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DYNAMIC PLASTIC BEHAVIOUR OF OVERHANGING CIRCULAR PLATE WITH VARIOUS SUPPORT CONDITIONS

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This paper considers the problem of dynamic plastic response of a rigid, perfectly plastic circular plate with overhang subjected to rectangular pressure pulse. The complete analytical solution is obtained by using the yield condition of Tresca. This problem is of particular interest because it contains the simply supported and the built-in circular plates as two limiting cases, and is likely to represent more accurately real support conditions occurring in practice. This theory may be considered as a general approach to the problem of a circular plate satisfying various boundary conditions and subjected to dynamic pressure.

NOTATION

H plate thickness,
$H(t)$ Heaviside unit step function,
m mass per unit plate area,
M_0 $\dfrac{\sigma_0 H^2}{4}$,
M_r, M_θ radial and circumferential bending moments per unit length,
q_0 static collapse load,
q uniformly distributed pressure per unit area,
Q_r transverse shear force per unit length of plate,
r radial coordinate of plate,
R radius of the supporting circle,
t_f response time,
t^* $\dfrac{t_f}{T}$,
T duration of pulse,
w transverse deflection,
w^* $\dfrac{W_{\max}}{\dfrac{qT^2}{m}}$,
α ratio of the radius of plate to that of the supporting circle,
η $\dfrac{q_0 R^2}{6M_0}$,

θ circumferential coordinate,
κ_r, κ_θ radial and circumferential curvatures,
λ $\frac{q}{q_0}$,
ξ $\frac{r_1}{R}$,
σ_0 yield stress,
$(\,)^{\bullet}$ $\frac{\partial}{\partial t}(\,)$,
$(\,)'$ $\frac{\partial}{\partial r}(\,)$.

1. Introduction

The study of the existing literature on the dynamic plastic deformation of circular plates reveals that most of the papers were concerned with either simply supported or clamped plates. HOPKINS and PRAGER [1] studied the dynamic response of simply supported plate to a rectangular pressure pulse. The plate is made of rigid-perfectly plastic material, which is assumed to obey the Tresca yield condition and the associated flow rule. In Ref. [2] WANG and HOPKINS investigated the behaviour of a circular plate with transverse velocity imparted to the entire plate except at the built-in outer edge where the velocity is zero. PERZYNA [3] examined the influence of a pulse of arbitrary shape by developing further the theory of [1] to show that for a given impuse the character of the pressure-time function has little influence on the final shape of the plate. FLORENCE [4] solved the problem of a clamped circular plate loaded by a central rectangular pulse. JONES [5] considered the simultaneous influence of membrance forces and bending moments of a simply supported circular plate subjected to impulsive loading. In Ref. [7], ZHAO and HSUEH analyzed the influence of a damping medium on the large dynamic plastic deflection of a simply supported circular plate subjected to rectangular pressure pulse. Based on the energy equlibrium, YU and CHEN [10] developed a new procedure called Membrane Factor Method to analyze the dynamic plastic response of simply supported circular plates with finite deflections under impulsive loading.

The problem treated in this paper is the response of a circular plate with overhang (Fig. 1) subjected to a rectangular pressure pulse. The plate is of radius αR, simply supported on circle of radius R and the pressure is uniformly distributed over the area interior to the circle of support. The plate is made of rigid-perfectly plastic material, which is assumed to obey the Tresca yield condition and the associated flow rule.

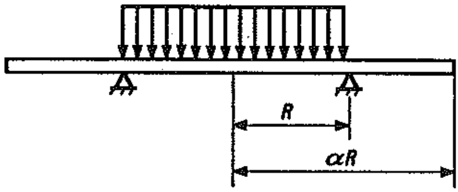

FIG. 1. Circular plate with overhang.

The objective of the present paper is an attempt to link the two distinct boundary conditions (i.e. simply supported or built-in) and it is believed that this theoretical analysis should facilitate the interpretation and understanding of the dynamic characteristics of the plates having complex support conditions.

2. Equations of motion and yield condition

Assume the Tresca yield condition to be satisfied; the corresponding yield locus, drawn in the (M_θ, M_r) plane, is shown in Fig. 2.

