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PARAMETRIC OPTIMIZATION OF A ROD SUBJECT TO FORCED TORSIONAL VIBRATION

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Forced torsional vibration of a rod made of a Kelvin-Voigt viscoelastic material is analyzed. The rod has the form of a truncated cone. One end of the rod is loaded by a harmonically variable torque, the other end is rigidly fixed. Vibrational amplitude of the cross-section subject to external excitation is the objective function; its minimum determines the optimum shape of the rod. The results derived are based on the solution consisting of the first term of its expansion due to the Galerkin method; the results are illustrated by graphs.

1. Introduction

The problems of optimization of the shape of elastic rods performing free torsional vibrations were analyzed by M.H.S. Elvany and A.D.S. Barr [1, 2]. Similar problems concerning torsional vibration of elastic rods subject to harmonic excitation were presented in [3]. The present paper deals with the problem of optimization of a visco-elastic rod performing torsional vibration under the action of harmonically time-dependent torque applied to one end of the rod, the other end being rigidly fixed. An analogous boundary-value problem for a prismatic rod was solved in [4]. The rod to be considered in the present paper has the form of a truncated cone. The optimality condition consists in the requirement that the vibration amplitude of the end of the rod subject to excitation should reach a minimum; rod's resistance to forced vibration reach the maximum. The analysis of the results derived (limited to the first term of the corresponding series expansions) yields the conclusion that (among all rods of the considered form) the optimum result is achieved when the truncated cone is fixed at the larger basis. The results are illustrated by graphs.

2. Formulation of the problem

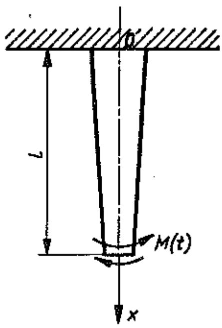

FIG. 1.

A right conical rod of circular cross-section (Fig.1) made of the Kelvin-Voigt-type visco-elastic material is fixed (built-in) at the end $x = 0$. The other end $x = l$ of the rod is acted on by torque, $M(t) = M_0 \sin \omega t$, where M_0 and ω are positive constants. Vector of the moment of force $\mathbf{M}(t)$ has the direction of rod's axis which coincides with the x-axis. Length l of the rod and its volume V are fixed. Radius $R(x)$ of the cross-section is given by the formula

$$R(x) = R_0(1 + \varepsilon \frac{x}{l}) \equiv R_0 \Phi(x)\,, \tag{2.1}$$

with parameters $R_0 > 0$ and $\varepsilon \in (-1, \infty)$ determining the shape of the rod; for a prismatic rod $\varepsilon = 0$. Parameters R_0 and ε satisfy the relation (cf. [5])

$$R_0 = \sqrt{\frac{V}{\pi l (1 + \varepsilon + \varepsilon^2/3)}}\,. \tag{2.2}$$

Thus, only one of them is independent. In what follows, ε will be treated as the optimization parameter.

The equation of torsional vibration of the rod is written in the form [6]

$$\Phi(x)\left(G\frac{\partial^2 \varphi}{\partial x^2} + \eta \frac{\partial^3 \varphi}{\partial x^2 \partial t}\right) + \frac{4\varepsilon}{l}\left(G\frac{\partial \varphi}{\partial x} + \eta \frac{\partial^2 \varphi}{\partial x \partial t}\right) = \rho \Phi(x) \frac{\partial^2 \varphi}{\partial t^2}\,, \tag{2.3}$$

where $\varphi(x, t)$ is the angle of twist of the cross-section x at time t, G is the modulus of rigidity, η is the internal damping coefficient and ρ – density of the material.

The boundary conditions assumed have the form

$$\varphi(0,t) = 0\,, \tag{2.4}$$

$$I_0(l)\left(G\frac{\partial\varphi}{\partial x} + \eta\frac{\partial^2\varphi}{\partial x\partial t}\right)(l,t) = M_0\sin\omega t\,, \tag{2.5}$$

$I_0(x)$ denoting the cross-sectional moment of rotary inertia; $I_0(x) = \frac{1}{2}\pi R_0^4 \times\Phi^4(x)$. The initial conditions are not formulated here since the problem considered is stationary.

