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DISCRETE MODEL OF WAVE PROPAGATION IN A ROD WITH RIGID UNLOADING CHARACTERISTIC

Z. SZCZEŚNIAK (WARSZAWA)

A principle of construction and action of a discrete model of a rod with rigid characteristic of unloading is proposed. Its applications are shown for loading that varies arbitrarily with time. Propagating waves in this case cause multiple effects of rigid unloading and reloading in each cross-section of the rod. These effects are highly nonlinear and very difficult to describe analytically. Numerical algorithm is given. Its effectiveness is demonstrated on an example and solution errors are discussed.

1. Introduction

Problems of plastic wave propagation must be analysed with the use of properly formulated physical relationships in which the loading and unloading processes are distinguished. Experimental evidence has shown that the stress-strain unloading branches for soils and other bodies can be successfully approximated with rigid behaviour [1–3].

Deformation models of materials that become rigid on unloading are of considerable practical importance [4–10, 17, 18]. Many engineering problems have been found to have relatively simple solutions with the use of such a model. Good agreement with experiments have also been found [4, 6–8, 11].

However, the obtained solutions deal with particular, simplified loading patterns [4, 6, 7, 10, 11]. Solutions for arbitrary time-dependent applied loadings are fraught with serious difficulties. The reason is that the unloading process is generally described by a nonlinear differential equation with nonstationary boundary conditions, e.g. [17, 18]. Accounting for layered structure of a body makes the problem practically unsolvable.

In this paper a one-dimensional problem of wave propagation will be dealt with. Boundary loads can vary in an arbitrary manner and the unloading characteristic is assumed to be rigid. A discrete method leading to effective solutions will be presented. Its foundations as applied to linearly elastic loading and unloading behaviour were given in [12] and developed in [13]. The method consists in a discrete model whose performance is based on a finite difference approximation that generates no errors. Similar model is put forward in this paper to cover situations in which unloading can be treated as rigid. This type of approximation works well in the loading processes but cannot be applied to the rigid unloading processes, in which waves cease to be propagated.

The proposed discrete model, together with the model described in [12], forms a basis to construct complex discrete models of wave propagation when the stress-strain relationships are nonlinear. In this case these relationships ought to be approximated in a piece-wise linear manner.

2. General physical characteristics of the problem

Consider a semi-infinite rod with constant cross-section made of a material whose $\sigma - \varepsilon$ relationship is shown in Fig.1. The material behaves as linearly elastic on loading whereas its internal constraints make it impossible to deform on unloading – thus it stays rigid. Some soils have been found to behave in such a manner. Various models for soils can be found in the literature [3, 4, 14–20]. Some soils are treated as three-phase media, not infrequently strain-rate sensitive ones. The physical law adopted in this paper does not allow for viscous effects.

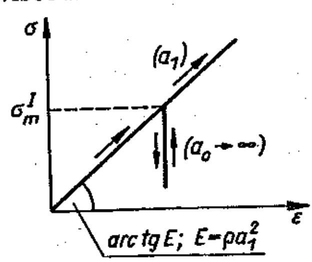

Fig. 1.

A free end of a rod is subjected to an arbitrary, piece-wise monotonic load $p(t)$, Fig.2a. Two regions are generated in the rod: a loading region I and a rigid unloading region II, Fig.2d. The regions are determined by two time-dependent interfaces $a - a$ and $c - c$.

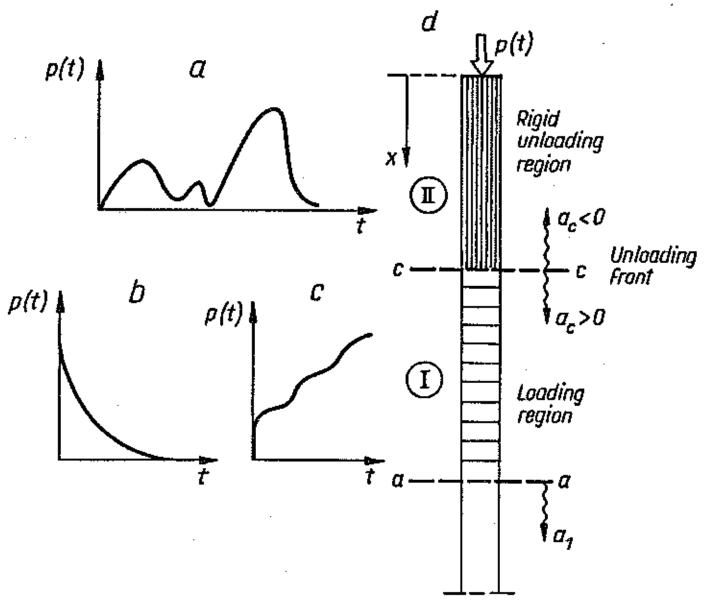

FIG. 2.

