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SECOND-ORDER STRAINS OF LIGNOSTONE UNDER CREEP AND CONSTITUTIVE EQUATIONS

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Specimens of the beech wood lignostone are tested for creep under simple tension along the grain and torsion, respectively. The second-order strains (interrelated strains) $2\bar{l}_{12}$ and $\bar{l}_{11}$ generated in result of pure tension and torsion are found to be nonlinear. To describe that nonlinear behaviour, the nonlinear theory of visco-elasticity has been employed and the similarity of the creep curves is used. Correct description of the considered phenomena is obtained as a result.

1. Introduction

The aim of the paper is the associated strains measurement [1] (ancillary [2]), second-order [3] and an attempt to formulate the constitutive equations under pure tension and torsion.

2. Description of experiment and results

The associated strains in the form of the twisting angle were measured under tensile test. Related strains l_{11} coinciding with the specimen axis were measured under torsion. Specimens for the creep tests were made of the beach wood lignostone with the density $\varrho = 990\,\mathrm{kg/m^3}$ and the compression ratio $n = 1.45$. The hollow cylinders had the outer diameter $d_z = 19.2\,\mathrm{mm}$, the inner diameter $d_v = 16.0\,\mathrm{mm}$ and the measuring length 70 mm. The specimen axis coincided with the grain direction.

The creep tests were performed under the relative air humidity $(65 \pm 2)\%$ and the temperature (293 ± 3)K. The tensile creep tests were made for three stress levels: $\sigma_{11}/R_{11} = 0.2,\ 0.297,\ 0.35$, where $R_{11} = 195.1\,\mathrm{MPa}$ denotes the tensile strength parallel to the grains. The torsional creep tests were performed for three stress levels: $\sigma_{12}/R_{12} = 0.4;\ 0.5;\ 0.6$, where $R_{12} = 15.56\,\mathrm{MPa}$ denotes the technical shearing strength. The results of strain measurement $\bar{l}_{12}$ obtained by measuring the angle of twist, where $\bar{l}_{12}$ denotes the average non-dilatational strains l_{12} and l_{13} in the anisotropic material are seen in Fig. 1, while strains l_{11} are shown in Fig. 2.

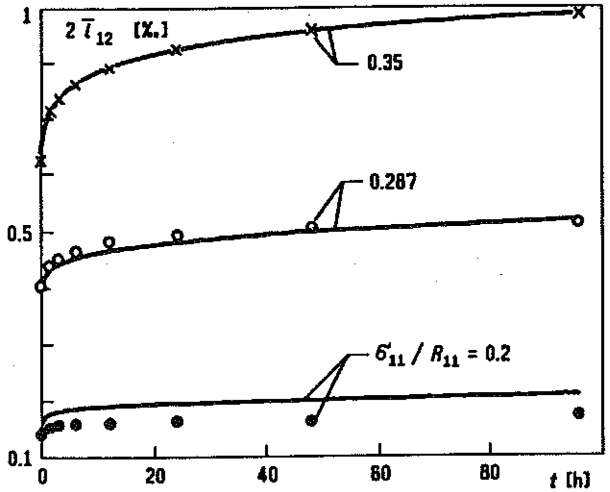

FIG. 1. Creep curves $2\bar{l}_{12}$: dots, circles and crosses represent experimental data according to (3.2), (3.11) and (3.16).

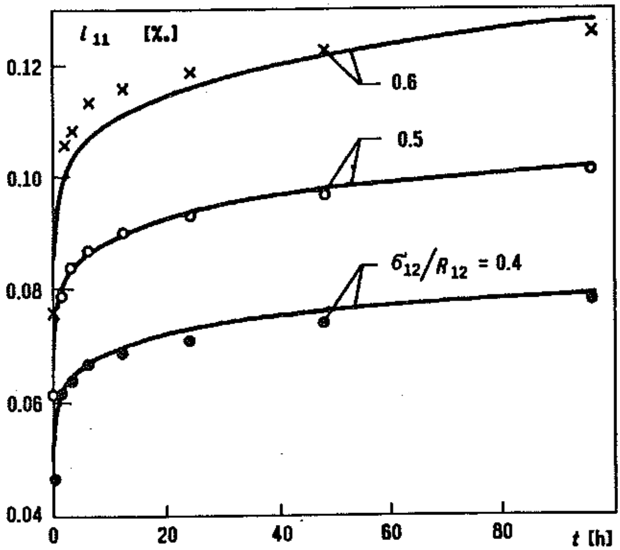

FIG. 2. Creep curves l_{11}: dots, circles and crosses represent experimental data according to (3.18), (3.28) and (3.29).

