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SHRINKAGE STRESSES IN DRIED MATERIALS

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The aim of the paper is to analyse the shrinkage stresses in isotropic material when they do not exceed its strength. The model established in author's previous work is used for describing the problem undertaken. The model relates stresses with strains, moisture content and temperature. The state of dried material is described by a system of five differential equations with double coupling. Their solution must satisfy additionally the compatibility relations. The problem of convectively dried plate is solved as an example. The evolution of both the moisture content and the shrinkage stresses distributions as well as the deformation of the plate during drying process were determined. The finite difference method and the method of separation of variables were used and good agreement of the results obtained on the basis of these two methods were stated. They are presented on graphs.

1. Notation

x, y, z [m]	position coordinates,
t [s]	time,
u_x, u_y, u_z [m]	components of the displacement vector,
A [N/m^2]	bulk modulus for the porous solid,
M [N/m^2]	shear modulus for the porous solid,
f_v [1]	porosity ratio,
$\sigma_{i,j}$ [N/m^2]	total stress tensor,
p_m [N/m^2]	true pore pressure,
b_i [N/kg]	body force,
$\varepsilon_{i,j}$ [N/m^2]	strain tensor,
T [°K]	absolute temperature,
ϑ [deg]	relative temperature,
Θ [1]	specific moisture content,
c_ϑ [J/m^2 deg]	temperature coefficient of the moisture potential,
c_Θ [J/m^3]	moisture content of the moisture potential,
α_ϑ [deg^{-1}]	coefficient of the linear thermal expansion,

α_Θ [1]	coefficient of the linear humidity expansion,
α_m [kg s/m^4]	coefficient of the convective mass exchange,
q_i [W/m° k]	heat flux,
η_i [kg/s m^2]	moisture flux,
μ [J/kg]	moisture potential density,
Λ_T [W/m° K]	thermal conductivity,
Λ_m [kg s/m^3]	moisture conductivity,
ρ_o [kg/m^3]	bulk mass density of the porous body (dry body).

2. Introduction

A change of body shape due to shrinkage accompanies most drying processes of moist porous materials. This change may induce shrinkage stresses which, like thermal stresses, are caused by internal deformations occurring in the material because of a non-uniform distribution of the moisture content. Their existence is therefore independent of the external mechanical forces.

In fact, three reasons for internal stresses to arise in dried materials can be distinguished. They are: nonuniform distribution of moisture content, nonuniform distribution of temperature and mechanical field due to external forces (if they exist).

The non-uniformity of moisture content distribution increases along with the rate of drying process, since the evaporation of moisture from the boundary surface proceeds faster than the flow of moisture from the interior out to the surface of the dried material. For this reason the shrinkage of material close to the boundary is considerably larger than that in the remaining part, and the shrinkage stresses are first to arise there. The shrinkage stresses at the boundary often exceed the limit values of strength of the material so that the surface cracks or even cracks within the material can occur. These affect the quality of the dried products or even make them useless.

The premise of this work is to analyse the shrinkage stresses in dried material when they do not exceed its strength. Only the elastic strains of dried material are allowed in the present considerations. The model given by Kowalski [5,6] is used for describing the problem undertaken. The model relates stresses to the strains, moisture content and temperature. The state of dried material is described by a system of five differential equations with double couplings. Their solutions must additionally satisfy the compatibility relations. The couplings mean that a change of one of the three fields, i.e. strains, moisture content and temperature, causes changes in the remaining fields. The coupling effect between temperature and moisture is found to be most significant, particularly when the dried medium undergoes a sudden change in surface temperature while the surface moisture concentration is kept constant (see e.g. [13, 14]).

Other models of drying presented in the literature up to now (see e.g. [2, 3, 9, 16] did not take into consideration the deformability of the dried material and its influence on the drying process.

In particular, the conditions of existence of shrinkage stresses in dried materials are analysed, and numerical illustration of applicability of the drying model given in [5] to evaluation of stresses presented in this paper is given. The problem of convective dried plate is solved as an example. To focus our attention mainly on the shrinkage stresses due to changes of the moisture content, the considerations are limited to the constant drying rate period in which the temperature of the dried material is constant and equal to the wet-bulb temperature. The evolution of shrinkage stresses distribution, the moisture potential distribution, and the deformation of the plate are investigated. The results are presented on graphs.