The problem consists now in determining the solution of Eq.(2.3) with the boundary conditions (2.4), (2,5), i.e. the solution describing stationary vibration. Then the torsional vibration amplitude of the cross-section subject to excitation will be determined. Under fixed material constants this amplitude — the objective function-depends on the optimality parameter ε. The value of ε at which the objective function reaches a minimum determines the optimal shape of the rod.

3. Solution of the problem

Following the procedure outlined in [4], introduce the notation

$$F(t) = \frac{\partial\varphi}{\partial x}(l,t)\,, \tag{3.1}$$

what makes it possible to write the boundary condition (2.5) in the form

$$I_0(l)\left[G\,F(t) + \eta\dot{F}(t)\right] = M_0\sin\omega t\,, \tag{3.2}$$

hence

$$F(t) = \frac{M_0}{I_0(l)\,(G^2+\eta^2\omega^2)}(G\sin\omega t - \eta\omega\cos\omega t) \equiv a\sin\omega t + b\cos\omega t\,, \tag{3.3}$$

where the term which decreases exponentially with time has been disregarded, the considerations being limited to stationary vibration.

Solution of the boundary-value problem (2.3)-(2.5) is sought for in the form ($\varepsilon \neq 0$)

$$\varphi(x,t) = \varphi_1(x,t) + \frac{(1+\varepsilon)^4 l}{3\varepsilon}\left[1-\Phi^{-3}(x)\right]F(t)\,, \tag{3.4}$$

$\varphi_1(x,t)$ being the unknown function which, due to Eq.(2.3), satisfies the equation

$$\Phi(x)\left(G\frac{\partial^2\varphi_1}{\partial x^2}+\eta\frac{\partial^3\varphi_1}{\partial x^2\partial t}\right)+\frac{4\varepsilon}{l}\left(G\frac{\partial\varphi_1}{\partial x}+\eta\frac{\partial^2\varphi_1}{\partial x\partial t}\right) = \rho\Phi(x)\frac{\partial^2\varphi_1}{\partial t^2}+\rho\frac{(1+\varepsilon)^4 l}{3\varepsilon}\Big[\Phi(x)-\Phi^{-2}(x)\Big]\ddot{F}(t) \tag{3.5}$$

and, in view of Eqs. (2.4), (2.5), the boundary conditions

$$\varphi_1(0,t)=0\,,\qquad \frac{\partial\varphi_1}{\partial x}(l,t)=0\,. \tag{3.6}$$

The solution of the case $\varepsilon=0$ is given in the paper [4].

The boundary-value problem (3.5), (3.6) will be solved by means of the Galerkin method. Assume that function $\varphi_1(x,t)$ may be expanded into a series of eigenfunctions occurring in the solution of an analogous problem concerning prismatic rod [4],

$$\varphi_1(x,t)=\sum_{m=1}^{N}q_m(t)\sin\lambda_m x\ , \tag{3.7}$$

where $\lambda_m=\dfrac{(2m-1)\pi}{2l}$, $m=1,2,\dots,N$, and $q_m(t)$ are the functions of time to be determined. Substitution of (3.7) into (3.5) yields

$$\sum_{m=1}^{N}(\rho\ddot{q}_m+\eta\lambda_m^2\dot{q}_m+G\lambda_m^2q_m)\Phi(x)\sin\lambda_m x - \frac{4\varepsilon}{l}\sum_{m=1}^{N}(\eta\dot{q}_m+Gq_m)\lambda_m\cos\lambda_m x = -\frac{\rho(1+\varepsilon)^4 l}{3\varepsilon}\Big[\Phi(x)-\Phi^{-2}(x)\Big]\ddot{F}(t)\,. \tag{3.8}$$

