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ON THE SOLUTION OF A CONCENTRATED TORQUE PROBLEM FOR A NON–HOMOGENEOUS ORTHOTROPIC HALF-SPACE

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The exact expression for displacement and stresses in a non-homogeneous orthotropic half-space subjected to a concentrated torque acting in the interior or on a surface are obtained by means of Hankel transforms.

1. Introduction

For most applications in geomechanics the shear modulus of natural soil deposits varies continuously with depth according to the geologic and loading history of the soil deposit (WROTH *et al.* [1]). In addition experimental investigation of natural soil deposits confirms the presence of anisotropy (DAHAN *et al.* [2]). The incorporation of both non-homogenity and anisotropy into the load transfer analysis would enhance the practicality of the solution and its usefulness in engineering practice. In general, the studies involving non-homogeneous and anisotropic elastic media are rather limited. Some boundary-value problems involving surface loading of a non-homogeneous half-space have been considered by KASSIR [3]; CHUAPRASERT and KASSIR [4]; ERGÜVEN [5] and SELVADURAI, SINGH and VRBIK [6]. In the present study the exact solution, within the assumption of classical elasticity theory, for axisymmetric torsional displacement and stresses in a non-homogeneous orthotropic half-space subjected to concentrated internal torque is obtained through the application of Hankel integral transform techniques.

2. Basic equations

Consider an orthotropic non-homogeneous half-space, with a cylindrical polar coordinate system (r, θ, z) chosen such that the z-axis is normal to the surface. The stress-displacement relations for axisymmetrical torsion problem are

$$\sigma_{r\theta} = G_r(z)\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right), \qquad \sigma_{\theta z} = G_z(z)\frac{\partial v}{\partial z}, \tag{2.1}$$

where $\sigma_{r\theta}$ and $\sigma_{\theta z}$ are the stress components, v is the displacement and $G_r(z), G_z(z)$ are the shear moduli which vary continuously with depth according to the following functional forms:

$$G_z(z) = G_0 e^{\beta z}, \qquad G_r = s^2 G_z(z) \tag{2.2}$$

or

$$G_z(z) = G_0(1 + mz)^\alpha, \qquad G_r(z) = s^2 G_z(z), \qquad m > 0. \tag{2.2'}$$

In Eqs.(2.2) $\alpha = \beta = 0$ represents a homogeneous orthotropic solid; $\alpha > 0$ or $\beta > 0$ represents a situation where shear moduli increase nonlinearly with depth; $\alpha < 0$ and $\beta < 0$ represents a situation where shear moduli decrease with depth and $\alpha = 1$ corresponds to an elastic medium with linearly increasing shear moduli. The constant s is a measure of orthotropy and $s = 1$ represents an isotropic solid. It is evident that Eqs. (2.2) could represent a variety of practical situations for non-homogenity. For most applications in geomechanics the shear moduli of natural soil deposits increase with depth due to the increase in effective overburden pressure and degree of consolidation. In addition, the homogeneous condition could be justified in the radial direction. Under these assumptions shear moduli G_z and G_r are functions of the z-coordinate only. It is assumed above that the ratio of the shear moduli G_z and G_r is independent of z. With these assumptions the problem is solved by means of integral transform and exact analytical solution or approximate solution, respectively, are obtained.

Substituting Eq.(2.1) into equation of equilibrium

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{r\theta} = 0 \tag{2.3}$$

and considering the appropriate variation of shear modulus as given by Eq.(2.2) or (2.2′), one obtains

$$s^2\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) + \beta\frac{\partial v}{\partial z} + \frac{\partial^2 v}{\partial z^2} = 0, \tag{2.4}$$

$$s^2\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) + \frac{\alpha m}{1 + mz}\frac{\partial v}{\partial z} + \frac{\partial^2 v}{\partial z^2} = 0 \tag{2.5}$$

for exponential and power variations of the shear moduli, respectively.