In the region I the wave propagation problem is described by the equation

$$\frac{\partial^2 u}{\partial t^2} - a_1^2 \frac{\partial^2 u}{\partial x^2} = 0, \tag{2.1}$$

where $u = u(x,t)$ denote displacement of section x and suitably formulated boundary conditions. Decreasing load $p(t)$ causes an unloading process in the region II. However, due to the assumed constraints no decrease in strains is possible and the waves begin to fade in the region II which becomes completely rigid and undergoes a rigid translation with the velocity v_{II}. The motion equation in the region II has the form

$$\rho \frac{\partial v_{\mathrm{II}}}{\partial t} = \frac{\partial \sigma}{\partial x}. \tag{2.2}$$

The interface $c-c$ moves with variable velocity a_c according to the dynamic states of neighbouring regions. The signs of velocity a_c characterize two different situations.

SITUATION 1. $a_c \geq 0$

Decreasing load $p(t)$ is accompanied by decreasing velocity v_{II} and a rigid region grows at the cost of shrinking loading region. The process depends on the current values of dynamic parameters that characterize the loading region.

SITUATION 2. $a_c \leq 0$

Increasing load $p(t)$ is accompanied by increasing velocity v_{II} and the rigid region contracts. The loading region expands in accordance with the velocities a_1 and a_c. This effect will be called an activation of the rigid region. A certain part of this region becomes activated depending on its state reached on prior loading. Basic parameters of this state are: maximum stress σ_m^{I}, Fig.1, and an associated maximum mass velocity v_m^{I}.

An arbitrary load, Fig.2a, can result in that the same region of the rod becomes activated and rigid again in a cyclic way. For a particular type of loading shown in Fig.2b it is only the rigid unloading region that is propagated along the rod and $a_c = a_1$. Load $p(t)$ shown in Fig.2c generates the loading region alone. From the above it follows that Eq.(2.2) contains two unknowns: the velocity v_{II} and the length of region II. The continuity condition at the interface $c-c$ is

$$v_c^{\mathrm{II}} = v_c^{\mathrm{I}} \qquad \text{or} \qquad \sigma_c^{\mathrm{II}} = \sigma_c^{\mathrm{I}}. \tag{2.3}$$

For a rigid process v_c^{I} and σ_c^{I} denote current parameters corresponding to the loading region. For an activation process the following equalities should be assumed: $v_c^{\mathrm{I}} = v_m^{\mathrm{I}}$, $\sigma_c^{\mathrm{I}} = \sigma_m^{\mathrm{I}}$.

Accounting for a suitable continuity condition leads to a nonlinear differential equation. It is only for very simple loadings $p(t)$ that closed form solutions can be arrived at.

3. DISCRETE MODEL OF A ROD WITH RIGID UNLOADING CHARACTERISTIC

A discrete model of a considered rod is similar to that described in [12]. It consists of a series of lumped masses Δm, Fig.3, whose motions are controlled by suitable weightless constraints, sensitive to the sign of a displacement increment (or rate). These ratchet-type constraints can be visualized as "pike's teeth". On loading the teeth deform elastically whereas on unloading they lock.

Performance of the model upon loading process results from the criterion of an exact difference approximation [12]. This means that spatial discretization Δx and temporal discretization Δt satisfy a relation $\Delta x = a_1 \Delta t$. Performance of the discrete model upon rigid unloading is different since the unloading part of the rod becomes an undeformable body. The length of