This equation is now multiplied by $\sin\lambda_n x$ and integrated with respect to x between the limits $0,l$, what leads to the set of ordinary differential equations

$$\ddot{q}_n+\frac{\eta}{\rho}\lambda_n^2\dot{q}_n+\frac{G}{\rho}\lambda_n^2q_n+\frac{2\varepsilon}{\rho l^2}\sum_{m=1}^{N}I_{mn}(G\lambda_m^2q_m+\eta\lambda_m^2\dot{q}_m+\rho\ddot{q}_m) - \frac{8\varepsilon}{\rho l^2}\sum_{m=1}^{N}J_{mn}\lambda_m(Gq_m+\eta\dot{q}_m) = -\frac{2(1+\varepsilon)^4}{3\varepsilon}M_n\ddot{F}(t), \qquad n=1,2,\dots,N, \tag{3.9}$$

with the notations

$$(3.10)\qquad \begin{aligned} I_{mn} &= \int_0^l x \sin \lambda_m x \sin \lambda_n x \, dx \,, \\ J_{mn} &= \int_0^l \sin \lambda_n x \cos \lambda_m x \, dx \,, \\ M_n &= \int_0^l \Big[\Phi(x) - \Phi^{-2}(x)\Big] \sin \lambda_n x \, dx \,. \end{aligned}$$

The results known from the analysis of prismatic rod indicate that the first term of expansion (3.7) is of principal importance, since the succeeding terms are proportional to $(2m-1)^{-4}$ (cf. Eqs.(4.18) in paper [4]). It is assumed that in the present case the situation is similar.

The solution of the set of differential Eqs.(3.9) will be presented in the first approximation under the assumption that only $q_1(t) \neq 0$. Retaining the first term of expansion (3.7) and using Eq.(3.9), we obtain

$$(3.11)\qquad \ddot{q}_1 + \frac{\eta}{\rho}\lambda_1^2\dot{q}_1 + \frac{G}{\rho}\lambda_1^2 q_1 + \frac{2\varepsilon}{l^2} I_{11}\left(\frac{G}{\rho}\lambda_1^2 q_1 + \frac{\eta}{\rho}\lambda_1^2 \dot{q}_1 + \ddot{q}_1\right) - \frac{8\varepsilon}{\rho l^2} J_{11}\lambda_1 (G q_1 + \eta \dot{q}_1) = -\frac{2(1+\varepsilon)^4}{3\varepsilon} M_1 \ddot{F}(t).$$

Following the assumption of stationary forced vibration, the solution of the above equation is written in the form

$$(3.12)\qquad q_1(t) = c \sin \omega t + d \cos \omega t \,.$$

Further calculations yield the constants c and d

$$(3.13)\qquad c = \frac{W_c}{W}\,, \qquad d = \frac{W_d}{W}\,,$$

where the following notations have been introduced:

$$(3.14)\qquad \begin{aligned} W = {} & \left(\frac{G}{\rho}\lambda_1^2 - \omega^2\right)^2 + \frac{\eta^2\omega^2}{\rho^2}\lambda_1^4 \\ & + \frac{4\varepsilon}{l^2}\left\{\left(\frac{G}{\rho}\lambda_1^2 - \omega^2\right)\left[I_{11}\left(\frac{G}{\rho}\lambda_1^2 - \omega^2\right) - \frac{4G}{\rho}\lambda_1 J_{11}\right] + \frac{\eta^2\omega^2}{\rho^2}\lambda_1^2\left(\lambda_1^2 I_{11} - 4\lambda_1 J_{11}\right)\right\} \\ & + \frac{4\varepsilon^2}{l^4}\left\{\left[I_{11}\left(\frac{G}{\rho}\lambda_1^2 - \omega^2\right) - \frac{4G}{\rho}\lambda_1 J_{11}\right]^2 + \frac{\eta^2\omega^2}{\rho^2}\left(\lambda_1^2 I_{11} - 4\lambda_1 J_{11}\right)^2\right\}, \end{aligned}$$