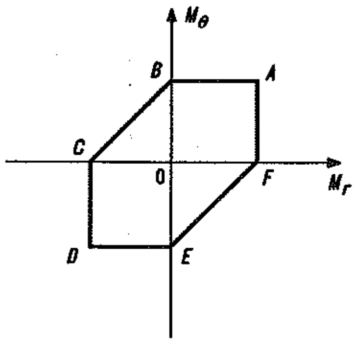

FIG. 2. Tresca yield condition.

The two limiting cases of a circular plate with overhang are the simply supported plate ($\alpha = 1$), and the built-in plate (α sufficiently large, namely $\alpha \geq 2.718$). During deformation the plate is divided into some regions in each of which certain plastic regime exists defined by the vertex or side of the Tresca yield hexagon (Fig. 2). The correct form of the static collapse load may be expressed by the following equation [5]:

$$\eta - \frac{1}{3}\log\eta = 1 + \frac{2}{3}\log\alpha, \tag{2.1}$$

where $\eta = q_0 R^2/(6M_0)$, q_0 and M_0 are the static collapse load and the fully plastic bending moment per unit length of the circular plate of thickness H, respectively.

The fundamental dynamic equation of the plate can be written as

$$(2.2)\qquad (rM_r)' - M_\theta = rQ_r = \int_0^r [-q(r,t) + m\ddot{w}]\, r\, dr,$$

provided the rotary inertia effect and the membrane force are disregarded.

Meanwhile, let

$$(2.3)\qquad \dot{k}_r = -\dot{w}'', \qquad \dot{k}_\theta = -\frac{1}{r}\dot{w}'$$

be the radial and circumferential curvature rates, respectively.

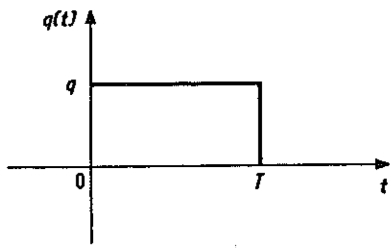

FIG. 3. Rectangular pressure pulse.

The rectangular pressure pulse illustrated in Fig. 3 can be described by

$$(2.4)\qquad q(t) = q\,[H(t) - H(t-T)],$$

where $H(t)$ and T are the Heaviside unit step function and the duration of pulse, respectively.

3. Mechanisms of deformation and solution

The plastic deformation is such that the plate middle surface is divided into three regions, each of them being in a different plastic regime. For simplicity, the present paper considers only the moderate load; for such a load the position of these regions does not vary with time. The flow rule results are required only for the plastic regimes AB and BC, which can be expressed as

$$(3.1)\quad \text{Regime}\begin{cases} AB: & 0 \le M_r \le M_0, & M_\theta = M_0, & \dot{k}_r = 0, \\ BC: & -M_0 \le M_r \le 0, & M_\theta - M_r = M_0, & \dot{k}_\theta > 0. \end{cases}$$

The various mechanical quantities within all the regions must satisfy the appropriate relations (3.1), and certain continuity or discontinuity relations at the boundaries between any two regions mentioned above. The condition of continuity we used here is that the displacement, velocity and radial moment are continuous in r and t. In addition, let the plate be flat and at rest at $t = 0$, then

$$(3.2)\quad \begin{aligned} w(r,0) &= 0 \quad \text{(for the whole plate)}, \\ \dot{w}\,(r,0) &= 0 \quad \text{(for the whole plate)}. \end{aligned}$$

Symmetry demands that at the centre of the plate

$$(3.3)\quad M_r = M_\theta = M_0.$$

For $0 < r < R$ it may be shown that there exists a circle $r = r_1$ at which the radial moment $M_r(r_1, t) = 0$. The relevant plastic regime for $0 \le r \le r_1$ is AB, its associated flow rule requires

$$\dot{w} = \dot{W}_0 + Ar,$$

where $\dot{W}_0$ is the velocity of the centre of the plate, A is an undetermined constant.

The plastic regime for $r_1 \le r \le R$ is BC and its velocity profile satisfies the equation

$$\dot{w}\,(r,t) = B \log\left(\frac{r}{R}\right),$$

where B is an integration constant to be determined by the condition of continuity at $r = r_1$.