3. Formulation of the constitutive equations

3.1. *The constitutive equations for tension*

In the theory of the micropolar linear elasticity (i.e. the moment theory), the following constitutive equations are formulated for the anisotropic body

$$(3.1) \qquad \sigma_{kl} = A_{klmn}\varepsilon^e_{mn} + C_{klmn}l^e_{mn}, \qquad m_{kl} = C_{mnkl}\varepsilon^e_{mn} + B_{klmn}l^e_{mn}.$$

The unknown material constants and moment stresses m_{kl} create essential difficulties in the formulation of the constitutive equations (3.1). The nonlinear anisotropic body model (complete anisotropy) was applied to construct the constitutive equations in order to overcome those difficulties.

Averaged micropolar strains $2\bar{l}_{12}$ under tension are presented in the form of the sum of the immediate strains and the creep strain, i.e.

$$(3.2) \qquad 2\bar{l}_{12} = 2\bar{l}^e_{12}(0,\sigma_{11}) + 2\bar{l}^c_{12}(t,\sigma_{11}).$$

In order to determine $2\bar{l}^e_{12}(0,\sigma_{11})$ let us employ the general relationship for nonlinear anisotropic body in the form

$$(3.3) \qquad 2l'_{12} = 2(a'_{12kl}\sigma'_{kl} + a'_{12klmn}\sigma'_{kl}\sigma'_{mn} + a'_{12klmnop}\sigma'_{kl}\sigma'_{mn}\sigma'_{op} + \ldots).$$

Taking the first and the third term for an anisotropic body, it is obtained

$$(3.4) \qquad 2l'_{12} = 2\left(a'_{1211}\sigma_{11} + a'_{12111111}\sigma^3_{11}\right).$$

Incremental angle of twist has, in the presence of Eq. (3.4), the form

$$(3.5) \qquad d(2l'_{12}) = 2\left(a'_{1211} + 3a'_{12111111}\sigma^2_{11}\right)d\sigma_{11}.$$

The deformation of $d\sigma_{11}$ as a function of angle α can be expressed in view of Fig. 3, by

$$(3.6) \qquad d\sigma_{11} = \frac{dF}{S} = \frac{r\,d\alpha \cdot g \cdot \sigma_{11}}{2\pi r g} = \frac{\sigma_{11}\,d\alpha}{2\pi}.$$

The tensorial transformation formulae for the anisotropic body leads to

$$(3.7) \qquad \begin{aligned} a'_{1211} &= \alpha_{1i}\alpha_{2j}\alpha_{1k}\alpha_{1l}a_{ijkl}, \\ a'_{12111111} &= \alpha_{1i}\alpha_{2j}\alpha_{1k}\alpha_{1l}\alpha_{1m}\alpha_{1n}\alpha_{1o}\alpha_{1p}a_{ijklmnop}. \end{aligned}$$

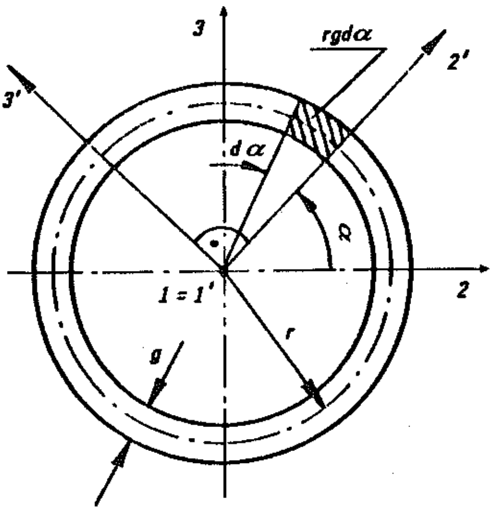

FIG. 3. Cross-section of a specimen and the coordinate system.