The boundary and continuity (or discontinuity) conditions are:

$$\sigma_{z\theta}(r,0) = 0, \tag{2.6}$$

$$\left[\!\left| v(r,z') \right|\!\right] = 0, \tag{2.7}$$

$$\left[\!\left| \sigma_{z\theta}(r,z') \right|\!\right] = -\lim_{\alpha\to 0}\left[T\delta(r-a)/2\Pi r^2\right], \tag{2.8}$$

where it is assumed that the twisting moment T acts in the interior of a half-space on the plane $z = z'$, $\delta(r-a)$ is Dirac's delta function and the symbol $[\![\;]\!]$ denotes the jump of the function, defined as follows

$$\left[\!\left| f(z) \right|\!\right]\Big|_{z=z'} = \lim_{\Delta\to 0}[f(z'+\Delta) - f(z'-\Delta)]. \tag{2.9}$$

The solution of Eqs.(2.4) and (2.5) with the conditions (2.6) to (2.8) could be obtained through the application of Hankel integral transforms, with respect to r, defined as

$$v^{\nu}(\xi) = \int_0^{+\infty} v(r)J_{\nu}(r\xi)r\,dr, \tag{2.10}$$

where J_ν is the Bessel function of the first kind and order ν.

The transform of a gradient, with respect to r, is given by

$$\int_0^{\infty}\frac{\partial v(r)}{\partial r}J_1(r\xi)r\,dr = -\xi v^0(\xi). \tag{2.11}$$

The form of equations indicates that the first-order Hankel transform is the proper one to use on both v and $\sigma_{\theta z}$.

Applying the Hankel transforms, Eqs.(2.4) to (2.8) become

$$\frac{\partial^2 v^1}{\partial z^2} + \beta\frac{\partial v^1}{\partial z} - \xi^2 s^2 v^1 = 0, \tag{2.12}$$

$$\frac{\partial^2 v^1}{\partial z^2} + \frac{\alpha m}{1+mz}\frac{\partial v^1}{\partial z} - \xi^2 s^2 v^1 = 0, \tag{2.13}$$

$$\frac{\partial v^1}{\partial z}\Big|_{z=0} = 0, \tag{2.14}$$

$$\left[\!\left| v^1 \right|\!\right]\Big|_{z=z'} = 0, \tag{2.15}$$

$$G_z\left[\!\left|\frac{\partial v^1}{\partial z}\right|\!\right]\Big|_{z=z'} = -\frac{T}{4\Pi}\xi. \tag{2.16}$$

Applying the foregoing transforms to Eqs.(2.1), we obtain

$$\sigma^2_{r\theta} = -G_r(z)v^1\xi, \qquad \sigma^1_{\theta z} = G_z(z)\frac{\partial v^1}{\partial z}, \tag{2.17}$$

where the superscripts 2 and 1 denote the second- and first-order Hankel transforms of $\sigma_{r\theta}$ and $\sigma_{\theta z}$, respectively.

3. Solution

The solutions of Eqs.(2.12) and (2.13) are

$$\begin{aligned} v^1(\xi,z) &= e^{-\beta z/2}[A(\xi)e^{-zc} + B(\xi)e^{zc}], \quad 0 \le z \le z', \\ v^1(\xi,z) &= e^{-\beta z/2}C(\xi)e^{-zc}, \quad z \ge z' \end{aligned} \tag{3.1}$$

or

$$\begin{aligned} v^1(\xi,z) &= (m^{-1}+z)^p D(\xi)K_p[(m^{-1}+z)\xi s] + E(\xi)I_p[(m^{-1}+z)\xi s], \\ &\qquad 0 \le z \le z', \\ v^1(\xi,z) &= (m^{-1}+z)^p F(\xi)K_p[(m^{-1}+z)\xi s], \quad z \ge z', \end{aligned} \tag{3.2}$$

where

$$c = [(\beta/2)^2 + \xi^2 s^2]^{1/2}, \tag{3.3}$$

$$2p = 1-\alpha \tag{3.4}$$

and I_p and K_p denote the modified Bessel functions of the first and second kind of order p, respectively. The functions $A(\xi)$, $B(\xi)$, $C(\xi)$, $D(\xi)$, $E(\xi)$ and $F(\xi)$ are obtained from the boundary conditions (2.14) to (2.16) and assume the forms:

$$\begin{aligned} B(\xi) &= \frac{Te^{z'\beta/2}}{8\varPi G_z(z')}\frac{\xi e^{-z'c}}{c}, \\ A(\xi) &= B(\xi)\frac{c-(\beta/2)}{c+(\beta/2)}, \\ C(\xi) &= A(\xi) + B(\xi)e^{2z'c}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} E(\xi) &= \frac{T(m^{-1}+z')}{4\varPi G_z(z')(m^{-1}+z')^p}\xi K_p[(m^{-1}+z')\xi s], \\ D(\xi) &= E(\xi)\frac{I_{p-1}(\xi s/m)}{K_{p-1}(\xi s/m)}, \\ F(\xi) &= D(\xi) + E(\xi)\frac{I_p[(m^{-1}+z')\xi s]}{K_p[(m^{-1}+z')\xi s]}. \end{aligned} \tag{3.6}$$

Equations (2.17), (3.1) or (3.2) and (3.5) or (3.6) yield the Hankel transforms of the displacement and stresses.

For the medium with $G_z(z) = G_0 e^{\beta z}$

$$
\begin{aligned}
v^1(\xi,z) &= \frac{T}{8\Pi G_z(z)} e^{\beta(z-z')/2}\xi\left[\frac{c-(\beta/2)}{c+(\beta/2)}\frac{e^{-c(z+z')}}{c}+\frac{e^{-c|z-z'|}}{c}\right],\\
(3.7)\quad \sigma^1_{\theta z}(\xi,z) &= -\frac{T}{8\Pi} e^{\beta(z-z')/2}\xi\left[(c-(\beta/2))\frac{e^{-c(z+z')}}{c}+(\delta c+(\beta/2))\frac{e^{-c|z-z'|}}{c}\right],\\
\sigma^2_{r\theta}(\xi,z) &= -\frac{Ts^2}{8\Pi} e^{\beta(z-z')/2}\xi^2\left[\frac{c-(\beta/2)}{c+(\beta/2)}\frac{e^{-c(z+z')}}{c}+\frac{e^{-c|z-z'|}}{c}\right],
\end{aligned}
$$

$$z \geq 0,$$

where $\delta = 1$ if $z > z'$ and $\delta = -1$ if $z < z'$.

For the medium with $G_z(z) = G_0(1+mz)^\alpha$

$$
\begin{aligned}
v^1(\xi,z) &= \frac{T}{4\Pi G_z(z)s}\left(\frac{1+mz}{1+mz'}\right)^{\alpha/2}\xi K_p(\xi u')K_p(\xi u)\sqrt{uu'}\\
&\quad\times\begin{bmatrix} E_{p-1}(\xi u_0)+E_p(\xi u)\\ E_{p-1}(\xi u_0)+E_p(\xi u')\end{bmatrix}, \qquad \begin{matrix} 0\leq z\leq z',\\ z\geq z',\end{matrix}\\
(3.8)\quad \sigma^1_{\theta z}(\xi,z) &= -\frac{T}{4\Pi}\left(\frac{1+mz}{1+mz'}\right)^{\alpha/2}\xi^2 K_p(\xi u')K_{p-1}(\xi u)\sqrt{uu'}\\
&\quad\times\begin{bmatrix} E_{p-1}(\xi u_0)-E_{p-1}(\xi u)\\ E_{p-1}(\xi u_0)+E_p(\xi u')\end{bmatrix}, \qquad \begin{matrix} 0\leq z< z',\\ z> z',\end{matrix}\\
\sigma^2_{r\theta}(\xi,z) &= -\frac{Ts}{4\Pi}\left(\frac{1+mz}{1+mz'}\right)^{\alpha/2}\xi^2 K_p(\xi u')K_p(\xi u)\sqrt{uu'}\\
&\quad\times\begin{bmatrix} E_{p-1}(\xi u_0)+E_p(\xi u)\\ E_{p-1}(\xi u_0)+E_p(\xi u')\end{bmatrix}, \qquad \begin{matrix} 0\leq z\leq z',\\ z\geq z',\end{matrix}
\end{aligned}
$$