3. The essence of shrinkage stresses

To explain the essence of the shrinkage stresses let us consider wetting of a thin porous rod (Fig.1). An increase of the moisture content in the rod causes its elongation. If the dry rod of length l_0 elongates freely to $l_0 + \Delta l_0$ due to wetting, then the drying process will cause its return to the primary length l_0. No stresses will appear during wetting and drying of the free rod. The length of free elongation Δl_0 is proportional to moisture content Θ and to initial length l_0 (see, for instance, [4] p.280).

$$\Delta l_0 = \alpha_\Theta \, \Theta \, l_0, \tag{3.1}$$

where α_Θ is the coefficient of the linear humidity expansion (analogy to the coefficient of the linear thermal expansion), and $\Theta = \rho_m/\rho_0$ is the specific moisture content (ratio of partial moisture density ρ_m to partial density of dry body ρ_0).

If the distance of the free elongation of the rod was limited to the value $\Delta l_1 < \Delta l_0$ (see Fig.1), then stress σ would arise. Its value, assuming elastic deformations, would be

$$\sigma = E\frac{\Delta l_0 - \Delta l_1}{l_0}, \tag{3.2}$$

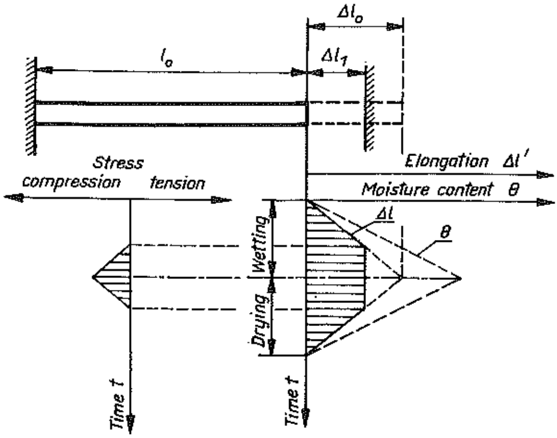

FIG. 1. Deformation of a wetted rod.

where E denotes Young's modulus. In the course of drying process the moisture content of the rod decreases and the stress σ tends to vanish. It vanishes entirely when $\alpha_\Theta \, \Theta l_0 = \Delta l_1$.

It was an example of stresses arising in a body during its wetting (or drying) caused by limited freedom of its expansion (or contraction).

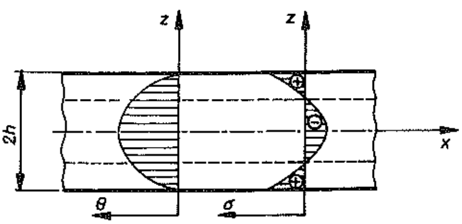

FIG. 2. Example of a moisture content distribution Θ and stress distribution σ in a dried plate.

Another reason for the stresses to arise is a nonuniform distribution of the moisture content in the body. A simple example of symmetrically drying plate (Fig.2) is a good illustration of this phenomenon. For the sake of clarity, we assume that we analyse the constant drying rate period in which the temperature of dried material is kept constant in the whole body. Let us imagine that the plate consists of many very thin separate layers. If these layers were deformed independly, without any interaction between them, then the shape and the size of the plate would be determined by the moisture content distribution because of the fact that the change of size is proportional to the moisture content

$$l(z) = l_0 + l_0 \alpha_\Theta \, \Theta(z). \tag{3.3}$$

However, the individual (imagined) layers are not free and mutually independent, and some forces of interaction between them exist. This induces a field of internal stresses which can produce cracks in the material, especially at the surface of the drying body where the strongest shrinkage occurs.

4. Conditions of arising of shrinkage stresses

The physical relation between the stress tensor σ_{ij} and the strain tensor ε_{ij}, relative temperature ϑ and relative moisture content $(\Theta - \Theta_r)$ is [5]

$$\sigma_{ij} = 2M\varepsilon_{ij} + [A\varepsilon_{kk} - \gamma_\vartheta \, \vartheta - \gamma_\Theta \, (\Theta - \Theta_r)] \, \delta_{ij}, \tag{4.1}$$

where $\gamma_\vartheta = \alpha_\vartheta \, (2M + 3A)$, $\gamma_\Theta = \alpha_\Theta \, (2M + 3A)$ and α_ϑ, α_Θ are the coefficients of the linear thermal and humidity expansion, respectively. M and A are the mechanical constants of the medium, and Θ_r is the reference moisture content.

