr}\mathbf{A}^3$, $\mathrm{tr}\mathbf{A}\mathbf{M}_{22}$, $\mathrm{tr}\mathbf{A}^2\mathbf{M}_{22}$, $\mathrm{tr}\mathbf{A}\mathbf{N}_1^2$, $\mathrm{tr}\mathbf{A}^2\mathbf{N}_1^2\mathbf{A}\mathbf{N}_1$, $\mathrm{tr}\mathbf{A}^2\mathbf{M}_{22}\mathbf{N}_1$
5) rhombic system	a) pyramidal class, $\mathbf{e}_1$, $\mathbf{M}_{22}$	$\mathrm{tr}\mathbf{A}$, $\mathrm{tr}\mathbf{A}^2$, $\mathrm{tr}\mathbf{A}^3$, $\mathrm{tr}\mathbf{A}\mathbf{M}_{22}$, $\mathrm{tr}\mathbf{A}^2\mathbf{M}_{22}$, $\mathbf{e}_1 \cdot \mathbf{A}\mathbf{e}_1$, $\mathbf{e}_1 \cdot \mathbf{A}^2\mathbf{e}_1$
	b) dipyramidal class	as in row 2)

$$
\begin{aligned}
\mathbf{e}_1 \cdot \mathbf{A}\mathbf{e}_1 &= A_{11}, \quad \mathrm{tr}\mathbf{A}\mathbf{M}_{22} = A_{22}, \quad \mathrm{tr}\mathbf{A}\mathbf{M}_{22}\mathbf{N}_1 = A_{33}, \\
\mathbf{e}_1 \cdot \mathbf{A}^2\mathbf{e}_1 &= A_{11}^2 + A_{12}^2 + A_{13}^2 = \tilde{A}_{11}, \\
\mathrm{tr}\mathbf{A}^2\mathbf{M}_{22} &= A_{12}^2 + A_{22}^2 + A_{23}^2 = \tilde{A}_{22}, \\
\mathrm{tr}\mathbf{A}^2\mathbf{M}_{22}\mathbf{N}_1 &= A_{13}^2 + A_{23}^2 + A_{33}^2 = \tilde{A}_{33}, \\
\mathrm{tr}\mathbf{A}^2\mathbf{N}_1^2\mathbf{A}\mathbf{N}_1 &= A_{23}(\tilde{A}_{22} - \tilde{A}_{33}) \\
&+ (A_{12}A_{13} + A_{22}A_{23} + A_{23}A_{33})(A_{33} - A_{22}), \\
\mathrm{tr}\mathbf{A}^3 &= A_{11}^3 + A_{22}^3 + A_{33}^3 + 3A_{11}(A_{12}^2 + A_{13}^2) \\
&+ 3A_{22}(A_{23}^2 + A_{12}^2) + 3A_{33}(A_{13}^2 + A_{23}^2) + 6A_{12}A_{23}A_{13}.
\end{aligned}
\tag{3.2}
$$

The remaining invariants are reducible since they can be expressed in terms of Eq. (3.2). These are:

$$
\begin{aligned}
&\mathrm{tr}\mathbf{A} = \mathbf{e}_1 \cdot \mathbf{A}\mathbf{e}_1 + \mathrm{tr}\mathbf{A}\mathbf{M}_{22} + \mathrm{tr}\mathbf{A}\mathbf{M}_{22}\mathbf{N}_1,\\
&\mathrm{tr}\mathbf{A}\mathbf{N}_1^2\mathbf{M}_{22}\mathbf{N}_1 = -\mathrm{tr}\mathbf{A}\mathbf{M}_{22}\mathbf{N}_1,\\
&\mathrm{tr}\mathbf{A}^2 = \mathbf{e}_1 \cdot \mathbf{A}^2\mathbf{e}_1 + \mathrm{tr}\mathbf{A}^2\mathbf{M}_{22} + \mathrm{tr}\mathbf{A}^2\mathbf{M}_{22}\mathbf{N}_1,\\
&\mathrm{tr}\mathbf{A}\mathbf{N}_1^2 = -(\mathrm{tr}\mathbf{A}\mathbf{M}_{22} + \mathrm{tr}\mathbf{A}\mathbf{M}_{22}\mathbf{N}_1),\\
&\mathrm{tr}\mathbf{A}^2\mathbf{N}_1^2 = -(\mathrm{tr}\mathbf{A}^2\mathbf{M}_{22} + \mathrm{tr}\mathbf{A}^2\mathbf{M}_{22}\mathbf{N}_1).
\end{aligned}
\tag{3.3}
$$

