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SOLUTION FOR AN IRREGULAR PERTURBATION PROBLEM OF VISCOPLASTIC SPHERICAL CONTAINER UNDER INTERNAL PRESSURE

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An irregular perturbation problem around a quasi-static solution for a motion of a thick-walled spherical container made from viscoplastic material and subjected to a time-dependent internal pressure is dealt with. Only the first perturbations are taken into account. A practically important case of a power excess stress function above the static yield stress is considered.

1. Introduction

Theoretical analysis of a dynamic problem for a structure made of a viscoplastic material is fraught with difficulties since, apart from a few exceptions, a nonlinear initial-boundary value problem must be solved. In the applications of the viscoplasticity theory these difficulties have to some extent been overcome by introducing various simplifying assumptions and approximations discussed in [4]. One of the techniques to solve approximately a viscoplastic problem is the perturbation method.

Perturbation procedure around a plastic solution was proposed in [6,10] in order to solve the motion of a thick-walled sphere made of a viscoplastic material described by PERZYNA'S constitutive relationship [5]. The motion was caused either by an ideal impulse or by a time-independent internal pressure. In both cases solutions for a power function of the stress excess was found by using a modified Linstedt–Poincaré method [4]. This modification was presented in detail in [10].

A concept of perturbation around a plastic solution was also employed successfully in [7] and [8] where motions of a single span built-in viscoplastic beam and of a clamped circular plate under an ideal impulse were considered. To obtain a solution with separated variables the material properties were described with the help of a homogeneous constitutive relationship of a viscous type, and the perturbation solutions themselves were arrived at by using the Rayleigh–Schrödinger method. A perturbation problem similar to that solved in [7] was presented in [9]. However, the beam had properties described by a Perzyna's nonhomogeneous constitutive equation. In order to obtain, as before, a solution with separated variables, Galerkin's weighted method was used followed by a perturbation technique of a generalized averaging.

In the present paper a perturbation solution is presented for a quasi-static problem of a thick-walled spherical container subjected to a time-dependent internal pressure. The rigid-viscoplastic properties of the material are described by Perzyna's constitutive equation with a nonlinear function of the stress excess above the state of plastic yielding. An approximate solution to within an accuracy of the first perturbation is constructed by using the method of matched asymptotic expansions. A practically important case of a power function of the stress excess is considered together with a linearly decreasing pressure.

2. The problem and its reduction to the initial value problem

Consider a thick-walled spherical container made of a strain rate sensitive incompressible rigid-perfectly plastic material whose properties are formulated by Perzyna. The aim is to describe a motion of this structure under an internal pressure under the assumption that strains remain small.

Let us denote by r, a, b the current, inner and outer radii, respectively, and by t the time. The radial stresses are denoted by σ_r, the circumferential ones by σ_θ and σ_φ, the internal pressure by p. Due to the spherical symmetry of both the container and its loading, the stresses σ_r and $\sigma_\theta = \sigma_\varphi$ are the principal ones. The radial displacement, its rate and the mass density are denoted by u, v and ϱ, respectively.

With the above notation, the problem is formulated by the incompressibility equation

$$\frac{\partial v}{\partial r} + 2\frac{v}{r} = 0, \tag{2.1}$$

the motion equation

$$\frac{\partial \sigma_r}{\partial r} + 2\frac{\sigma_r - \sigma_\varphi}{r} = \varrho \frac{\partial v}{\partial t}, \tag{2.2}$$

the constitutive relationship

$$(2.3)\qquad \frac{v}{r}=\frac{\gamma}{\sqrt{3}}\Phi\left(\frac{\sigma_\varphi-\sigma_r}{\sqrt{3}k}-1\right),$$

and the kinematic formula

$$(2.4)\qquad \frac{\partial u}{\partial t}=v.$$

Let the initial conditions be the following:

$$(2.5)\qquad v(r,0)=0,\qquad u(r,0)=0,\qquad a\le r\le b,$$

and the boundary conditions take the form:

$$(2.6)\qquad \sigma_r(a,t)=-p(t),\qquad \sigma_r(b,t)=0,\qquad 0\le t\le t_k,$$

where t_k is an instant of time at which the container ceases to move any more. If the motion continues, t_k corresponds to a time instant for which an assumption of small displacements is no longer valid.