We shall use the relation (4.1) written in the inverse form

$$\varepsilon_{ij} = 2M'\sigma_{ij} + [A'\sigma_{kk} + \alpha_\vartheta \, \vartheta + \alpha_\Theta \, (\Theta - \Theta_r)] \, \delta_{ij}, \tag{4.2}$$

where

$$2M' = 1/(2M), \qquad A' = -A/[2M(2M + 3A)].$$

If there were no stresses in a dried material, then the strains would depend on the temperature and the moisture content only

$$\varepsilon_{ij} = [\alpha_\vartheta \, \vartheta + \alpha_\Theta \, (\Theta - \Theta_r)] \, \delta_{ij}. \tag{4.3}$$

The strains have to fulfil the compatibility relations [11]

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0, \tag{4.4}$$

where comma denotes the differentiation with respect to the spatial coordinate.

Substituting Eq.(4.3) to Eq.(4.4) we get

$$\begin{aligned} \alpha_\vartheta \, [\vartheta_{,kl}\delta_{ij} + \vartheta_{,ij}\delta_{kl} - \vartheta_{,jl}\delta_{ik} - \vartheta_{,ik}\delta_{jl}] & \\ + \alpha_\Theta \, [\Theta_{,kl}\delta_{ij} + \Theta_{,ij}\delta_{kl} - \Theta_{,jl}\delta_{ik} - \Theta_{,ik}\delta_{jl}] &= 0. \end{aligned} \tag{4.5}$$

Since α_ϑ and α_Θ are generally independent of each other, Eq.(4.5) holds if the expressions in square brackets are equal to zero. This is of course a sufficient condition to satisfy Eq.(4.5), but not necessary. The equations obtained in this way will be satisfied if

$$\vartheta_{,ij} = 0, \qquad \Theta_{,ij} = 0. \tag{4.6}$$

Linear functions

$$\begin{aligned} \vartheta &= a_0 + a_1 x + a_2 y + a_3 z, \\ \Theta &= b_0 + b_1 x + b_2 y + b_3 z, \end{aligned} \tag{4.7}$$

are the solutions of (4.6), with a_i and b_i being some arbitrary constants. Thus if ϑ and Θ are linear functions of spatial coordinates x, y, z and the deformations of the body are not limited (the body is free), then the compatibility relations will be fulfilled without existence of stresses. But, generally, the fields of both the temperature and the moisture content do not satisfy the linearity conditions (4.7). The internal stresses arise in such a case. The strains produced by the stresses, together with the strains caused by the temperature and the moisture content fields, have to satisfy the compatibility conditions (4.4). Of course, both functions ϑ and Θ must be bounded in an infinite space.

It implies certain constraints on the coefficients a_i and b_i in formulae (4.7).

5. Model describing shrinkage stresses

Let us now construct the model which can be used for describing the shrinkage stresses arising during drying of moist porous materials. The model is discussed in detail in Kowalski [5] and [6]. Its linear version is a sum of partial stresses in the porous matrix (skeleton) and in the fluid,

$$\sigma_{ij} = \sigma_{o\,ij} + \sigma_m \delta_{ij}, \tag{5.1}$$

where $\sigma_{o\,ij}$ is the partial stress in porous matrix and $\sigma_m = -p_m f_v$ means the partial pore pressure, whereas p_m is the true pore pressure and f_v denotes the porosity ratio.

To calculate the distribution of shrinkage stresses we must first determine the deformation, the temperature and the moisture content fields. In absence of phase transitions inside the dried material, that is when the evaporation of the moisture proceeds on the boundary surface as it is during the constant drying rate period, the fields mentioned above can be determined on the basis of the following system of equations [5]:

mechanical equilibrium equations:

$$\sigma_{ij,j} + \rho_0 \, (1 + \Theta) \, b_i \cong 0, \qquad \rho_0 = \text{const}; \tag{5.2}$$

heat balance equation:

$$c_\vartheta \dot{T} + T(\gamma_\vartheta \, \dot{\varepsilon}_{kk} - c_\Theta \dot{\Theta}) = -q_{k,k}; \tag{5.3}$$

moisture mass balance equation:

$$\rho_0 \dot{\Theta} = -\eta_{k,k}; \tag{5.4}$$

heat transport equation:

$$q_i = -\Lambda_T T_{,i}; \tag{5.5}$$

moisture mass transport equation:

$$\eta_i = -\Lambda_m \mu_{,i}; \tag{5.6}$$

physical relations for stresses (4.1):

$$\sigma_{ij} = 2M\varepsilon_{ij} + [A\varepsilon_{kk} - \gamma_\vartheta \, \vartheta - \gamma_\Theta(\Theta - \Theta_r)] \, \delta_{ij}; \tag{5.7}$$

physical relation for moisture potential μ:

$$\mu = [c_\vartheta \, \vartheta - \gamma_\Theta \, \varepsilon_{kk} + c_\Theta \, (\Theta - \Theta_r)] \, / \rho_0; \tag{5.8}$$

geometrical relations:

$$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2. \tag{5.9}$$

The above set of equations can be reduced to five equations: three displacement equations, heat transfer equation, and mass transfer equation (see [5,6]).

Usually it is convenient to determine the self-stresses by means of the stress equations which can be derived by substitution of the physical relations (4.2) into the compatibility relations (4.4). Assuming the body forces to be of minor importance in our consideration, we obtain

$$\nabla^2 \sigma_{ij} + \frac{2(A+M)}{3A+2M} \sigma_{kk,ij} + 2M \left[\frac{2M+3A}{2M+A} \nabla^2 \, \psi \, \delta_{ij} + \psi_{,ij} \right] = 0, \tag{5.10}$$

where

$$\psi = \alpha_\vartheta\, \vartheta + \alpha_\Theta\, (\Theta - \Theta_r). \tag{5.11}$$

These equations have to be supplemented with the equilibrium equations (5.2) (without body forces), and with the boundary conditions

$$\sigma_{ij}\, n_j = p_i. \tag{5.12}$$

The set of stress equations will be complete if the heat and mass transfer equations (5.3)–(5.6) as well as the physical relation for the matrix dilatation resulting from Eq.(5.7), i.e.

$$\varepsilon_{kk} = \frac{1}{2M + 3A}\sigma_{kk} + 3\psi, \tag{5.13}$$

and the physical relation (5.8) are added to the system mentioned above. To render the solutions of these equations unique, the boundary conditions describing heat and mass transfer and the initial conditions for the temperature and the moisture content must be given.

Each drying process consists of some stages (preheating, constant drying rate period, decreasing drying rate period), and for each stage of drying different model should be used. It is a consequence of the fact that each stage of drying is governed by a different mechanism of heat and mass transfer (see [6]). We shall not discuss all models here, but rather restrict ourselves to an example of shrinkage stresses in a convectively dried plate during the constant drying rate period.

6. Shrinkage stresses in a plate dried by convection

To fix our attention on shrinkage stresses caused by the moisture content changes only, we confine our considerations to constant drying rate period in which the temperature is constant in the whole body and equal to the wet-bulb temperature. Thermal stresses are absent in this stage of drying.

Let us consider a plate of constant thickness 2h and of arbitrary shape in x, y plane (Fig.3). The plate is free of loading, what means that there are no external forces involving stresses. The moisture content distribution is a function of coordinate z and time t, i.e. $\Theta = \Theta(z, t)$. This implies the stress components to be as follows:

$$\begin{aligned} \sigma_{xx} &= \sigma_{yy} = f(z,t), \\ \sigma_{zz} &= \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0. \end{aligned} \tag{6.1}$$

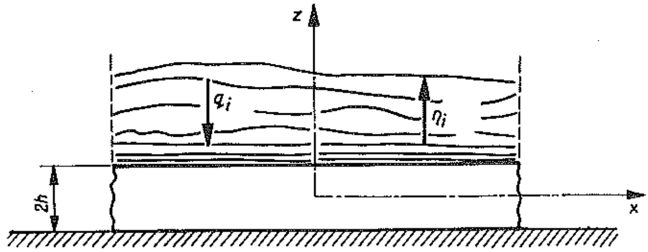

FIG. 3. The considered plate resting on an impermeable surface.