The obtained nonpolynomial representations of anisotropic scalar functions can be used in the formulation of, for instance, yield criteria for anisotropic continua or elastic potentials as bases for the derivations of constitutive equations for nonlinearly elastic Green's materials exhibiting the material symmetry groups (1.1). Then the variable A should be identified with the symmetric stress or strain tensors.

4. Example

As a simple example of using one of the results given in Table 4, a procedure will be shown to obtain the constitutive equations for nonlinear elasticity and the yield criterion for an isotropic matrix reinforced with two orthogonal straight families of bars, each having different mechanical properties.

4.1. Nonlinear elasticity

Let versors $\mathbf{e}_2$ and $\mathbf{e}_3$ coincide with the directions of reinforcement. The elastic potential is assumed to exist and have the form

$$W = f(\boldsymbol{\varepsilon}, \mathbf{M}_{22}, \mathbf{M}_{33}), \tag{4.1}$$

where $\boldsymbol{\varepsilon}$ is the small strain tensor. Function (4.1) is, according to row 2 of Table 4, dependent on seven invariants I_k, $k = 1, \ldots, 7$. If the two families of bars are nonorthogonal then the elastic potential is determined in row 4a, Table 4. The stress tensor $\boldsymbol{\sigma}$ can be derived from the rule

$$\boldsymbol{\sigma} = \frac{\partial W(I_k)}{\partial \boldsymbol{\varepsilon}} = \frac{\partial W}{\partial I_k}\frac{\partial I_k}{\partial \boldsymbol{\varepsilon}}. \tag{4.2}$$

On finding that

$$
\begin{aligned}
\frac{\partial I_1}{\partial \boldsymbol{\varepsilon}} &= \mathbf{I}\,, \qquad \frac{\partial I_2}{\partial \boldsymbol{\varepsilon}} = 2\boldsymbol{\varepsilon}\,, \qquad \frac{\partial I_3}{\partial \boldsymbol{\varepsilon}} = 3\boldsymbol{\varepsilon}^2\,,\\
\frac{\partial I_4}{\partial \boldsymbol{\varepsilon}} &= \mathbf{e}_2 \otimes \mathbf{e}_2\,, \qquad \frac{\partial I_5}{\partial \boldsymbol{\varepsilon}} = \mathbf{e}_2 \otimes \boldsymbol{\varepsilon}\mathbf{e}_2 + \mathbf{e}_2\boldsymbol{\varepsilon} \otimes \mathbf{e}_2\,,\\
\frac{\partial I_6}{\partial \boldsymbol{\varepsilon}} &= \mathbf{e}_3 \otimes \mathbf{e}_3\,, \qquad \frac{\partial I_7}{\partial \boldsymbol{\varepsilon}} = \mathbf{e}_3 \otimes \boldsymbol{\varepsilon}\mathbf{e}_3 + \mathbf{e}_3\boldsymbol{\varepsilon} \otimes \mathbf{e}_3,
\end{aligned}
\tag{4.3}
$$