where

$$(3.8')\qquad u = (1+mz)s/m, \qquad u' = (1+mz')s/m, \qquad u_0 = s/m.$$

and quantities $E_n(x)$ represent the ratios of the I_n and K_n Bessel functions

$$(3.8'')\qquad E_n(x) = \frac{I_n(x)}{K_n(x)}, \quad n = p-1, p, \quad x = \xi u_0,\ \xi u,\ \xi u'.$$

The inverse Hankel transforms for (3.7) may be obtained in an analytical closed form according to the integrals

$$
(3.9)\quad
\begin{aligned}
&\int_0^\infty \frac{\xi^{p+1}}{\sqrt{\xi^2+\beta'^2}} e^{-Z\sqrt{\xi^2+\beta'^2}} J_p(r\xi)\, d\xi = \sqrt{\frac{2}{\Pi}} r^p \left(\frac{|\beta'|}{R}\right)^{p+1/2} K_{p+1/2}(|\beta'|R),\\
&\int_0^\infty \frac{1}{\sqrt{\xi^2+\beta'^2}} e^{-Z\sqrt{\xi^2+\beta'^2}} J_1(r\xi)\, d\xi = \frac{1}{r|\beta'|}\left(e^{-|\beta'|Z} - e^{-|\beta'|R}\right),
\end{aligned}
$$

and the known property of the Bessel function

$$J_2(\xi r) = (2/\xi r)J_1(\xi r) - J_0(\xi r),$$

where (in our notation)

$$R^2 = Z^2 + r^2, \quad \beta' = \beta/2s, \quad Z = s(z+z') \text{ or } Z = s|z-z'|.$$

The exact solutions are:

$$
(3.10)\quad
\begin{aligned}
v(r,z) &= \frac{T}{8\Pi G_z(z) s} e^{\beta' z_1}\Bigg[r\sum_{i=1}^{2} \frac{1}{R_i^3}(1+|\beta'|R_i)e^{-|\beta'|R_i}\\
&\quad + \frac{2}{r}\left((|\beta'|-\beta')e^{-|\beta'|z_2} - \left(|\beta'| - \beta'\frac{z_2}{R_2}\right) e^{-|\beta'|R_2}\right)\Bigg],\\
\sigma_{\theta z}(r,z) &= -\frac{3Tr}{8\pi} e^{\beta' z_1} \sum_{i=1}^{2} \Bigg[\frac{z_i}{R_i^5}\left(1+|\beta'|R_i + \frac{1}{3}\beta'^2 R_i^2\right)\\
&\quad - (-1)^i \frac{\beta'}{3R_i^3}(1+|\beta'|R_i)\Bigg] e^{-|\beta'|R_i},\\
\sigma_{r\theta}(r,z) &= -\frac{3Ts}{8\pi} e^{\beta' z_1}\Bigg[r^2 \sum_{i=1}^{2} \frac{1}{R_i^5}\left(1+|\beta'|R_i + \frac{1}{3}\beta'^2 R_i^2\right) e^{-|\beta'|R_i}\\
&\quad + \frac{2\beta'}{3R_2}\left(\frac{z_2}{R_2}\left(|\beta'| + \frac{1}{R_2}\right) - \beta'\right) e^{-|\beta'|R_2}\\
&\quad + \frac{4}{3r^2}\left((|\beta'|-\beta')e^{-|\beta'|z_2} - \left(|\beta'| - \beta'\frac{z_2}{R_2}\right) e^{-|\beta'|R_2}\right)\Bigg],
\end{aligned}
$$

for the medium with $G_z(z) = G_0 e^{\beta z}$, and

$$(3.11)\quad R_i^2 = z_i^2 + r^2, \quad z_i = s[z + (-1)^i z'], \quad (i=1,2), \quad \beta' = \beta/2s.$$