In Eq. (2.3) k stands for the yield stress in pure shear, γ is a constant responsible for the viscosity of material and $\Phi(*)$ represents the stress excess function above the static yield stress; $\Phi(0)=0$. The value of γ together with the form of the function $\Phi(*)$ are determined by means of suitable tests on dynamic behaviour of materials

The system of Eqs. (2.1)–(2.4) together with the conditions (2.5) and (2.6) constitutes an initial-boundary value problem for the sought functions $v(r,t)$, $u(r,t)$, $\sigma_r(r,t)$, and $\sigma_\varphi(r,t)$.

Since the solution of the incompressibility Eq. (2.1) can be obained independently of the remaining equations in the form with separated variables r and t, the above formulated initial-boundary value problem can be reduced (see [10]) to the initial value problem for the ordinary differential equations

$$(2.7)\qquad \varrho a(1-\eta)\frac{dv_a}{dt}+2\sqrt{3}k\int\limits_a^b\Phi^{-1}\left(\frac{\sqrt{3}a^2v_a}{\gamma r^3}\right)\frac{dr}{r}=p(t)-2\sqrt{3}k\ln\eta^{-1},$$

$$(2.8)\qquad \frac{du_a}{dt}=v_a,$$

with respect to the radial displacement rate $v_a(t)\equiv v(a,t)$ and the radial displacement $u_a(t)\equiv u(a,t)$ of points at the inner surface of the container in the presence of the conditions

$$(2.9)\qquad v_a(0)\equiv v_{a0}=0,\qquad u_a(0)\equiv u_{a0}=0,$$

where

$$\eta = \frac{a}{b}. \tag{2.10}$$

Further, the spatial distributions of the displacements and their rates are dependent on v_a and u_a according to the formulae

$$v(r,t) = \left(\frac{a}{r}\right)^2 v_a(t), \tag{2.11}$$

$$u(r,t) = \left(\frac{a}{r}\right)^2 u_a(t), \tag{2.12}$$

and, finally, the radial and the circumferential stresses are given by

$$\frac{\sigma_r}{\sqrt{3}\,k} = -\frac{p(t)}{\sqrt{3}\,k}\frac{\frac{a}{r}-\eta}{1-\eta} + A + C, \tag{2.13}$$

$$\frac{\sigma_\varphi}{\sqrt{3}\,k} = 1 - \frac{p(t)}{\sqrt{3}\,k}\frac{\frac{a}{r}-\eta}{1-\eta} + A + C + \Phi^{-1}\left(\frac{\sqrt{3}\,a^2 v_a}{\gamma r^3}\right), \tag{2.14}$$

where A and C are expressed by

$$\frac{1}{2}A = \ln\frac{r}{b} + \frac{\frac{a}{r}-\eta}{1-\eta}\ln\eta^{-1}, \tag{2.15}$$

$$\frac{1}{2}C = \int_r^b \Phi^{-1}\left(\frac{\sqrt{3}\,a^2 v_a}{\gamma r^3}\right)\frac{dr}{r} - \frac{\frac{a}{r}-\eta}{1-\eta}\int_a^b \Phi^{-1}\left(\frac{\sqrt{3}\,a^2 v_a}{\gamma r^3}\right)\frac{dr}{r}. \tag{2.16}$$