The stresses have to fulfill:

equilibrium conditions

$$\sigma_{ij,j} = 0; \tag{6.2}$$

boundary conditions

$$\sigma_{ij} n_j = p_i = 0; \tag{6.3}$$

compatibility relations (5.10) which for the one-dimensional problem is reduced to

$$\frac{d^2}{dz^2}\left[\sigma_{xx} + \gamma_\Theta \frac{2M}{2M+A}(\Theta - \Theta_r)\right] = 0. \tag{6.4}$$

In the above equation ψ was eliminated through Eq.(5.11) under the assumption that $\vartheta = \text{const}$ (constant drying rate period). It is easy to see that stress components (6.1) satisfy the equilibrium conditions (6.2).

Boundary conditions (6.3) are fulfilled approximately in a sense of Saint-Venant's principle. It means that some local values of stresses exist in directions x and y but the resultant force and the resultant moment involved by the stresses are equal to zero,

$$\int_{-h}^{h} \sigma_{xx}\, dz = 0 \qquad \text{and} \qquad \int_{-h}^{h} \sigma_{xx}\, z\, dz = 0. \tag{6.5}$$

The following form of integral is obtained on the basis of Eq.(6.4)

$$\sigma_{xx} = \sigma_{yy} = -\gamma_\Theta \frac{2M}{2M+A}(\Theta - \Theta_r) = c_1 z + c_2, \tag{6.6}$$

where c_1 and c_2 are constants which have to fulfill conditions (6.5). After their estimation, we can rewrite Eq. (6.6) as follows:

$$\sigma_{xx} = \sigma_{yy} = 2\frac{2M}{2M+A}\left[-\gamma_\Theta \frac{M}{A+M}(\Theta - \Theta_r) + \frac{1}{2h}N_\Theta + \frac{3z}{2h^3}M_\Theta\right], \tag{6.7}$$

where

(6.7')
$$N_\Theta = \gamma_\vartheta \frac{M}{A+M} \int_{-h}^{h} (\Theta - \Theta_r) dz,$$
$$M_\Theta = \gamma_\vartheta \frac{M}{A+M} \int_{-h}^{h} (\Theta - \Theta_r) z dz.$$

Strain components can be calculated by means of relations (4.2),

(6.8)
$$\varepsilon_{xx} = \varepsilon_{yy} = \frac{A+M}{M(2M+3A)} \left[\frac{1}{2h} N_\Theta + \frac{3z}{2h^3} M_\Theta \right],$$
$$\varepsilon_{zz} = \frac{\gamma_\Theta}{2M+A} (\Theta - \Theta_r) - \frac{2M(A+M)}{M(2M+A)(2M+3A)} \times \left(\frac{1}{2h} N_\Theta + \frac{3z}{2h^3} M_\Theta \right),$$
$$\varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0.$$

The components of the displacement vector referred to the origin of the coordinate system (without regarding the motion of the plate as a rigid body) are:

(6.9)
$$u_x = \frac{A+M}{M(2M+3A)} \left[\frac{1}{2h} N_\Theta + \frac{3z}{2h^3} M_\Theta \right] x,$$
$$u_y = \frac{A+M}{M(2M+3A)} \left[\frac{1}{2h} N_\Theta + \frac{3z}{2h^3} M_\Theta \right] y,$$
$$u_z = \frac{\gamma_\Theta}{2M+A} \int_0^z (\Theta - \Theta_r) dz - \frac{2M(A+M)}{M(2M+A)(2M+3A)} \times \left(\frac{1}{2h} N_\Theta + \frac{3z}{2h^3} M_\Theta \right) + c(x,y,t).$$

Function $c(x,y,t)$ describes the deflection of the middle plane ($z = 0$) of the plate caused by an asymmetrical moisture content distribution across the plate thickness. Quantity M_Θ is responsible here for the asymmetrical deformations, as it disappears when the moisture content distribution $\Theta(z,t)$ is symmetrical. For that reason $c(x,y,t)$ has to be dependent on M_Θ.

Making use of the known solution of the problem of symmetric bending of a circular plate by uniformly distributed moments acting on its edge (see e.g.[7]), we write

$$c(x,y,t) = -\frac{3(A+M)}{4h^3M(2M+3A)}M_\Theta(x^2+y^2). \tag{6.9'}$$

Note that if the moisture content distribution is symmetrical, that means if $\Theta(z,t) = \Theta(-z,t)$, the quantity $M_\Theta = 0$ but $N_\Theta \neq 0$. Inversely, if moisture content distribution is antisymmetric, i.e. $\Theta(z,t) = -\Theta(-z,t)$, then $M_\Theta \neq 0$ but $N_\Theta = 0$. For linear moisture distribution $\Theta(z,t) = a(t)+b(t)z$ the stresses disappear but the strains remain different from zero.