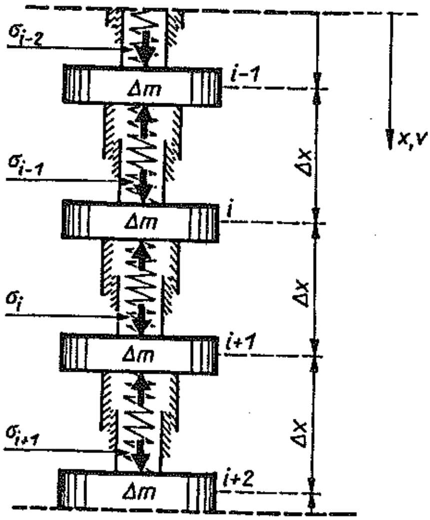

FIG. 3. 1) $\Delta v_{i,i+1} = v_i - v_{i+1}$; 2) $\Delta v_{i,i+1} \geq 0$ – loading; 3) $\Delta v_{i,i+1} < 0$ – does not apply – rigid unloading.

this part can increase or decrease, in accordance with the dynamic conditions of a problem in question. Thus a procedure is necessary to determine the length of rigid part and the transmitted stresses.

A discrete model for the case of growing rigid region is depicted in Fig.4.

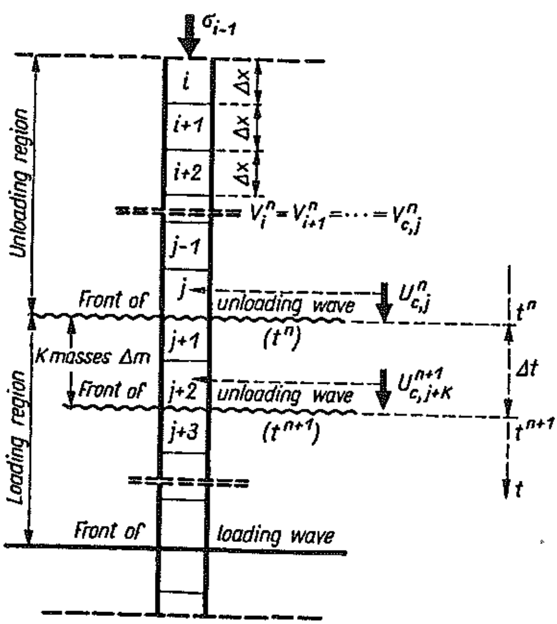

FIG. 4.

Dynamic parameters of a front of unloading region, located at the mass $\Delta m = \rho\Delta x$ numbered j, will be designated with a subscript c, j. These are: velocity of the rigid region front $a^n_{c,j}$, mass velocity $v^n_{c,j}$ (constant within the whole rigid region) and the stress $\sigma^n_{c,j}$; they constitute basic unknowns in the problem. Velocity of the unloading region front is variable and depends on the current dynamic parameters. Its variation and, at the same time, the length of rigid region are characterized by a parameter $K = 1,2,3\ldots$ denoting a distance $K\Delta x$ covered by the front over the time Δt. Parameter K will vary, from one time step to another, in an irregular manner as a result of variations in the mass velocity $v^n_{c,j}$. Velocity $v^{n+1}_{c,j}$ at an instant t^{n+1} can be calculated with the use of momentum conservation principle which can be expressed, for both rigid unloading and activation, as follows:

$$v^{n+1}_{c,j+K}[M + \Delta M(K)] = Mv^n_{c,j} + \Delta\Pi^n(K) + \Delta\sigma^n_x(K)\Delta t, \tag{3.1}$$

where $M = \Delta m(j - i + 1)$. The remaining terms in Eq.(3.1) assume the form:

For Situation 1 – rigid unloading ($a^n_{c,j} \geq 0$), the mass velocity $v^n_{c,j}$ decreases

$$\begin{aligned} \Delta M(K) &= K\Delta m, \\ \Delta\Pi^n(K) &= \sum_{s=j+1}^{s=j+K} \Delta m v^n_s, \\ \Delta\sigma^n_x(K) &= \sigma^n_{i-1} - \sigma^n_{j+K}. \end{aligned} \tag{3.2}$$

The stress σ^n_{j+K} and velocities v^n_s correspond to the current region of loading.

For Situation 2 – activation ($a^n_{c,j} \leq 0$), the mass velocity $v^n_{c,j}$ increases:

$$\begin{aligned} \Delta M(K) &= -K\Delta m, \\ \Delta\Pi^n(K) &= -K\Delta m v^n_{c,j}, \\ \Delta\sigma^n_x(K) &= \sigma^n_{i-1} - \sigma^n_{m,j-K}. \end{aligned} \tag{3.3}$$

The stress $\sigma^n_{m,j-K}$ attains the maximum value that was determined on the loading of the cross-section $j - K$.