Once both A and B are determined, the motion of the plate can be described by

$$\dot{w}(r,t) = \begin{cases} \left[1 - \dfrac{r}{r_1\left(1+\log\dfrac{R}{r_1}\right)}\right]\dot{W}_0, & r \in [0, r_1], \\ \dfrac{\log\dfrac{R}{r}}{1+\log\dfrac{R}{r_1}}\dot{W}_0, & r \in [r_1, R], \\ -\dfrac{\dot{W}_0(r-R)}{R\left(1+\log\dfrac{R}{r_1}\right)}, & r \in [R, \alpha R], \end{cases} \tag{3.4}$$

It is obvious that if $\alpha = 1$, we have $r_1 = R$, and relations (3.4) are immediately reduced to the velocity profile of a simply supported circular plate; on the other hand, if α is sufficiently large ($\alpha \geq 2.718$), relations (3.4) are the velocity profile of the built-in circular plate. For a rectangular pulse of moderate intensity, the motion of the plate is divided into two phases.

1. PHASE I. $0 \leq t \leq T$

1.1. $r \in [0, r_1]$

The acceleration profile in this region is

$$\ddot{w} = \left[1 - \frac{r}{r_1\left(1+\log\dfrac{R}{r_1}\right)}\right]\ddot{W}_0 .$$

Substitution of the above relation into Eq. (2.2) yields

$$rM_r = rM_0 - \frac{qr^3}{6} + \frac{m}{6}r^3\ddot{W}_0 - \frac{m\ddot{W}r^4}{12r_1\left(1+\log\dfrac{R}{r_1}\right)} + C.$$

The unknown constant C must be zero since the value of radial moment at the centre of the plate is bounded; therefore we have

$$M_r = M_0 - \frac{qr^2}{6} + \frac{m}{6}r^2\ddot{W}_0r^2 - \frac{m\ddot{W}r^3}{12r_1\left(1+\log\dfrac{R}{r_1}\right)}, \qquad r \in [0, r_1]. \tag{3.5}$$

Since $M_r(r_1, t) = 0$, thus

$$\ddot{W}_0 = \frac{1+\log\dfrac{R}{r_1}}{1+2\log\dfrac{R}{r_1}}\frac{2}{m}\left(q - \frac{6M_0}{r_1^2}\right). \tag{3.6}$$

1.2. $r \in [r_1, R]$

According to the D'Alembert's principle, the shear force per unit length Q_r satisfies the equation

$$2\pi r Q_r + \pi r^2 q - m \int_0^r 2\pi r \ddot{w}\, dr = 0.$$

This gives

$$(3.7) \qquad rQ_r = -\frac{qr^2}{2} + \frac{m\ddot{W}_0 r_1^2}{6} \frac{1 + 3\log\frac{R}{r_1}}{1 + \log\frac{R}{r_1}} + \frac{m\ddot{W}_0}{1 + \log\frac{R}{r_1}} \left[\frac{r^2 \log\frac{R}{r} - r_1^2 \log\frac{R}{r_1}}{2} + \frac{r^2 - r_1^2}{4} \right].$$

In view of Eqs. (2.2), $(3.1)_2$ and (3.7), it is found that

$$(3.8) \qquad M_r = M_0 \log\frac{r}{r_1} - \frac{q}{4}(r^2 - r_1^2 + \frac{m\ddot{W}_0}{6} r_1^2 \frac{1 + 3\log\frac{R}{r_1}}{1 + \log\frac{R}{r_1}} \log\frac{r}{r_1} + \frac{m\ddot{W}_0}{1 + \log\frac{R}{r_1}} \left[\frac{r^2}{4}\left(1 + \log\frac{R}{r}\right) - \frac{r_1^2}{4}\left(1 + \log\frac{R}{r_1}\right) - \frac{r_1^2}{2} \log\frac{R}{r_1} \log\frac{r}{r_1} - \frac{r_1^2}{4} \log\frac{r}{r_1} \right].$$

1.3. $r \in [R, \alpha R]$

Similarly, the transverse shear force per unit length may be determined by the application of D'Alembert' principle. Thus

$$2\pi r Q_r + m \int_r^{\alpha R} 2\pi r \ddot{w}\, dr = 0.$$

Hence

$$(3.9) \qquad rQ_r = \frac{m\ddot{W}_0}{R\left(1 + \log\frac{R}{r_1}\right)} \left[\frac{(\alpha R)^3 - r^3}{3} - R^2 \frac{(\alpha R)^2 - r^2}{2} \right].$$