the nonlinear elasticity law assumes the form

$$\begin{aligned}(4.4)\qquad \boldsymbol{\sigma} &= \alpha_1\mathbf{I} + 2\alpha_2\boldsymbol{\varepsilon} + 3\alpha_3\boldsymbol{\varepsilon}^2 + \alpha_4\mathbf{e}_2\otimes\mathbf{e}_2 \\ &+\alpha_5(\mathbf{e}_2\otimes\boldsymbol{\varepsilon}\mathbf{e}_2 + \mathbf{e}_2\boldsymbol{\varepsilon}\otimes\mathbf{e}_2) + \alpha_6\mathbf{e}_3\otimes\mathbf{e}_3 + \alpha_7(\mathbf{e}_3\otimes\boldsymbol{\varepsilon}\mathbf{e}_3 + \mathbf{e}_3\boldsymbol{\varepsilon}\otimes\mathbf{e}_3),\end{aligned}$$

where $\alpha_l = \alpha_l(I_k), \quad k,l = 1,\ldots,7.$

In addition, the following property must hold good:

$$\frac{\partial\alpha_l}{\partial I_k} = \frac{\partial\alpha_k}{\partial I_l}.$$

To linearize Eqs. (4.4) with respect to $\boldsymbol{\varepsilon}$, we have to assume: $\alpha_3 = 0$; α_2, α_5, α_7 as constants; α_1, α_4, α_6 as linear polynomials in the invariants I_1, I_4, I_6. Assuming the existence of neutral state $\boldsymbol{\varepsilon} = \mathbf{0} \Rightarrow \boldsymbol{\sigma} = \mathbf{0}$, we get

$$(4.5)\qquad \begin{aligned}\alpha_1 &= \lambda\mathrm{tr}\boldsymbol{\varepsilon} + a_1\mathbf{e}_2\cdot\boldsymbol{\varepsilon}\mathbf{e}_2 + a_2\mathbf{e}_3\cdot\boldsymbol{\varepsilon}\mathbf{e}_3\,,\\ \alpha_2 &= \mu, \qquad \alpha_3 = 0,\\ \alpha_4 &= a_1\mathrm{tr}\boldsymbol{\varepsilon} + b_1\mathbf{e}_2\cdot\boldsymbol{\varepsilon}\mathbf{e}_2 + b_2\mathbf{e}_3\cdot\boldsymbol{\varepsilon}\mathbf{e}_3,\\ \alpha_5 &= c_1, \qquad \alpha_7 = c_2,\\ \alpha_6 &= a_2\mathrm{tr}\boldsymbol{\varepsilon} + b_2\mathbf{e}_2\cdot\boldsymbol{\varepsilon}\mathbf{e}_2 + b_3\mathbf{e}_3\cdot\boldsymbol{\varepsilon}\mathbf{e}_3,\end{aligned}$$

where λ, μ are Lamé's constants for the matrix, $a_1, a_2, b_1, b_2, b_3, c_1, c_2$ are parameters to be determined from suitably planned experiments.

Equation (4.4), linearized with the help of Eqs. (4.5), turns out to be identical with the physical relationships obtained by SPENCER [14] for the same type of composite material.

4.2. *Perfect plasticity*

In the scope of the ideal plasticity theory it has recently been realized that the yield criterion can be arrived at by insisting that the relation between the stress and strain rate tensors must be zero-degree homogeneous with respect to the strain rates. Thus the yield criterion appears to be an additional scalar relationship for the stress invariants. In our case the seven invariants of the stress tensors are involved, namely

$$(4.6)\qquad f(J_k) = 0, \qquad k = 1,\ldots,7,$$

where J_k are determined in row 2, Table 4.

Assuming quadratic form of the invariants and replacing the trace of the second power of stress tensor by the corresponding expression for stress deviator $\mathbf{S}$, $\mathbf{S} = \boldsymbol{\sigma} - \frac{1}{3}(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I}$, we obtain:

$$
\begin{aligned}
(4.7) \qquad & \mathrm{tr}\mathrm{S}^2 + c_1 \mathrm{tr}^2\sigma + c_2(\mathrm{e}_2 \cdot \sigma \mathrm{e}_2)^2 + c_3(\mathrm{e}_3 \cdot \sigma \mathrm{e}_3)^2 \\
& + c_4 \mathrm{e}_2 \cdot \sigma^2 \mathrm{e}_2 + c_5 \mathrm{e}_3 \cdot \sigma^2 \mathrm{e}_3 + c_6 \mathrm{tr}\sigma(\mathrm{e}_2 \cdot \sigma \mathrm{e}_2) + c_7 \mathrm{tr}\sigma(\mathrm{e}_3 \cdot \sigma \mathrm{e}_3) \\
& + c_8(\mathrm{e}_2 \cdot \sigma \mathrm{e}_2)(\mathrm{e}_3 \cdot \sigma \mathrm{e}_3) - 2k^2 = 0,
\end{aligned}
$$