It can be seen that, if $\Delta x \to 0$ and $\Delta t \to 0$, then $\Delta M(K) \to 0$, $\Delta\Pi^n(K) \to 0$, $\Delta\sigma^n_x(K) \to \dfrac{\partial\sigma}{\partial x}$, and Eq.(3.1) will be transformed to assume an appropriate differential form, see Eq.(2.2). This means that both the difference and the differential approaches to the problem coincide. The value of paraneter K in the formulae (3.1)–(3.3) should be found in an iterative manner. Starting with $K = 1$, subsequent locations of the rigid region front are analysed till the calculated stress $\sigma^{n+1}_{c,j+k}$ or $\sigma^{n+1}_{c,j-K}$ satisfies the following condition for the first time:

in the rigid region:

$$\sigma^{n+1}_{c,j+K} \geq \sigma^{\mathrm{I},n+1}_{j+K}, \tag{3.4}$$

in the activated region:

$$\sigma^{n+1}_{c,j-K} \geq \sigma^{\mathrm{II},n+1}_{m,j-K}. \tag{3.5}$$

The stress $\sigma^{n+1}_{c,j+K}$ on the rigid region face can be calculated from the following set of equations

$$\begin{aligned} \ddot{u}^{n,n+1}_{c,j+K} &= \frac{v^{n+1}_{c,j+K} - v^{n}_{j+K}}{\Delta t}, \\ \Delta u^{n,n+1}_{c,j+K} &= \Delta u^{n-1,n}_{c,j+K} + \ddot{u}^{n,n+1}_{c,j+K}\Delta t^2, \\ u^{n+1}_{c,j+K} &= u^{n}_{c,j+K} + \Delta u^{n,n+1}_{c,j+K}, \\ \sigma^{n+1}_{c,j+K} &= \frac{u^{n+1}_{c,j+K} - u^{n+1}_{j+K+1}}{\Delta x}E, \end{aligned} \tag{3.6}$$

where $\ddot{u}^{n,n+1}_{c,j+K}$, $\Delta u^{n,n+1}_{c,j+K}$, $u^{n+1}_{c,j+K}$ denote acceleration, displacement increment and displacement itself of a considered mass designated $j + K$. In the case of activation the subscripts $j + K$ should be replaced by $j - K$ and in Eq.$(3.6)_1$ v^{n}_{j+K} by $v^{n}_{c,j-K}$. It must also be remembered that u^{n+1}_{j-K+1} in Eq.$(3.6)_4$ applies to the activation process, hence maximum stresses σ^{I}_{m} should be used determined in the loading process.

Stresses $\sigma^{n+1}_{i} \div \sigma^{n+1}_{j+K-1}$, in the rigid region, shown in Fig.4, can be calculated from the formula

$$\sigma^{n+1}_{l} = \sigma^{n+1}_{i-1} - \Delta m \frac{v^{n+1}_{c,j+K} - v^{n}_{l}}{\Delta t}, \tag{3.7}$$

where $i \leq l \leq j + K - 1$.

In the loading region the described model performs according to the rules given in [12]. Similar discrete model can be constructed for a nonprismatic bar.

Both boundary and initial conditions are formulated as in [12]. General form of the boundary condition is written for a layered rod in this cross-section that separates its segments with different impedances $a_r\rho_r$ and

$a_{r+1}\rho_{r+1}$. The condition was shown in [12] to be identically satisfied provided the boundary mass Δm_b amounts to

$$\Delta m_b = 0.5\Delta t(a_r\rho_r + a_{r+1}\rho_{r+1})\,. \tag{3.8}$$

From the above it follows that a free edge of the rod should be modelled with the mass $\Delta m_{br} = 0.5\Delta t\, a_r\rho_r$ since $a_{r+1}\rho_{r+1} = 0$. Remaining interior masses will be $\Delta m_r = a_r\rho_r\Delta t$. The edge mass cannot be directly acted upon by a discrete form $P_\Delta(t^n)$ of the applied load $p(t)$. A specific form of the load $P_z(t^n)$ must be here employed, namely

$$P_z(t^n) = \begin{array}{ll} 0.5\,P_\Delta(t^0), & \text{for} \quad n=0, \\ P_\Delta(t^{n-1}) - 0.5\left[P_\Delta(t^{n-1}) - P_\Delta(t^n)\right], & \text{for} \quad n = 1,2,3,\ldots. \end{array} \tag{3.9}$$

When the transmission of edge load is considered with no effects of wave reflections from the other edge of the rod, the edge mass can be equal to Δm_r and acted upon directly by the load $P_\Delta(t^n)$.