Introducing factors a_{ik} into Eqs. (3.7) we obtain

$$
\begin{aligned}
a'_{1211} &= a_{1211}\cos\alpha + a_{1311}\sin\alpha, \\
a'_{12111111} &= a_{12111111}\cos\alpha + a_{13111111}\sin\alpha.
\end{aligned} \tag{3.8}
$$

From formulae (3.5), (3.6) and (3.8) it follows that

$$
2\bar{l}^{e}_{12}(0,\sigma_{11}) = \frac{\sigma_{11}}{\pi}(a_{1211} + a_{1311}) + \frac{3\sigma_{11}^{3}}{\pi}(a_{12111111} + a_{13111111}), \tag{3.9}
$$

or introducing the notations

$$
\begin{aligned}
\tilde{a}_{1211} &= \frac{(a_{1211} + a_{1311})R_{11}}{\pi}, \\
\tilde{a}_{12111111} &= \frac{3(a_{12111111} + a_{13111111})R_{11}^{3}}{\pi},
\end{aligned} \tag{3.10}
$$

it leads to

$$
2\bar{l}^{e}_{12}(0,\sigma_{11}) = \tilde{a}_{1211}\sigma_{11}/R_{11} + \tilde{a}_{12111111}(\sigma_{11}/R_{11})^{3}. \tag{3.11}
$$

The creep deformation is described by the formulae resulting from the similarity of the creep curves

$$
2\bar{l}^{c}_{12}(t,\sigma_{11}) = f_{12}(\sigma_{11})\int_{0}^{t} K_{1211}(\tau)\,d\tau. \tag{3.12}
$$

The creep kernel is assumed in the form

$$K_{1211}(\tau) = t_0^{-d_{1211}} \bar{b}_{1211} \tau^{d_{1211}-1} e^{-c_{1211}(\tau/t_0)^{d_{1211}}}, \tag{3.13}$$

$$t_0 = 1\,\mathrm{h}, \qquad \bar{b}_{1211} = b_{1211} c_{1211} d_{1211}.$$

Substituting Eqs. (3.13) into Eq. (3.12) and integrating, we get

$$2\bar{l}^c_{12}(t, \sigma_{11}) = f_{12}(\sigma_{11}) b_{1211} \left[1 - e^{-c_{1211}(t/t_0)^{d_{1211}}}\right]. \tag{3.14}$$

The function $f_{12}(\sigma_{11})$ is expressed by the formula of the type

$$f_{12}(\sigma_{11}) = \tilde{e}_{1211}\sigma_{11}/R_{11} + \tilde{e}_{12111111}(\sigma_{11}/R_{11})^3. \tag{3.15}$$

Substituting Eq. (3.15) into Eq. (3.14) we obtain

$$2\bar{l}^c_{12}(t, \sigma_{11}) = \left[\tilde{e}_{1211}\sigma_{11}/R_{11} + \tilde{e}_{12111111}(\sigma_{11}/R_{11})^3\right] \times b_{1211} \left[1 - e^{-c_{1211}(t/t_0)^{d_{1211}}}\right]. \tag{3.16}$$

The constants appearing in the formulae (3.11) and (3.16) are determined by means of the least square procedure, and they are

$$\begin{aligned} \tilde{a}_{1211} &= 0.3179 \cdot 10^{-3}, & \tilde{a}_{12111111} &= 12.26 \cdot 10^{-3}, \\ b_{1211} &= 0.436 \cdot 10^{-3}, & c_{1211} &= 0.205, \\ d_{1211} &= 0.334, & \tilde{e}_{1211} &= 0, \qquad \tilde{e}_{12111111} = 22.22. \end{aligned} \tag{3.17}$$

The description of creep by the formula (3.2), taking into account (3.11) and (3.16) with the constants (3.17), is visualized in Fig. 1.