Note that $\sigma_{\theta z}(r,0) = 0$ and that the conditions

$$(3.12)\quad 2\pi \int_0^\infty G_{\theta z}(r,z) r^2 dr = \begin{cases} -T, & \text{for } z > z', \\ 0 & \text{for } z < z', \end{cases}$$

are identically satisfied for any values of z and β since, for the analytic function in expression $(3.10)_2$ the improper integral in Eq.(3.12) is equal 4/3 for $z > z'$ and $z \to z' + 0$, or 0 for $z < z'$ and $z \to z' - 0$. Equations (3.12) are the equilibrium conditions and we may expect that the resultant stress $\sigma_{\theta z}$ acting on the cross-section for arbitrary $z \geq 0$ should be equivalent to a concentrated torque for $z \geq z'$ and to zero for $z < z'$. Thus Eqs.(3.12) confirm the validity of the derived expressions. The asymptotic behaviour of the solution $(3.10)_1$ is as follows: $v(z) = 0[(z-z')^{-2}e^{-z'\beta}]$ for $\beta < 0$ and $v(z) = 0[(z-z')^{-2}e^{-z\beta}]$ for $\beta > 0$, and $v(z) = 0[(z-z')^{-3}]$ for $\beta = 0$, as $z \to \infty$, for fixed $r > 0$. For all bounded values of β and z' the regularity conditions, $v, \sigma_{\theta z}, \sigma_{r\theta} \to 0$ for $r \geq 0$, $z \to \infty$, are satisfied. It is evident that $v \sim (1/r^2)e^{-|\beta|r/2s}$ for all β and large r, so that the energy crossing the sphere at infinity is zero.

The inverse Hankel transforms of equations (3.8) yield integration which is adjusted by adding and substracting the terms, respectively

$$
\begin{aligned}
\frac{1}{2}\int_0^\infty \xi e^{-\xi|z_i|} J_1(\xi r)\, d\xi &= \frac{r}{2R_i^3}, \\
\frac{1}{2}\int_0^\infty \xi^2 e^{-\xi|z_i|} J_1(\xi r)\, d\xi &= \frac{3r|z_i|}{2R_i^5}, \qquad (3.13)\\
\frac{1}{2}\int_0^\infty \xi^2 e^{-\xi|z_i|} J_2(\xi r)\, d\xi &= \frac{3r^2}{2R_i^5},
\end{aligned}
$$

with the multiplier $\delta = \mathrm{sign}(z-z')$ for the stress $\sigma_{\theta z}$ and z_1, where z_i and R_i $(i = 1, 2)$ are defined by Eqs.(3.11).

After simplification, the inverse Hankel transforms of Eq.(3.8) reduce to

$$
\begin{aligned}
(3.14)\qquad v(r,z) = \frac{T}{8\Pi G_z(z)s}\left(\frac{1+mz}{1+mz'}\right)^{\alpha/2}\Bigg[\frac{r}{R_1^3} + \frac{r}{R_2^3} \\
+ \int_0^{\xi_0} \xi\left[F_1(\xi) - e^{-\xi z_2}\right] J_1(\xi r)\, d\xi + \int_0^{\xi_0} \xi\left[F_3(\xi) - e^{-\xi|z_1|}\right] J_1(\xi r)\, d\xi \\
+ \sum_{k=1}^{\infty}\Bigg[a_k(z,z')\int_{\xi_0}^{\infty} \xi^{-k+1} e^{-\xi z_2} J_1(\xi r)\, d\xi \\
+ \eta_k b_k(z,z')\int_{\xi_0}^{\infty} \xi^{-k+1} e^{-\xi|z_1|} J_1(\xi r)\, d\xi\Bigg]\Bigg],
\end{aligned}
$$

(3.14) [cont.]