It may be shown, using Eqs. $(3.1)_2$ and (2.2) and being aware of the boundary condition $M_r(\alpha R, t) = 0$ that

$$M_r = M_0 \log \frac{r}{\alpha R} + \frac{m\ddot{W}_0}{R\left(1+\log\frac{R}{r_1}\right)}\left[\frac{(\alpha R)^3}{3}\log\left(\frac{r}{\alpha R}\right) - \frac{r^3-(\alpha R)^3}{9} - \frac{\alpha^2 R^3}{2}\log\left(\frac{r}{\alpha R}\right) + \frac{Rr^2-\alpha^2R^3}{4}\right]. \tag{3.10}$$

1.4. The velocity and displacement solution in phase I

Since it is required that the radial moments should be continuous at $r = R$, thus

$$M_r\Big|_{3.8} = M_0 \log \frac{r}{\alpha R} + \frac{m\ddot{W}_0}{R\left(1+\log\frac{R}{r_1}\right)}\left[\frac{(\alpha R)^3}{3}\log\left(\frac{r}{\alpha R}\right) - \frac{r^3-(\alpha R)^3}{9} - \frac{\alpha^2 R^3}{2}\log\left(\frac{r}{\alpha R}\right) + \frac{Rr^2-\alpha^2R^3}{4}\right]. \tag{3.11}$$

Combining Eqs. (3.6) and (3.11), we can obtain the dimensionless equation in r_1, viz

$$\frac{2}{1-2\log\xi}\left[\lambda\eta - \frac{1}{\xi^2}\right]\left[-\frac{\xi^2}{3}\log\xi - \frac{\alpha^2-\xi^2}{4} - \frac{1-\alpha^3}{9} + \frac{\alpha^2}{6}(3-2\alpha)\log\alpha\right] = \frac{1}{6}\log\left(\frac{\alpha}{\xi}\right) - \frac{\lambda\eta}{4}(1-\xi^2), \tag{3.12}$$

where

$$\xi = \frac{r_1}{R}, \qquad \lambda = \frac{q}{q_0}, \qquad \lambda\eta = \frac{qR^2}{6M_0}.$$

As a nonlinear equation, (3.12) can be solved numerically. Once r_1 is determined by Eq. (3.12) for the applied pulse, the acceleration field of the plate is given by Eq. (3.6).

Using Eqs. (3.4) and the initial condition (3.2), it may be shown that the velocity distribution of the whole plate is given by

$$\dot{w}(r,t) = \begin{cases} \dfrac{1-\log\xi-\dfrac{r}{r_1}}{1-2\log\xi}\dfrac{2}{m}\left(q-\dfrac{6M_0}{r_1^2}\right)t, & r\in[0,\, r_1], \\[2ex] \dfrac{\log\dfrac{R}{r}}{1-2\log\xi}\dfrac{2}{m}\left(q-\dfrac{6M_0}{r_1^2}\right)t, & r\in[r_1,\, R], \\[2ex] -\dfrac{\dfrac{r}{R}-1}{1-\log\xi}\dfrac{2}{m}\left(q-\dfrac{6M_0}{r_1^2}\right)t, & r\in[R,\, \alpha R]. \end{cases} \tag{3.13}$$

Integrating Eqs. (3.13) with respect to time we obtain

$$w(r,t) = \begin{cases} \dfrac{1 - \log\xi - \dfrac{r}{r_1}}{1 - 2\log\xi} \dfrac{1}{m} \left(q - \dfrac{6M_0}{r_1^2}\right) t^2, & r \in [0, r_1], \\ \dfrac{\log \dfrac{R}{r}}{1 - 2\log\xi} \dfrac{1}{m} \left(q - \dfrac{6M_0}{r_1^2}\right) t^2, & r \in [r_1, R], \\ -\dfrac{\dfrac{r}{R} - 1}{1 - \log\xi} \dfrac{1}{m} \left(q - \dfrac{6M_0}{r_1^2}\right) t^2, & r \in [R, \alpha R]. \end{cases} \tag{3.14}$$

The analysis of phase I is complete.

2. PHASE II. $T \le t \le t_f$

In this phase $q(t) = 0$, so the acceleration of the centre of the plate is

$$\ddot{W}_0 = -\frac{12M_0}{mr_1^2} \frac{1 - \log\xi}{1 - 2\log\xi}. \tag{3.15}$$

Integrating Eq. (3.15) once and twice with respect to time, we obtain

$$\dot{W}_0(t) = \frac{1 - \log\xi}{1 - 2\log\xi} \frac{2}{m} \left(qT - \frac{6M_0}{r_1^2} t\right), \tag{3.16}$$

$$W_0(t) = \frac{1 - \log\xi}{1 - 2\log\xi} \frac{1}{m} \left[qT(2t - T) - \frac{6M_0}{r_1^2} t^2\right], \tag{3.17}$$

where the constants of integration have been evaluated from the requirements that W_0 and $\dot{W}_0$ are continuous at $t = T$.