3. Limiting quasi-static problem

Denote by f an arbitrarily chosen function from among $v(r,t)$, $u(r,t)$, $v_a(r,t)$, $u_a(r,t)$, $\sigma_r(r,t)$, $\sigma_\varphi(r,t)$ that describe the motion in the formulated dynamic problem. Let f^0 be a counterpart of f taken from the set of functions $v^0(r,t)$, $u^0(r,t)$, $v_a^0(r,t)$, $u_a^0(r,t)$, $\sigma_r^0(r,t)$, $\sigma_\varphi^0(r,t)$ that describe the motion when the mass density of the material is negligibly small, i.e. $\varrho \to 0$. To obtain the limiting problem for $\varrho \to 0$, ϱ in the Eqs. (2.1)–(2.6) must vanish and the functions f are replaced by their limiting counterparts f^0. As a consequence, the motion equation (2.2) becomes an equilibrium equation

$$\frac{\partial \sigma_r^0}{\partial r} + 2\frac{\sigma_r^0 - \sigma_\varphi^0}{r} = 0, \tag{3.1}$$

and the remaining Eqs. (2.1), (2.3), (2.4), together with the initial conditions (2.5) and boundary conditions (2.6), remain unchanged. Thus, the problem is reduced to a quasi-static one which – as before – is expressed by

$$2\sqrt{3}\,k\int_a^b \Phi^{-1}\left(\frac{\sqrt{3}\,a^2v_a^0}{\gamma r^3}\right)\frac{dr}{r} = p(t) - 2\sqrt{3}\,k\ln\eta^{-1}, \tag{3.2}$$

$$\frac{du_a^0}{dt} = v_a^0, \tag{3.3}$$

$$v_a^0(0) \equiv v_{a0}^0 = 0, \qquad u_a^0(0) \equiv v_{a0}^0 = 0, \tag{3.4}$$

in the variables: displacement rate $v_a^0(t)$ and displacement $u_a^0(t)$. These can be also obtained by transformation of the relations (2.7)–(2.9) to the limiting case, i.e. for $\varrho = 0$. In addition, let us notice that the form of the relations (2.11)–(2.16) remains unchanged during the limiting procedure. Finally, let us add that the value

$$p_s = 2\sqrt{3}\,k\ln\eta^{-1} \tag{3.5}$$

appearing in (3.2) denotes the pressure at which in the quasi-static process the full yielding of a rigid-ideally plastic container takes place.

A particular property of the discussed limiting case is its singularity (irregularity) consisting in that the ordinary differential equation (2.7) becomes an algebraic equation (3.2) in $v_a^0(t)$. Thus, the obtained rate $v_a^0(t)$ cannot, in general, satisfy the condition $(3.4)_1$ and, for the same reason, the rate distribution $v^0(r,t)$ determined from (2.11) cannot satisfy the initial condition $(2.5)_1$. As a consequence, while formulating a quasi-static problem the condition $(2.5)_1$ (and therefore also $(3.4)_1$) has to be rejected as an initial one. If not rejected, it must be treated as a constraint on an admissible class of functions $p(t)$ of the applied load process. On specifying the Eq. (3.2) for an initial instant $t = 0$ and then using $(3.4)_1$ we can conclude, in the presence of the property of the excess function $\Phi^{-1}(0) = 0$, that an admissible class of the pressure functions must be constrained by the condition $p(0) = p_s$.

With sufficiently small value of ϱ a solution of a quasi-static problem is known to play an essential role since it is close to the exact solution of a dynamic problem except for a small time interval in the vicinity of an initial time instant. In this so-called limiting layer the exact solution changes rapidly to obtain the properties imposed by the initial conditions. The smaller the mass density, the thinner the limiting layer and the solutions are closer to each other. Thus, apart from the quasi-static solution, the knowledge of a solution near the exact one is also necessary in the limiting layer.

The quasi-static solution plays also another important role, namely, it allows to choose such comparative values while reducing the problem to a nondimensional form which enable us to apply successfully a suitable perturbation technique for the considered problem.