3.2. The constitutive equations for torsion

The second-order creep curves l_{11} accompanying torsion are described similarly as in the formula (3.2). The global strain l_{11} is the sum of the immediate strain and the creep strain

$$l_{11}(t, \sigma_{12}) = l^e_{11}(0, \sigma_{12}) + l^c_{11}(t, \sigma_{12}). \tag{3.18}$$

In order to determine $l_{11}(0, \sigma_{12})$ the general relationship for a nonlinear anisotropic body is employed in the form

$$l_{11} = a'_{11kl}\sigma_{kl} + a'_{11klmn}\sigma_{kl}\sigma_{mn} + a'_{11klmnop}\sigma_{kl}\sigma_{mn}\sigma_{op} + \dots . \tag{3.19}$$

Taking the first and the third term, it can be written

$$l_{11} = a'_{1112}\sigma_{12} + a'_{11121212}\sigma_{12}^3. \tag{3.20}$$

On the other hand, the incremental elongation has, in the presence of Eq. (3.20), the form

$$dl_{11}(0,\sigma_{12}) = (a'_{1112} + 3a'_{11121212}\sigma_{12}^2)\, d\sigma_{12}. \tag{3.21}$$

In order to determine $d\sigma_{12}$ as a function of angle α (Fig. 3), it can be written

$$d\sigma_{12} = \frac{dM \cdot r}{J_0} = \frac{\sigma_{12} r d\alpha \cdot g \cdot r \cdot r}{2\pi r g r^2} = \frac{\sigma_{12} d\alpha}{2\pi}. \tag{3.22}$$

From (3.21) and (3.22), as a result of integration, we obtain

$$l_{11}^e(0,\sigma_{12}) = 4 \int_0^{\pi/2} \left(a'_{1112}(\alpha) + 3a'_{11121212}(\alpha)\sigma_{12}^2\right) \frac{\sigma_{12}}{2\pi}\, d\alpha. \tag{3.23}$$

The tensorial transformation formulas for the anisotropic body lead to

$$\begin{aligned} a'_{1112} &= \alpha_{1i}\alpha_{1j}\alpha_{1k}\alpha_{2l}a_{ijkl}, \\ a'_{11121212} &= \alpha_{1i}\alpha_{1j}\alpha_{1k}\alpha_{2l}\alpha_{1m}\alpha_{2n}\alpha_{1o}\alpha_{2p}a_{ijklmnop}. \end{aligned} \tag{3.24}$$

Substituting factors α_{ik} into Eqs. (3.24), we get

$$\begin{aligned} a'_{1112} &= a_{1112}\cos\alpha + a_{1113}\sin\alpha, \\ a'_{11121212} &= a_{11121212}\cos^3\alpha + 3a_{11121213}\cos^2\alpha\sin\alpha \\ &\quad + 3a_{11121313}\sin^2\alpha\cos\alpha + a_{11131313}\sin^3\alpha. \end{aligned} \tag{3.25}$$

From Eqs. (3.23) and (3.25) we obtain

$$\begin{aligned} l_{11}^e(0,\sigma_{12}) &= \frac{2}{\pi}(a_{1112} + a_{1113})\sigma_{12} \\ &\quad + \frac{6}{\pi}\left(\frac{2}{3}a_{11121212} + a_{11121213} + a_{11121313} + \frac{2}{3}a_{11131313}\right)\sigma_{12}^3, \end{aligned} \tag{3.26}$$

or introducing the notation

$$\begin{aligned} \tilde{a}_{1112} &= \frac{2(a_{1112} + a_{1113})R_{12}}{\pi}, \\ \tilde{a}_{11121212} &= \frac{6}{\pi R_{12}^3}\left(\frac{2}{3}a_{11121212} + a_{11121213} + a_{11121313} + \frac{2}{3}a_{11131313}\right), \end{aligned} \tag{3.27}$$

we get Eq. (3.28).