(3.14) [cont.]
$$W_c = \frac{4\pi l^2 M_0 \omega^2 M_1}{3\varepsilon V^2(G^2+\eta^2\omega^2)}\left(1+\varepsilon+\varepsilon^2/3\right)^2\left\{G\left(\frac{G}{\rho}\lambda_1^2-\omega^2\right) - \frac{\eta^2\omega^2}{\rho}\lambda_1^2 + \frac{2\varepsilon}{l^2}\left[G\left(I_{11}\left(\frac{G}{\rho}\lambda_1^2-\omega^2\right)-\frac{4G}{\rho}\lambda_1 J_{11}\right) - \frac{\eta^2\omega^2}{\rho}(\lambda_1^2 I_{11} - 4\lambda_1 J_{11})\right]\right\},$$

$$W_d = \frac{4\pi l^2 M_0 \omega^3 M_1 \eta}{3\varepsilon V^2(G^2+\eta^2\omega^2)}(1+\varepsilon+\varepsilon^2/3)^2\left[\omega^2-\frac{2G}{\rho}\lambda_1^2 - \frac{2\varepsilon}{l^2}\left(\frac{2G}{\rho}\lambda_1^2 I_{11} - I_{11}\omega^2 - \frac{8G}{\rho}\lambda_1 J_{11}\right)\right].$$

The first approximation of the solution describing the stationary vibration of the rod considered has, in view of Eqs.(3.4), (3.7), the form

$$\varphi(x,t) = \left\{c\sin\frac{\pi x}{2l} + \frac{a(1+\varepsilon)^4 l}{3\varepsilon}\left[1-\Phi^{-3}(x)\right]\right\}\sin\omega t + \left\{d\sin\frac{\pi x}{2l} + \frac{b(1+\varepsilon)^4 l}{3\varepsilon}\left[1-\Phi^{-3}(x)\right]\right\}\cos\omega t \equiv A(x)\sin[\omega t+\delta(x)], \tag{3.15}$$

where the vibration amplitude is

$$A(x) = \left\{\left[c\sin\frac{\pi x}{2l} + \frac{a(1+\varepsilon)^4 l}{3\varepsilon}\left(1-\Phi^{-3}(x)\right)\right]^2 + \left[d\sin\frac{\pi x}{2l} + \frac{b(1+\varepsilon)^4 l}{3\varepsilon}\left(1-\Phi^{-3}(x)\right)\right]^2\right\}^{\frac{1}{2}}, \tag{3.16}$$

and tangent of the phase shift angle equals

$$\mathrm{tg}\delta(x) = \frac{3\varepsilon d\sin\frac{\pi x}{2l} + b(1+\varepsilon)^4 l[1-\Phi^{-3}(x)]}{3\varepsilon c\sin\frac{\pi x}{2l} + a(1+\varepsilon)^4 l[1-\Phi^{-3}(x)]}. \tag{3.17}$$

Both the vibration amplitude and the phase shift angle depend substantially on the value of parameter ε determining the shape of the rod.

4. Parametric optimization

By assuming $x = l$ in the formula (3.16) we obtain the amplitude of forced vibration of the lower end of the rod, at which the excitation is applied,

$$A(l) = \left\{ \left[c + a(1+\varepsilon)\left(1 + \varepsilon + \frac{\varepsilon^2}{3}\right) l \right]^2 + \left[d + b(1+\varepsilon)\left(1 + \varepsilon + \frac{\varepsilon^2}{3}\right) l \right]^2 \right\}^{\frac{1}{2}} . \tag{4.1}$$

This formula is now used to perform the parametric optimization of the shape of the rod in consideration of minimum of the amplitude.

In order to perform the numerical analysis of the solution, let us define the dimensionless parameters (cf. [4])

$$\nu = \frac{2l}{\pi}\sqrt{\frac{\rho}{G}}\,\omega\,, \qquad \beta_1 = \frac{2l}{\pi\eta}\sqrt{\rho G}\,. \tag{4.2}$$

The first one plays the role of a dimensionless frequency. The analysis is performed for four different values of β_1 : 3, 5, 10 and 15, and under a fixed value of the optimization parameter $\varepsilon \in \langle -0.75, 0.75\rangle \backslash \{0\}$. In each of the cases considered, the quantity

$$\tilde{A} = \frac{\pi G V^2}{16 M_0 l} A(l) \tag{4.3}$$

proportional to $A(l)$ is evaluated as a function of the dimensionless frequency ν. Thus the maximal value of the amplitude occurring in the resonance is determined. In the case $\varepsilon = 0$ Eq.(4.21) of paper [4] is used.