The plate will come to rest when $\dot{W}(t_f) = 0$. Hence,

$$t_f = \lambda\eta\xi^2 T. \tag{3.18}$$

The permanent deflection at the centre of the plate is given by

$$W_{\max} = \frac{1 - \log\xi}{1 - 2\log\xi} \frac{qT^2}{m} \left[\lambda\eta\xi^2 - 1\right]. \tag{3.19}$$

It may be shown, using Eqs. (3.5) and (3.6), that

$$M_r''\Big|_{r=0} = -\frac{q}{3} + \frac{2}{3} \frac{1 - \log\xi}{1 - 2\log\xi} \left(q - \frac{6M_0}{r_1^2}\right). \tag{3.20}$$

Thus, the above analysis is correct if

$$q \le (1 - 2\log\xi) \frac{12M_0}{r_1^2}, \tag{3.21}$$

i.e.

$$\lambda\eta\xi^2 \le 2(1-2\log\xi). \tag{3.22}$$

If $q > (1-2\log\xi)\dfrac{12M_0}{r_1^2}$, then the yield condition given by Eqs. (3.1) will be violated and some alternative yield condition must be sought.

The analysis of phase II is complete.

4. Discussion

The relation between the two dimensionless parameters, α and η, in Eq. (2.1) is illustrated in Fig. 4. When the suddenly applied pressure pulse exceeds the static collapse load, the plate will move along the direction of the applied pulse.

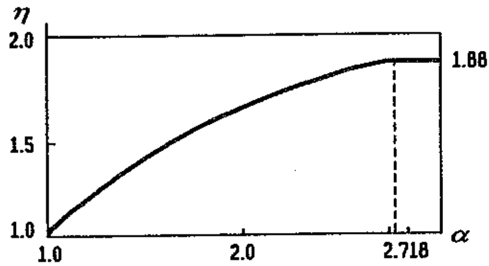

FIG. 4. $\eta \sim \alpha$ curve.

The dimensionless parameter ξ is of central importance in the present paper since it determines the boundary of different regions. To show explicitly the validity of the present method, let us consider again the problem studied by HOPKINS and PRAGER in 1954 [1]. Equation (3.12) will always give $\xi = 1.0$ when $\alpha = 1$ and $\lambda \le 2.0$, and inequality (3.31) holds true if $\alpha = 1.0$; the other formulae are reduced to the well-known relations.

Since the clamped circular plate is a limiting case of the circular plate with overhang, it is easy to understand that its solutions (when the plate is subjected to a rectangular pulse) can be directly obtained when α is equal to 2.718. It may be shown that then we have $\xi = 0.729$, which is exactly the same value as that in the static case [7], cf. Eq. (3.12) with $\alpha = 2.718$ and $\lambda = 1.0$. The curve in Fig. 5 shows the relation between the dimensionless radius ξ and the dimensionless ratio of the applied dynamic load to the static one. The curve in Fig. 5 also shows that the dimensionless radius ξ

decreases monotonically from the initial value $\xi_0 = 0.729$ with increasing dimensionless loading ratio. The velocity profile and the displacement field can be determined directly from the formulae given in the present paper once the dimensionless radius ξ is determined.

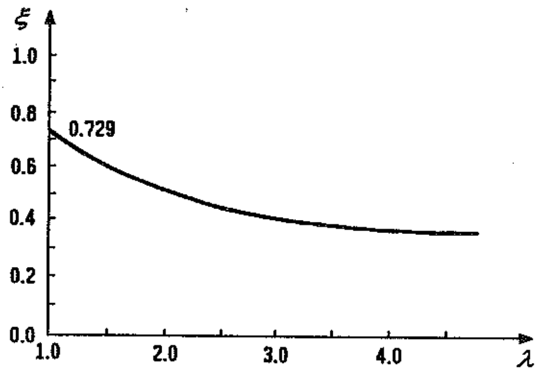

FIG. 5.

In practice, it is often the case that the plate is neither simply supported nor clamped and satisfies certain intermediate supporting conditions. The present theory represents more accurately real conditions enabling us to select appropriate dimensionless parameter α.