$$l^e_{11}(0,\sigma_{12}) = \tilde{a}_{1112}\sigma_{12}/R_{12} + \tilde{a}_{11121212}(\sigma_{12}/R_{12})^3. \tag{3.28}$$

An analogous approach leads to $l^c_{11}(t,\sigma_{12})$; using Eq. (3.16) we have

$$l^c_n(t,\sigma_{12}) = \left[\tilde{e}_{1112}\sigma_{11}/R_{11} + \tilde{e}_{11121212}(\sigma_{11}/R_{11})^3\right] \times b_{1112}\left[1 - e^{-c_{1112}(t/t_0)^{d_{1112}}}\right]. \tag{3.29}$$

The constants appearing in the formulae (3.28) and (3.29) are determined by means of the least square procedure.

$$\begin{aligned} &\tilde{a}_{1112} = 0.124\cdot 10^{-3}, && \tilde{a}_{11121212} = 0, && \\ &b_{1112} = 0.201\cdot 10^{-3}, && c_{1112} = 0.126, && d_{1112} = 0.210, \\ &\tilde{e}_{1112} = 1.028, && \tilde{e}_{11121212} = 1.612. && \end{aligned} \tag{3.30}$$

The creep described by the formula (3.18), taking into account Eqs. (3.28) and (3.29) with the constants (3.30) is illustrated by the Fig. 2.

4. Statistical verification of the mathematical models

To assess the accuracy of description, the mean absolute and relative square errors are calculated from the formulae

$$\begin{aligned} r_1 &= \left\{\left[\sum_{i=1}^{N}(y_i - y_t)^2\right]/N\right\}^{1/2}, \\ r_2 &= \left\{\left[\sum_{i=1}^{N}((y_i - y_t)/y_i)^2\right]/N\right\}^{1/2}, \end{aligned} \tag{4.1}$$

where y_i – measured deformation, y_t – theoretical deformation, N – number of measurements. The results of calculations are set in Table 1. In the table are also shown some auxiliary values, necessary to compute the confidence intervals of the specific observations.

The following formulae were applied to calculate the absolute error

$$\Delta_1 = t_{\alpha,\nu}\left\{\sum_{i=1}^{n}\sum_{j=1}^{m}(y_{ij} - \bar{y}_i)/\nu\right\}^{1/2}, \tag{4.2}$$

the relative error

$$\Delta_2^w = t_{\alpha,\nu}\left\{\frac{1}{\nu}\sum_{i=1}^{n}\sum_{j=1}^{m}[(y_{ij}-\bar{y}_i)/\bar{y}_i]^2\right\}^{1/2}, \tag{4.3}$$

where $\alpha - 0.05$, $\nu = nm - n$ – the number of degrees of freedom, n – number of stress levels, m – number of instants at which the strains were measured.

Table 1. Results of statistical calculations.

Statistical quantities	Kind of strain	
	$2\bar{l}_{12}$	l_{11}
r_1 [‰]	0.017	0.003
r_2	0.092	0.036
Δ_1 [‰]	0.067	0.011
Δ_2^w	0.145	0.143

Prior to these calculations, the Cochran test was used to check the variance homogeneity.

5. Conclusions

The second-order creep strains under tension $2\bar{l}_{12}$ and torsion l_{11} are found to be nonlinear. The creep curves are similar. Their correct description can be obtained by taking into account the nonlinear anisotropic body model (complete anisotropy). This is confirmed by Fig. 1 and Table 1 in which $r_1 < \Delta_1$ and $r_2 < \Delta_2^w$ – for $2\bar{l}_{12}$, as well as Fig. 2 and Table 1, where $r_1 < \Delta_1$ and $r_2 < \Delta_2^w$ for l_{11}.

References

1. A.L. Rabinovič, *On calculation of the orthotropic layered panels under tension, bending and shearing* [in Russian], Works of MAP, no. 675, Oborongiz 1948.
2. S.G. Lekhnickij, *Theory of elasticity of the anisotropic body* [in Russian], GITI, 1950.
3. A.N. Mitinskij, *Elastic characteristic of wood considered as orthotropic material* [in Russian], Works of LTA, 63, 1948.

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