4. Accuracy of the proposed model

The presented model performs in the loading process in the same manner as does the linearly elastic model described in [12]. In that paper a condition $\Delta x = a_1\Delta t$ was shown to ensure modelling of the loading process to within an accuracy of truncation errors. Such an accuracy is not, however, possible for the rigid unloading process. In general, the space-time course of unloading is nonlinear and that is why a location of the unloading wave front $l_{c,j+K} = (j+K)\Delta x$ does not necessarily have to agree with the exact solution. The above remark also applies to all other parameters referred to the front of the unloading wave. In this situation the truncation errors cannot be avoided. However, a numerical analysis of the problem shows that a finer discretization, with respect both to spatial and temporal coordinates, is an effective method to keep these errors as small as possible. Numerical algorithm of the method has turned out to be very effective. Thus a number of solutions for various discretizations can be readily obtained and suitable conclusions on their convergence can be drawn.

The errors will be assessed with the help of a problem whose exact solution is given in [6]. A semi-infinite rod made of a material with rigid unloading characteristic is considered. At an edge $x = 0$ a load is applied that increases linearly in time to suddenly disappear at an instant τ, Fig.5.

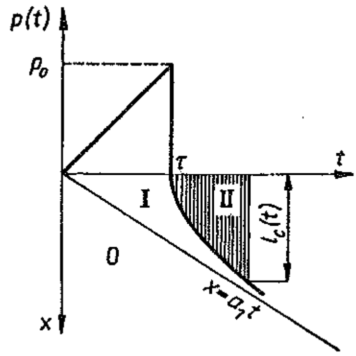

FIG. 5.

The unknowns of the problem are the changes in length of the rigid unloading region and the stress at its front. The following data are assumed: $p_0 = 0.095\,\text{MPa}$, $\tau = 0.125\,\text{s}$, $a_1 = 101.4\,\text{m/s}$, $\rho_1 = 1800\,\text{kg/m}^3$.

The exact solution of the problem [6] indicates that, for the assumed type of load $p(t)$, the front of rigidly unloaded region propagates with fast varying velocity, especially at the first stage beginning at $t = \tau$. From the viewpoint of high accuracy of numerical solution, this fact is very disadvantageous. The problem is now solved with the use of discrete model described in Sec.3.

In the solution for $0 \le t \le \tau$ (loading process) there will be the truncation errors only. For $t > \tau$ the error analysis will concern the length l_c of the unloaded region of the rod and the stress σ_c at its end. The errors are defined as follows:

1. Error in the length l_c

$$\delta^n(l_c) = \frac{l^n_{c,D} - l^n_{c,N}}{l^n_{c,D}} 100\,.$$

2. Error in the stress σ_c

$$\delta^n(\sigma_c) = \frac{\sigma^n_{c,D} - \sigma^n_{c,N}}{\sigma^n_{c,D}} 100\,,$$

where $l^n_{c,D}$, $\sigma^n_{c,D}$ denote values obtained from the exact solution for $t = n\,\Delta t$, $l^n_{c,N}$, $\sigma^n_{c,N}$ stand for the corresponding values taken from the numerical solution.

The obtained computational results for $\Delta x = 1.305\,\text{m}$ are depicted in Fig.6 from which it follows that the largest errors exist at the beginning of the unloading and refer to the length of unloaded part. However, the

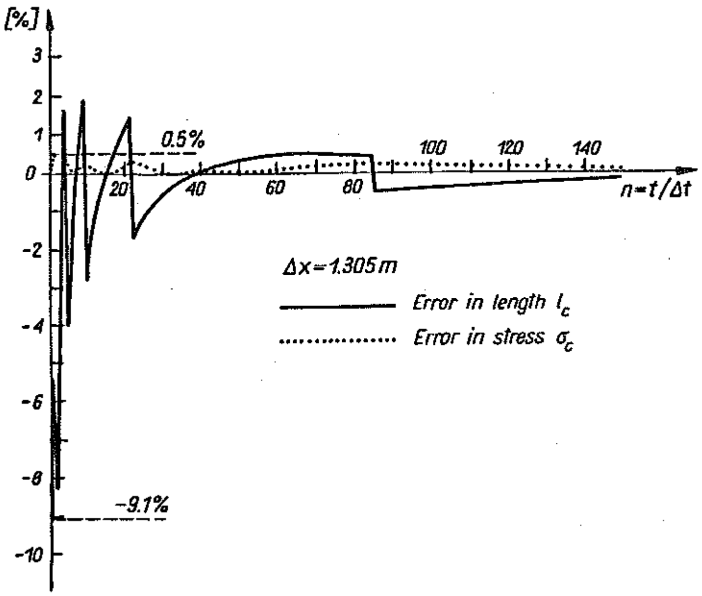

FIG. 6.