where $c_k, k = 1, \ldots, 8$ are material constants, k is the yield point in shear of the matrix. When the hydrostatic pressure is incapable of making the material yield plastically ($c_1 = c_6 = c_7 = 0$), we get the generalization of the well known Huber - Mises yield criterion for the considered composite material. The classical Huber - Mises yield criterion clearly results in insisting that all the c_k'-s vanish simultaneously.

5. Appendix

In his paper [6] I-Shih Liu used two parametric tensors M_{22} and M_{33} to determine orthotropic tensor-valued functions, whereas Boehler in his papers [2,3,4,5] employed additionally a parametric tensor M_{11}. The former author maintains that the two tensors he uses completely determine an orthotropic group of material symmetry and goes on to say that the tensors used by Boehler do not define this group in an appropriate manner. Telega in his paper [15] writes that I-Shih Liu's statement is not well grounded but he supplies no examples of orthotropic functions whose sets of invariants and/or generators are not equivalent.

In what follows it will be shown that for the functions (1.6) and (1.7) considered in this paper, dependent upon a finite number of symmetric second-order tensors, the assumption of the parametric tensors as done by I-Shih Liu in [6] leads to the equivalent set of invariants and generators in accordance with Boehler's results [2, 3, 4, 5].

With the use of the theorem on the isotropic scalar function, given in [1, 3, 17, 18], of arguments A_k, $k = 1, \ldots, a, \mathrm{M}_{22}, \mathrm{M}_{33}$, a certain set of tensorial invariants can be obtained in which the number of irreducible ones amounts only to $7a + 5\binom{a}{2} + \binom{a}{3}$. This appears to be a result of the following simple relationships:

$$
\mathrm{M}_{ll}^i = \mathrm{M}_{ll}, \quad \mathrm{M}_{ll}\mathrm{M}_{mm} = 0, \quad \mathrm{tr}\mathrm{M}_{ll}^i = 1, \quad i = 1, 2, 3, \quad l, m = 2, 3,
$$

Invariants of the type $\mathrm{tr}\mathrm{A}_k^2\mathrm{A}_n^2$ ($\mathrm{tr}\mathrm{A}_k^2\mathrm{A}_n^2 = \mathrm{tr}\mathrm{A}_n^2\mathrm{A}_k^2$) $k, m = 1, \ldots, a$, the number of which is $\binom{a}{2}$, prove to be reducible. This fact can be readily acknowledged by using the manner of reduction given by Boehler in [2]. Final results are shown in Table 5, column 2. As to the generators of a symmetric second-order tensor-valued function, let us employ, similarly as in Eqs. (2.7) and (2.8), an auxiliary orthotropic function together with the

Table 5. Functional basis and generators for orthotropy with parametric tensors of I-Shih Liu [6].

Variables	Invariants	Generators
1	2	3
–		I, M_{22}, M_{33}
A	$\mathrm{tr}A$, $\mathrm{tr}A^2$, $\mathrm{tr}A^3$, $\mathrm{tr}M_{22}$, $\mathrm{tr}M_{22}A^2$, $\mathrm{tr}M_{33}A$, $\mathrm{tr}M_{33}A^2$,	A, A^2 $M_{22}A + AM_{22}$ $M_{33}A + AM_{33}$
A_1,A_2,	$\mathrm{tr}A_1A_2$, $\mathrm{tr}A_1^2A_2$, $\mathrm{tr}A_1A_2^2$, $\mathrm{tr}M_{22}A_1A_2$, $\mathrm{tr}\, M_{33}A_1A_2$,	$A_1A_2 + A_2A_1$
A_1,A_2,A_3	$\mathrm{tr}A_1A_2\ A_3$	–

formula (2.9) and take suitable invariants from Table 5, column 2. The resulting generators are given in column 3 of the Table.