It may be shown by numerical calculation that the critical value between the medium and high load is 2.62 if $\alpha = 1.3436$. If $\alpha = 1.6872$, then the corresponding value is 12.09.

The permanent displacement at the center and the total time of response are two values that are perhaps most important for the practice. In order to illustrate the influence of α on $W_{\max}$ and t_f, two dimensionless parameters are introduced here, namely

$$w^* = \frac{W_{\max}}{\dfrac{qT^2}{m}}, \qquad t^* = \frac{t_f}{T}, \tag{4.1}$$

where w^* and t^* are dimensionless permanent displacement at the centre and the dimensionless time of response, respectively. The variation of w^* and t^* with the pressure ratio under certain α is shown in Fig. 6 and Fig. 7, respectively. Figures 6 and 7 show that the greater is the value of α, the smaller are the values of w^* and t^* for the same λ. This is true because larger values of α correspond to more rigid support condition.

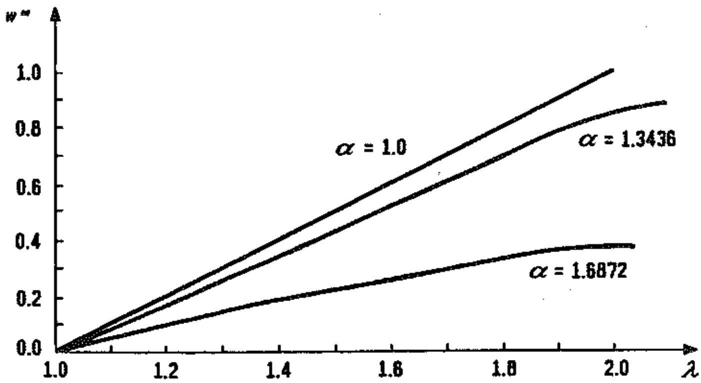

FIG. 6.

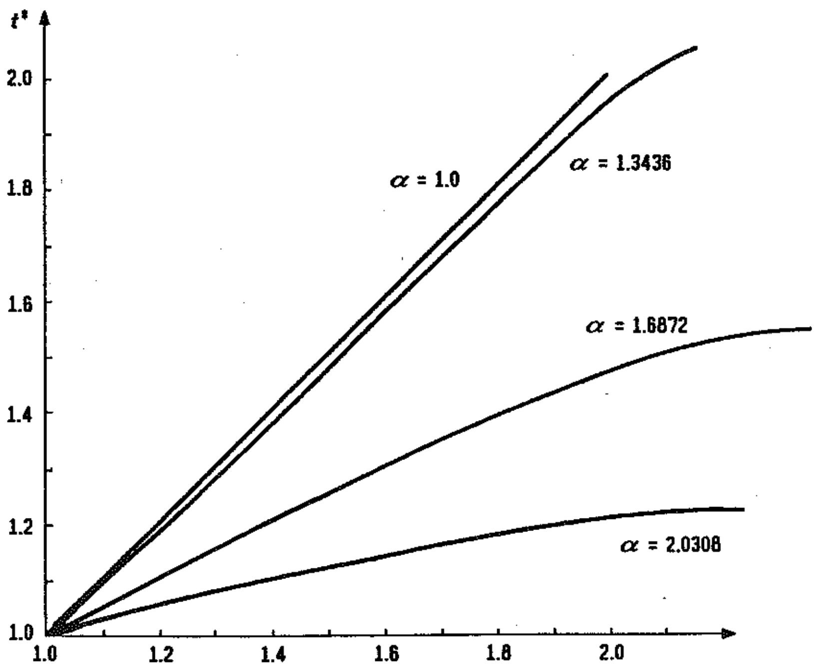

FIG. 7.

5. CONCLUSIONS

A general method has been developed for the analysis of the dynamic plastic behaviour of a circular plate having various support conditions. Simply supported and clamped circular plates are two limiting cases of the present model. The formulae derived in the present paper are reduced to the well-known results if the inertia term is disregarded (i.e. the static case) or when $\alpha = 1.0$ (the simply supported plate studied by Hopkins and Prager). The plastic response of the clamped circular plate subjected to a rectangular pressure pulse is also studied. The load considered is of a moderate size, the analysis concerning high loads will be published in another paper.

ACKNOWLEDGEMENT

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