$$\sigma_{\theta z}(r,z) = -\frac{T}{8\Pi}\left(\frac{1+mz}{1+mz'}\right)^{\alpha/2}\left[\frac{3rz_1}{R_1^5}+\frac{3rz_2}{R_2^5}\right.$$

$$+\int\limits_0^{\xi_0}\xi^2\left[F_2(\xi)-e^{-\xi z_2}\right]J_1(\xi r)\,d\xi+\delta\int\limits_0^{\xi_0}\xi^2\left[F_4(\xi)-e^{-\xi|z_1|}\right]J_1(\xi r)\,d\xi$$

$$+\sum_{k=1}^{\infty}\left[c_k(z,z')\int\limits_{\xi_0}^{\infty}\xi^{-k+2}e^{-\xi z_2}J_1(\xi r)\,d\xi\right.$$

$$\left.\left.+\delta\eta_k d_k(z,z')\int\limits_{\xi_0}^{\infty}\xi^{-k+2}e^{-\xi|z_1|}J_1(\xi r)\,d\xi\right]\right],$$

$$\sigma_{r\theta}(r,z) = -\frac{Ts}{8\Pi}\left(\frac{1+mz}{1+mz'}\right)^{\alpha/2}\left[\frac{3r^2}{R_1^5}+\frac{3r^2}{R_2^5}\right.$$

$$+\int\limits_0^{\xi_0}\xi^2\left[F_1(\xi)-e^{-\xi z_2}\right]J_2(\xi r)\,d\xi+\int\limits_0^{\xi_0}\xi^2\left[F_3(\xi)-e^{-\xi|z_1|}\right]J_2(\xi r)\,d\xi$$

$$+\sum_{k=1}^{\infty}\left[a_k(z,z')\int\limits_{\xi_0}^{\infty}\xi^{-k+2}e^{-\xi z_2}J_2(\xi r)\,d\xi\right.$$

$$\left.\left.+\eta_k b_k(z,z')\int\limits_{\xi_0}^{\infty}\xi^{-k+2}e^{-\xi|z_1|}J_2(\xi r)\,d\xi\right]\right],$$

for the medium with $G_z(z) = G_0(1+mz)^{\alpha}$; here ξ_0 is a finite value of ξ, and $\eta_k = 1$ for $0 \le z < z'$, and $\eta_k = (-1)^k$ for $z > z'$.

The functions $F_i(\xi)$ $(i = 1,2,3,4)$ denote the following combinations of Bessel functions:

$$\begin{bmatrix} F_1(\xi) \\ F_2(\xi) \end{bmatrix} = 2\xi\sqrt{uu'}\,K_p(\xi u')\frac{I_{p-1}(\xi u_0)}{K_{p-1}(\xi u_0)}\begin{bmatrix} K_p(\xi u) \\ K_{p-1}(\xi u) \end{bmatrix},$$

(3.15)
$$F_3(\xi) = 2\xi\sqrt{uu'}\begin{bmatrix} I_p(\xi u)\,K_p(\xi u') \\ I_p(\xi u')\,K_p(\xi u) \end{bmatrix}, \quad \begin{matrix} 0 \le z \le z', \\ z \ge z', \end{matrix}$$

$$F_4(\xi) = 2\xi\sqrt{uu'}\begin{bmatrix} K_p(\xi u')\,I_{p-1}(\xi u) \\ K_{p-1}(\xi u)\,I_p(\xi u') \end{bmatrix}, \quad \begin{matrix} 0 \le z < z', \\ z > z' \end{matrix}$$

and a_k, b_k, c_k, d_k are coefficients of the asymptotic expansions

$$
\begin{aligned}
F_1(\xi)e^{\xi z_2} - 1 &= \sum_{k=1}^{\infty} a_k(z,z')\xi^{-k}, \qquad z \geq 0, \\
F_3(\xi)e^{\xi|z_1|} - 1 &= \sum_{k=1}^{\infty} \eta_k b_k(z,z')\xi^{-k}, \qquad z \geq 0, \\
F_2(\xi)e^{\xi z_2} - 1 &= \sum_{k=1}^{\infty} c_k(z,z')\xi^{-k}, \qquad z \geq 0, \\
F_4(\xi)e^{\xi|z_1|} - 1 &= \sum_{k=1}^{\infty} \eta_k d_k(z,z')\xi^{-k}, \quad 0 \leq z < z', \ z > z'.
\end{aligned}
\tag{3.15'}
$$