To learn the basic properties of a quasi-static solution let us first note that, determining the rate $v_a^0(t)$ from (3.2), we can directly solve the initial value problem (3.3), $(3.4)_2$ and obtain

$$u_a^0(t) = \int_0^t v_a^0(\xi)\, d\xi. \tag{3.6}$$

The largest strain intensity in the wall of the container at an arbitrary instant t is given by(1)

$$\varepsilon_{ai}^0(t) = \frac{3}{a} \int_0^t v_a^0(\xi)\, d\xi. \tag{3.7}$$

Due to the fact that the deformation process is purely dissipative, $v_a^0 \geq 0$, and due to the properties of the stress excess function, $\Phi^{-1}(*) \geq 0$, hence the left-hand side of the relation (3.2) is always non-negative. That is we conclude that the motion takes place when $p(t) > p_s$ and, as seen from (3.6) and (3.7), both functions $u_a^0(t)$ and $\varepsilon_{ai}^0(t)$ are increasing.

Assume that the pressure function in the range $[0, t_1]$ is continuous and such that $p(t) > p_s$. If(2) $\varepsilon_{ai}(t_1) \leq \varepsilon_p$, the range of validity of the solution for the quasi-static problem (3.2), (3.3), $(3.4)_1$ terminates at the instant t_1. If, to the contrary, $\varepsilon_{ai} > \varepsilon_p$, the terminating time instant $t_\varepsilon^0 < t_1$ must be assumed in order to remain within the small deformation theory; t_ε^0 has to

(1)The largest strain is calculated from the formula

$$\varepsilon_i = \frac{1}{2}\left[(\varepsilon_r - \varepsilon_\varphi)^2 + (\varepsilon_\varphi - \varepsilon_\theta)^2 + (\varepsilon_\theta - \varepsilon_r)^2\right]^{\frac{1}{2}} = 3a^2 u_a^0(t) r^{-3},$$

for $r = a$, $\varepsilon_{ai}^0(t) \equiv \varepsilon_i(a, t)$.

(2)ε_p in this inequality denotes an admissible intensity of plastic strain (for the small deformation theory) satisfy the equation

$$\varepsilon_p = \frac{3}{a} u_a^0 \left(t_\varepsilon^0\right) = \frac{3}{a} \int_0^{t_\varepsilon^0} v_a^0(\xi)\, d\xi. \tag{3.8}$$

In the above cases both final rates $v_a^0(t_1)$ and $v_a^0(t_\varepsilon^0)$ cannot vanish.

The motion of the viscoplastic container in the quasi-static situation ceases at such an instant $t_f^0 \le t_1$ that – starting from the zero instant – the equation

$$p(t_f^0) - p_s = 0, \tag{3.9}$$

and the inequality

$$\varepsilon_{ai}^0 \left(t_f^0\right) \le \varepsilon_p, \tag{3.10}$$

are satisfied for the first time, i.e. when the internal pressure becomes equal to the full yielding pressure in the container. When the inequality (3.10) is not satisfied, the terminating instant is t_ε^0. In this case, in the range of validity of the solution of the problem (3.2), (3.3) and $(3.4)_2$, i.e. $[0, t_\varepsilon^0]$, the container continues to move.

Let us notice that the relationship (3.2) differentiated with respect to time yields the expression

$$\frac{6ka^2}{\gamma} \frac{dv_a^0}{dt} \int_a^b \left[\Phi^{-1}\left(\frac{\sqrt{3}\, a^2 v_a^0}{\gamma r^3}\right)\right]' \frac{dr}{r^4} = \frac{dp}{dt}, \tag{3.11}$$

where

$$\left[\Phi^{-1}\left(\frac{\sqrt{3}\, a^2 v_a^0}{\gamma r^3}\right)\right]' = \left.\frac{d\Phi^{-1}(x)}{dx}\right|_{x=\frac{\sqrt{3}\, a^2 v_a^0}{\gamma r^3}}, \tag{3.12}$$

is a positive, decreasing function. From this it follows that the derivatives of the rate v_a^0 and the pressure p have the same signs and their extremum values – if present – must appear at the same time instant t_e^0 such that

$$\left.\frac{dp}{dt}\right|_{t=t_e^0} = 0. \tag{3.13}$$

Since the function $\Phi^{-1}(*)$ increases monotonically with its argument, Eq.(3.2) shows that – if the rate does not reach its maximum value – its largest value

v^0_{an} occurs at the same instant t^0_n as the maximum value $p_n \equiv p(t^0_n)$ takes place.