The distribution of the moisture content across the plate thickness is determined on the basis of the mass balance equation (5.4) and the mass transport equations (5.6), in which the gradient of moisture potential μ is replaced by the one calculated from formula (5.8) with the dilatation

$$\varepsilon_{kk} = \frac{\gamma_\Theta}{2M+A}(\Theta-\Theta_r)+\frac{(A+M)(2M-A)}{M(2M+3A)(2M+A)}\left[\frac{1}{2h}N_\Theta+\frac{3z}{2h^3}M_\Theta\right]. \tag{6.10}$$

The last formula is determined on the basis of Eq.(6.8). After the prescribed operations we arrive at the diffusion equation of the form

$$\dot{\Theta} = K\frac{\partial^2\Theta}{\partial z^2}, \tag{6.11}$$

with the coefficient

$$K = \frac{\Lambda_m}{\rho_0^2}\left(c_\Theta - \frac{\gamma_\Theta^2}{2M+A}\right) \tag{6.11'}$$

containing material constants responsible also for the deformability of the porous matrix. The deformation term does not explicitly enter the diffusion equation in case when the deformation depends only on the moisture content Θ. The dilatation coupling effect is thus a constant quantity described here by parameter $\gamma_\Theta^2/(2M+A)$. Also temperature does not appear in Eq.(6.11) since we consider here the constant drying rate period.

Let us consider an example of a plate, one side of which is insulated (impermeable background), and the other is dried convectively. The boundary conditions are

$$-\Lambda_m\frac{\partial\mu}{\partial z}\bigg|_{z=h} = \alpha_m\left(\mu\Big|_{z=h} - \mu_a\right), \qquad \frac{\partial\mu}{\partial z}\bigg|_{z=-h} = 0, \tag{6.12}$$

where α_m is the coefficient of convective mass exchange and μ_a is the potential of drying medium. The moisture potential inside the dried material

μ (see Eq.(5.8)) depends on moisture content Θ, temperature ϑ and dilatation ε_{kk}. The last one depends in fact on the moisture content (see (6.10)). Thus, the final moisture potential μ is also a function of moisture content and of temperature (here constant)

$$\rho_0\mu = (c_\Theta - \frac{\gamma_\Theta^2}{2M+A})(\Theta - \Theta_r) + \alpha_\Theta \frac{(A+M)(2M-A)}{M(2M+A)} \left[\frac{1}{2h}N_\Theta + \frac{3z}{2h^3}M_\Theta\right] + c_\vartheta\, \vartheta. \tag{6.13}$$

Relation (6.13) allows us to express the boundary condition (6.12) by the moisture content function. The initial condition for the drying process is formulated as follows:

$$\mu(z,0) = \mu_0, \tag{6.14}$$

where μ_0 is determined on the basis of Eq.(6.13) by substituting $\Theta(z,t) = \Theta_0(z)$, i.e. the initial moisture content distribution.

7. Numerical example

We shall consider an infinite plate of thickness $2hl = 0.2m$ described by the following data (see [8]):

$$\begin{aligned} A &= 10^9 [\mathrm{N/m^2}], & M &= 6.25\cdot 10^8\ [\mathrm{N/m^2}],\\ \Lambda_m &= 3.02\cdot 10^{-5} [\mathrm{kg{\cdot}s/m^3}], & c_\Theta &= 6.6\cdot 10^6\ [\mathrm{J/m^3}],\\ \alpha_\Theta &= 2.4\cdot 10^{-3}, & \alpha_m &= 5\cdot 10^{-3}\ [\mathrm{kg\ s/m^4}],\\ \rho_0 &= 1200\ [\mathrm{kg/m^3}], & \mu_a &= 40\ [\mathrm{J/kg}],\\ \mu_0 &= 100\ [\mathrm{J/kg}]. \end{aligned}$$

Equation (6.11) with boundary conditions (6.12) and initial condition (6.14) is solved by means of the variable separation method. We represent the function Θ in the form of a series,

$$(\Theta - \Theta_r) = \sum_{i=1}^{N} \Theta_i\,(z,t) + \Theta_0\,(z). \tag{7.1}$$