The following notations have been introduced here:

$$
\begin{aligned}
a_1(z,z') &= -\frac{n-1}{4u_0} + \frac{l-1}{8}\left(\frac{1}{u'} + \frac{1}{u}\right), \\
a_2(z,z') &= \frac{(n-1)^2}{32u_0^2} - \frac{(l-1)(n-1)}{32u_0}\left(\frac{1}{u'} + \frac{1}{u}\right) \\
&\quad + \frac{(l-1)^2}{64}\frac{1}{u'u} + \frac{(l-1)(l-9)}{128}\left(\frac{1}{u'^2} + \frac{1}{u^2}\right), \\
b_1(z,z') &= -\frac{l-1}{8}\left(\frac{1}{u} - \frac{1}{u'}\right), \\
b_2(z,z') &= \frac{(l-1)(l-9)}{128}\left(\frac{1}{u^2} + \frac{1}{u'^2}\right) - \frac{(l-1)^2}{64}\frac{1}{uu'}, \\
c_1(z,z') &= -\frac{n-1}{4u_0} + \frac{n-1}{8u} + \frac{l-1}{8u'}, \\
c_2(z,z') &= \frac{(n-1)^2}{32u_0^2} - \frac{n-1}{32u_0}\left(\frac{n-1}{u} + \frac{l-1}{u'}\right), \\
d_1(z,z') &= -\frac{n-1}{8u} + \frac{l-1}{8u'}, \\
d_2(z,z') &= \frac{(n-1)(n-9)}{128u^2} - \frac{(l-1)(n-1)}{64uu'} + \frac{(l-1)(l-9)}{128u'^2}, \\
l &= (1-\alpha)^2, \qquad n = (1+\alpha)^2.
\end{aligned}
\tag{3.15''}
$$

where u, u' and u_0 are defined by Eqs.(3.8′).

The asymptotic behaviour of the solution (3.14) is as follows $v(z) = 0[(z-z')^{-3}(1+mz)^{-\alpha/2}]$, $\sigma_{\theta z}(z) = 0[(z-z')^{-4}(1+mz)^{\alpha/2}]$ as $z \to \infty$, for fixed $r > 0$.

The regularity condition requires that $-6 < \alpha < 8$ and the range of applicability of the solution (3.14) is restricted to these values of α. However, when compared with the existing analytical solutions of this problem (see, for example, [3], [5]), this restriction on the parameter α is weaker. In addition, $v \sim 1/r^2$ for large r, so that the energy crossing the cylinder at infinity is zero.

It is noted that the integrals appearing in Eqs.(3.14) are again well behaved and bounded, and create no difficulties when a numerical integration procedure is performed over the finite range $0 \le \xi \le \xi_0$ and infinite range $\xi_0 \le \xi < \infty$. For a prescribed large value of ξ_0 the integrands in the range $0 \le \xi \le \xi_0$ are of order $0[e^{-\xi_0|z_i|}\xi_0^{1/2}]$ or $0[e^{-\xi_0|z_i|}\xi_0^{3/2}]$, respectively if $\xi = \xi_0$, whereas the integrals in the range $\xi_0 \le \xi < \infty$ represent remainders with a small contribution to the total solution. However, on the surface $z = z'$ the first integral in the range $\xi_0 \le \xi < \infty$, for displacement, becomes increasingly large in the neighbourhood of the point loading and, in fact, this integral will be shown to contain the singularity which occurs in the region near the point of the applied concentrated load. For $\alpha = 0$ or $m \to 0$ the integrals or the coefficients $a_k, b_k, \ldots$ vanish (homogeneous solid). The stress and displacement equations derived can be readily reduced to: (i) an external concentrated torque, $z' = 0$, (ii) an orthotropic homogeneous material $\alpha = \beta = 0$, (iii) an isotropic non-homogeneous material, $s = 1$, (iv) an isotropic homogeneous material, $s = 1$ and $\alpha = \beta = 0$.

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