In turns out that the irreducible set of invariants for an orthotropic scalar function (Table 5, column 2) is equivalent to the set shown in Table 1. This is due to the following relationships:

$$\begin{aligned} \mathrm{tr}A &= \mathrm{tr}M_{11}A + \mathrm{tr}M_{22}A + \mathrm{tr}M_{33}A, \\ \mathrm{tr}A^2 &= \mathrm{tr}M_{11}A^2 + \mathrm{tr}M_{22}A^2 + \mathrm{tr}M_{33}A^2, \\ \mathrm{tr}A_1A_2 &= \mathrm{tr}M_{11}A_1A_2 + \mathrm{tr}M_{22}A_1A_2 + \mathrm{tr}M_{33}A_1A_2 \,. \end{aligned}$$

Moreover, these two sets have the same number of invariants.

Similarly, the set of generators for a symmetric orthotropic second-order tensor-valued function, given in Table 5, column 3, is equivalent to the set shown in Table 3. This is, in turn, due to simple linear relationships

$$\begin{aligned} I &= M_{11} + M_{22} + M_{33}, \\ 2A &= M_{11}A + AM_{11} + M_{22}A + AM_{22} + M_{33}A + AM_{33} \,. \end{aligned}$$

The obtained representations of the considered orthotropic functions are clearly equivalent.

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STRESZCZENIE

NIEREDUKOWALNA LICZBA NIEZMIENNIKÓW I GENERATORÓW W RÓWNANIACH KONSTYTUTYWNYCH

Zmodyfikowano sposób Pipkina-Rivlina wyznaczenia generatorów wielomianowej reprezentacji symetrycznej izotropowej funkcji tensorowej drugiego rzędu. Skonstruowano tą metodą generatory anizotropowej i ortotropowej symetrycznej funkcji tensorowej drugiego rzędu, zależnej od skończonej liczby symetrycznych tensorów drugiego rzędu. Uzyskano identyczne wyniki jak w pracach Boehlera. Wyznaczono także nieredukowalne niezmienniki anizotropowych funkcji skalarnych, zależnych od jednego symetrycznego tensora drugiego rzędu. Rozpatrzono te rodzaje anizotropii, których grupa symetrii materialnej opisana jest wektorami i symetrycznymi tensorami drugiego rzędu. Wyprowadzone anizotropowe funkcje skalarne mogą być wykorzystane do budowy równań konstytutywnych nieliniowej sprężystości materiałów Greena oraz potencjałów i warunków plastyczności. Przykładowo pokazano sposób otrzymania tych równań dla materiału zbrojonego dwiema rodzinami ortogonalnych włókien.

Резюме

НЕПРИВАДИМОЕ ЧИСЛО ИНВАРИАНТОВ И ГЕНЕРАТОРОВ В ОПРЕДЕЛЯЮЩИХ УРАВНЕНИЯХ

Описывается модификация метода Пипкина-Ривлина нахождения генераторов полиномиального представления симметрической изотропной тензорной функции второго ранга. Метод используется для построения генераторов анизотропной и ортотропной симметрический тензорных функций второго ранга, от конечного числа симметричных тензорных переменных того же ранга. Полученные результаты точно совпадают с приведенными в работах Белера. Найдены также неприводимые инварианты анизотропных скаларных функций одного симметричного тензора второго ранга. Рассмотрены те виды анизотропии, для которых группа материальной симметри описывается векторами и симметричными тензорами второго ранга. Полученные анизотропные скалярные функции могут использоваться для построения определяющих уравнений нелинейной упругости материалов типа Грина, а также пластических потенциалов и условий теукучести. В качестве примера показан путь получения этих уравнений для материала, армированного двумя семействами ортогональных волокон.

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