errors oscillate around zero and decrease very rapidly. For a large step $\Delta x = 1.305\,\mathrm{m}$ the greatest error is $-9.1\,\%$ and after 26 time steps Δt is contained within $1\,\%$. For $t = 150\Delta t$ the error is as small as $-0.17\,\%$. Errors in stresses are found not to exceed $0.5\,\%$. All the above errors can be diminished by decreasing a spatial step Δx or a corresponding time step Δt. For instance, $\Delta x_1 = 0.1\,\Delta x$ results in that an initial error $\delta^{n=1}(l_c)$ is reduced to $-1.25\,\%$.

It is worth noting that, due to the absence of wave propagation in the unloaded region, the errors are of a local character only.

Selection of suitable steps Δx or Δt, leading to a desired level of errors, does depend on the specific conditions and individual requirements of a problem in question.

It is often encountered in the engineering practice that the load $p(t)$ increases very abruptly at the very beginning and continually decreases according to a certain function, Fig.2b. Let a free edge of a semi-infinite rod be subjected to a load $p(t) = p_0(1 - t/\tau_1)$, $\tau_1 = 0.5\,\mathrm{s}$. Let material constants be $a_1 = 104.4\,\mathrm{m/s}$, $\rho_1 = 1800\,\mathrm{kg/m^3}$. It is only the rigid region that is present in the considered rod since the loading region shrinks to an edge cross-section and becomes, at the same time, a front of the rigid region. Velocity of propagation of this front is equal to a_1 which means that the length of rigid part is calculated with no error at all. Certain errors are only involved in the stresses σ_c^n. Table 1 shows that these errors are negligible.

Table 1.

$i = \dfrac{x}{\Delta x}$	$\sigma_c^n(x)/p_0$; $\Delta x = 1.305$ m	
	exact solution	numerical solution
1	0.987500	0.987500
2	0.975001	0.975001
3	0.962501	0.962501
⋮	⋮	⋮
38	0.525014	0.525014
39	0.512514	0.512515
40	0.500014	0.500015

5. Numerical example

Efficacy of the proposed method will now be demonstrated. Consider a semi-infinite rod made of a material whose stress-strain relationships is shown in Fig.1 and whose constants are $a_1 = 150\,\text{m/s}$, $\rho_1 = 1800\,\text{kg/m}^3$. A free edge of the rod is assumed to be subjected to the following time-dependent load $p(t)$, Fig.7a:

$$(5.1)\qquad p(t) = \begin{cases} p_1(2\xi - \xi^2), \quad \xi = \dfrac{t}{\tau_1}, & \text{for} \quad 0 \le t \le \tau_1, \\ p_1\left(\dfrac{\tau_2 - t}{\tau_2 - \tau_1}\right)^3, & \text{for} \quad \tau_1 \le t \le \tau_2, \\ p_2\left(\dfrac{t - \tau_2}{\tau_3 - \tau_2}\right)^2, & \text{for} \quad \tau_2 \le t \le \tau_3, \\ 0.0, & \text{for} \quad \tau_3 \le t \le \tau_4, \\ p_3\left(\dfrac{t - \tau_4}{\tau_5 - \tau_4}\right), & \text{for} \quad \tau_4 \le t \le \tau_5, \\ 0.0, & \text{for} \quad t \ge \tau_5. \end{cases}$$

The following data are to be substituted into Eqs.(5.1): $p_1 = 0.2\,\text{MPa}$, $p_2 = 0.3\,\text{MPa}$, $p_4 = 0.4\,\text{MPa}$, $\tau_1 = 4\,\text{ms}$, $\tau_2 = 6.7\,\text{ms}$, $\tau_3 = 14.6\,\text{ms}$, $\tau_4 = 18\,\text{ms}$, $\tau_5 = 21.3\,\text{ms}$.