Any function $\Theta_i(z,t)$ for $0 < i < N$ fulfills the uniform conditions

$$-\Lambda_m \frac{\partial\mu}{\partial z}\Big|_{z=h} = \alpha_m \mu\Big|_{z=h}, \tag{7.2}$$

$$\frac{\partial\mu}{\partial z}\Big|_{z=-h} = 0, \tag{7.3}$$

and function $\Theta_0(z)$ fulfills conditions (6.12). Function $\Theta_i(z,t)$, according to the variable separation method, is assumed in the form

$$\Theta_i(z,t) = \phi_i(z)\psi_i(t), \tag{7.4}$$

where

$$\psi_i(t) = C_i \exp(-\kappa_i t), \tag{7.5}$$

$$\phi_i(z) = X_i \sin(\omega_i z) + Y_i \cos(\omega_i z), \tag{7.6}$$

$$\omega_1^2 = K/\kappa_i \tag{7.7}$$

and κ_i is an arbitrary constant.

Knowing the form of function $\phi(z)$, it is possible to calculate N_Θ and M_Θ from Eq.(6.7′)

$$N_\Theta = \gamma_\Theta \frac{M}{A+M} \frac{1}{h\omega_i} Y_i \sin(\omega_i h), \tag{7.8}$$

$$M_\Theta = \gamma_\Theta \frac{M}{A+M} \frac{3X_i}{\omega_i^2} \left[\frac{1}{h^3} \sin(\omega_i h) - \frac{\omega_i}{h^2} \cos(\omega_1 h) \right]. \tag{7.9}$$

Introducing these into relations for the potential μ and its gradient, and then into condition (7.2) and (7.3), we obtain the system of equations enabling the evaluation of values ω_i, X_i, Y_i:

$$\begin{aligned} X_i \Bigg[& C_1 \omega_i \cos(\omega_i h) + 3\frac{C_2 \sin(\omega_i h)}{h^3 \omega_i^2} - 3\frac{C_3 \cos(\omega_i h)}{h^2 \omega_i} \\ & + C_3 \sin(\omega_i h) + 3\frac{C_4 \sin(\omega_i h)}{h^2 \omega_i^2} - 3\frac{C_4 \cos(\omega_i h)}{h\omega_i} \Bigg] \\ & + Y_i \left[-C_1 \omega_i \sin(\omega_i h) + \frac{C_4 \sin(\omega_i h)}{h\omega_i} + C_4 \cos(\omega_i h) \right] = 0, \end{aligned} \tag{7.10}$$

$$\begin{aligned} X_i \left[K_1 \omega_i \cos(\omega_i h) + 3\frac{K_2 \sin(\omega_i h)}{h^3 \omega_i^2} - 3\frac{K_2 \cos(\omega_i h)}{h^2 \omega_i} \right] & \\ + Y_i \left[K_1 \omega_i \sin(\omega_i h) \right] = 0. & \end{aligned} \tag{7.11}$$

Here $C_1, C_2, C_3, C_4, K_1, K_2$ are combinations of material constants.

Equations (7.10) and (7.11) can be treated as linear equations for unknowns X_i, Y_i with parameter ω_i. The principal determinant of this system of equations must be equal to zero if the solution should not be trivial. This requirement gives the equation for calculating ω_i. Making use of the Newton method we have calculated 60 values of ω_i. Quantities X_i and Y_i are related to each other as follows:

$$X_i = Y_i f(\omega_i), \tag{7.12}$$

where $f(\omega_i)$ is a function of ω_i. The initial condition (6.14) was used to determine 60 unknowns Y_i. These quantities were calculated by collocational fulfilling of (6.14) in 60 points $\xi_i \in [-h, h], i = 1, \ldots, 60$ by the potential (6.13) together with Eq.(7.1). The elimination method of Gauss was employed to solve the system of 60 equations. Thus we obtain a full set of quantities X_i, Y_i and ω_i for $i = 1, \ldots, 60$.

The relative moisture content $(\Theta - \Theta_r)$ can be obtained from relation (7.1) and Eqs.(7.4)-(7.7) for any time and any point of the plate. Next we can calculate stresses, moisture potential and strains using relations from Sets.6. The results are shown in figures placed below.