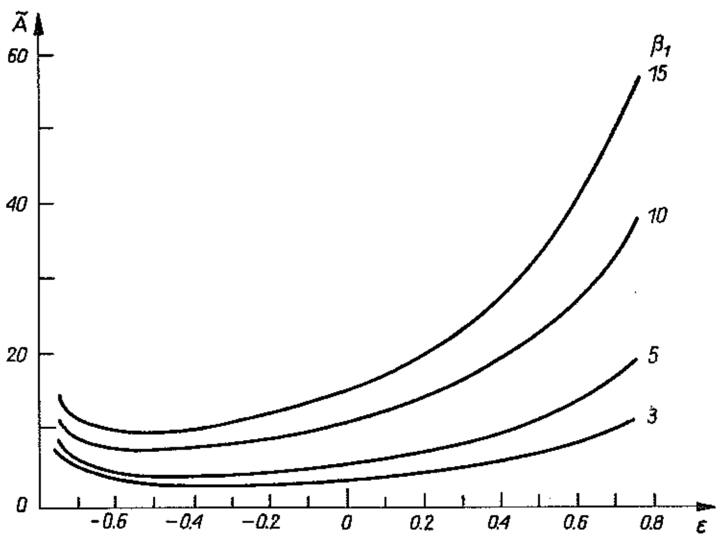

FIG. 2.

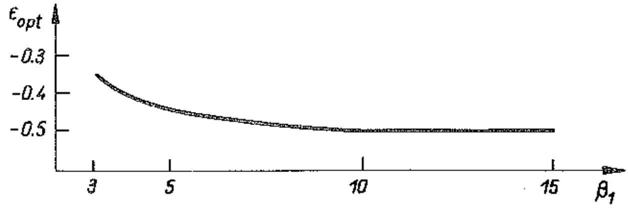

FIG. 3.

The maximum value of $\tilde{A}$ as a function of the optimization parameter ε is presented in Fig.2. Positions of minima of those functions depend on the value of β_1 and they determine the optimum value of the optimization parameter ε_{opt}, i.e. the optimal shape of the rod. For $\beta_1 = 3$, $\varepsilon_{\text{opt}} = -0.35$, for $\beta_1 = 5$, $\varepsilon_{\text{opt}} = -0.45$, for $\beta_1 = 10$ and $\beta_1 = 15$, $\varepsilon_{\text{opt}} = -0.5$. The results are shown in Fig.3. Moreover, Fig.4 represents the dependig of $\tilde{A}$ on the dimensionless frequency ν for the cases of four optimal shapes of the rod.

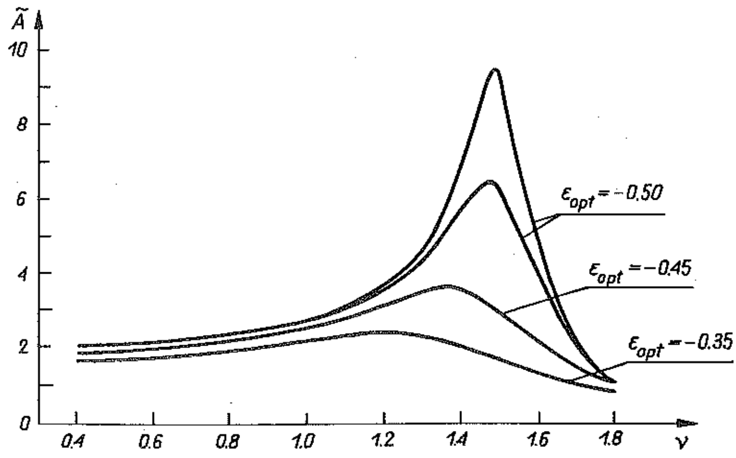

FIG. 4.

5. Conclusion

The results derived in the paper demonstrate that the optimal shape of the rod depends on the parameter β_1, i.e. on the material constants of the rod. For the rod made of a Kelvin-Voigt material of small viscosity coefficient, i.e. when the elastic properties of the material play the dominant role, $\varepsilon_{opt} = -0.5$. For the rod with great viscosity coefficient ε_{opt} increases to reach -0.35 for $\beta_1 = 3$. In both cases the truncated cone is built-in at the larger base, the smaller one being subject to excitation.

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