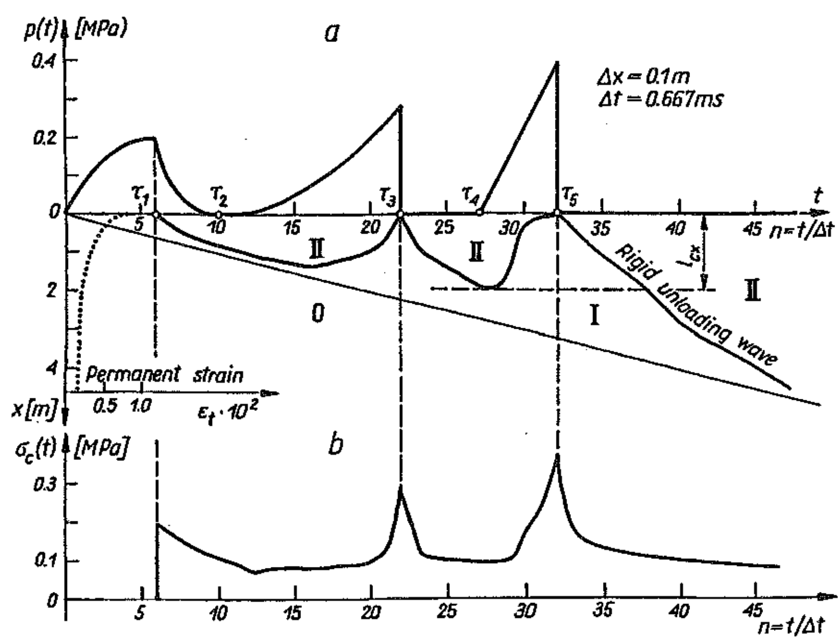

FIG. 7.

Let us analyse the deformation process in the rod by means of the discrete model described in Sec. 3. The steps along the rod are assumed $\Delta x = 0.1$ m. To obey the condition for an exact difference approximation in the loading process, they correspond to the time step $\Delta t = 0.6667$ ms. The solution is presented in the phase plane, Fig.7a. Characteristic regions are designated as follows: O – a region with no load, I – loading region, II – rigid unloading region. Regions I and II are separated by an interface which is here termed a rigid unloading wave and describes changes in the location of the front of rigid region. A diagram of rigid unloading wave is seen to be in accordance with the program of variable load.

It is not a single occurrence of rigid unloading of a certain part of the rod that is observed. Due to a complex time-dependence of $p(t)$ a certain length l_{cx} becomes rigid and activated in an alternating manner. No computational difficulties arise in the applied method of solution.

Permanent strains in the rod are shown with dotted line in Fig.7a.

Time-dependent stress $\sigma_c(t)$ at the front of the rigid unloading region is depicted in Fig.7b. The stress, together with the mass velocity of the growing rigid region, are both seen to decrease. The stress $\sigma_c(t)$ and the mass velocity are found to increase as the rigid region contracts.

Ten times shorter step Δx is not found to appreciably alter the solution.

6. Conclusions

In [12] a discrete model was proposed for one-dimensional wave propagation in the linearly elastic material insensitive to whether loading or unloading process takes place.

In this paper a similar model is put forward to reflect the unloading process which proceeds in an ideally rigid manner, i.e. with no recovery of strains. As shown in Sec.5, effective solutions to complex problems can be arrived at.

Models described in [12] and here enable such problems to be solved in which the stress-strain relationship is unilinear for both loading and unloading. They can be thus termed the basic elastic and plastic models, respectively. Automatic calculations enable to avoid the difficulties typical for analytical solutions so the models can be successfully employed in complex situations. It is also possible to combine the models in such a way that the wave propagation problems could be analyses for piece-wise multilinear approximations of the stress-strain relationships on both loading and unloading. This is particularly important for such media as soils subjected to arbitrary dynamic loading programs. Plane, cylindrical and spherical situations can be dealt with in which waves propagate and reflect. The proposed model seems also suitable for the discretization of the "plastic gas" problem, as described in [19]. It was also used in [20] to analyse the propagation of waves in soils under very high pressures. Another field of application can be found in the soil-structure interaction problems in which cavitation and layered structure of subgrade could be allowed for.

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MILITARY TECHNICAL ACADEMY, WARSZAWA.

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