Figure 4 shows the moisture potential distribution in the plate at some instants of time. At the beginning the potential is equal to the initial value $\mu_0 = 100$. In the course of the drying process its value decreases to the potential of drying medium $\mu_a = 40$. The potential distribution can be qualitatively identified with the moisture content distribution due to linear relation between them (see Eq.(6.13)).

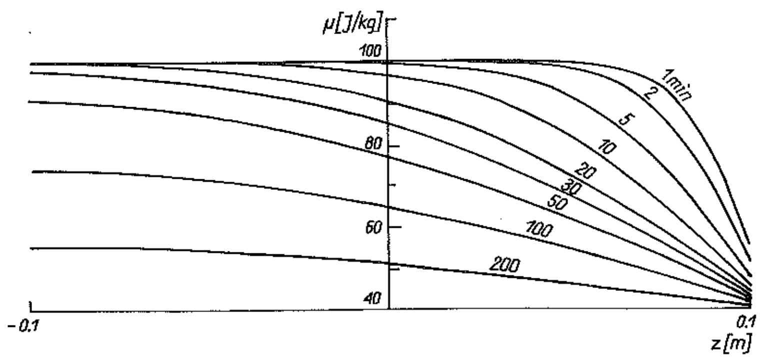

FIG. 4. The moisture potential distribution in the plate.

The stress distributions in the plate at some instants of time are shown in Fig.5. It is easy to see that maximal values of stresses occur close to the boundary surface at the initial stage of drying, i.e. when the moisture content distribution in the plate is mostly nonuniform. When the moisture content distribution tends to be uniform, the stresses decrease. Stresses

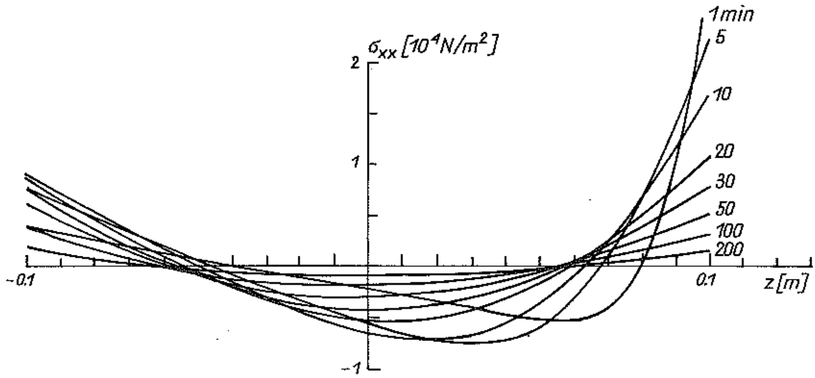

FIG. 5. The stress distribution in the plate.

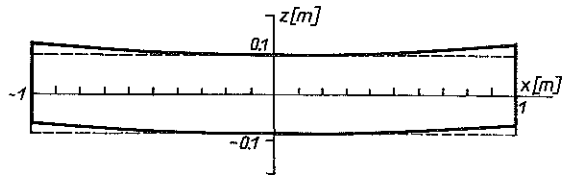

FIG. 6. The shape of the plate at the tenth minute of drying.

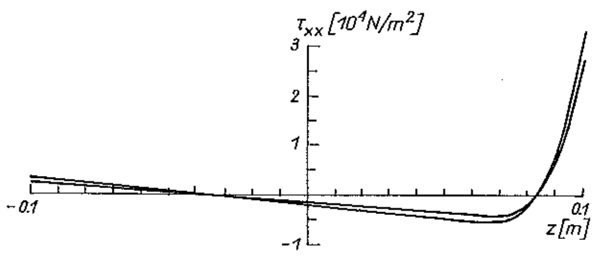

FIG. 7. Comparison of the methods used for solution of the problem.

close to the plate surfaces are extensional while inside the plate they are compressive. It is interesting that near the isolated surface of the plate, where the moisture content is the greatest, extensional stresses are also found. It is the result of bending of the plate due to strong contraction near the evaporation boundary. This is seen in Fig.6 where the shape of the plate at the tenth minute of drying is shown. The shape was calculated using relation (6.9). The values of displacement in Fig.6 are enlarged 300 times.

Figure 7 illustrates the stress distribution in the plate calculated by means of the finite difference method (denoted by $\circ$) and the method of separation of variables (denoted by $\times$) at time $t = 2$ minutes. A good agreement of the results has been established.

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