Consider now a particularly important practical case of a power function of the stress excess $\Phi(*) = (*)^n$ with a natural n. For such a function Eq. (3.2) takes the form

$$\frac{2kn}{\sqrt{3}}\left(1-\eta^{\frac{3}{n}}\right)\left(\frac{\sqrt{3}}{\gamma a}v^0_a\right)^{\frac{1}{n}} = p(t) - p_s, \tag{3.14}$$

from which the rate

$$v^0_a(t) = \frac{\gamma a}{\sqrt{3}}\left(\frac{\sqrt{3}\,(p(t)-p_s)}{2kn\left(1-\eta^{\frac{3}{n}}\right)}\right)^n \tag{3.15}$$

can be determined. After substituting it into (3.6), the displacement

$$u^0_a(t) = \frac{\gamma a}{\sqrt{3}}\left(\frac{\sqrt{3}}{2kn\left(1-\eta^{\frac{3}{n}}\right)}\right)^n \int\limits_0^t (p(t)-p_s)^n\,dt \tag{3.16}$$

is arrived at.

Finally, using (3.7) the strain intensity is found to be equal to

$$\varepsilon^0_{ai} = \gamma\sqrt{3}\left(\frac{\sqrt{3}}{2kn\left(1-\eta^{\frac{3}{n}}\right)}\right)^n \int\limits_0^t (p(t)-p_s)^n\,dt. \tag{3.17}$$

Moreover, let the function $p(t)$ in the interval $[0, t_1]$ decrease linearly according to the expression

$$p(t) = (1+h)\left(1-\frac{t}{t_1}\right)p_s, \tag{3.18}$$

where $h > 0$ describes the pressure excess above p_s at an initial time. If so, the formulae (3.15)–(3.17) take the form

$$v^0_a(t) = \frac{\gamma a}{\sqrt{3}}\left(\frac{\sqrt{3}\,p_s}{2kn\left(1-\eta^{\frac{3}{n}}\right)}\right)^n\left[h-(1+h)\frac{t}{t_1}\right]^n, \tag{3.19}$$

$$u^0_a(t) = \frac{\gamma a t_1}{\sqrt{3}(1+h)(n+1)}\left(\frac{\sqrt{3}\,p_s}{2kn\left(1-\eta^{\frac{3}{n}}\right)}\right)^n \times\left\{h^{n+1}-\left[h-(1+h)\frac{t}{t_1}\right]^{n+1}\right\}, \tag{3.20}$$

$$\varepsilon_{ai}^{0}(t)=\frac{\sqrt{3}\gamma t_1}{(1+h)(n+1)}\left(\frac{\sqrt{3}p_s}{2kn\left(1-\eta^{\frac{3}{n}}\right)}\right)^n \times\left\{h^{n+1}-\left[h-(1+h)\frac{t}{t_1}\right]^{n+1}\right\}. \tag{3.21}$$

From the equation (3.19) it is readily seen that the displacement rate attains its largest value v_{an}^{0} at an initial time,

$$v_{an}^{0}\equiv v_a^0(0)=\frac{\gamma a}{\sqrt{3}}\left(\frac{\sqrt{3}p_s h}{2kn\left(1-\eta^{\frac{3}{n}}\right)}\right)^n. \tag{3.22}$$

This rate becomes zero, according to (3.9), when

$$t_f^0=\frac{h}{1+h}t_1. \tag{3.23}$$

To this instant of time corresponds the strain intensity

$$\varepsilon_{ai}^{0}(t_f^0)=\frac{\sqrt{3}\gamma t_1}{(1+h)(n+1)}\left(\frac{\sqrt{3}p_s}{2kn\left(1-\eta^{\frac{3}{n}}\right)}\right)^n h^{n+1}. \tag{3.24}$$

From the above relationship it follows that, in order to satisfty the inequality (3.10) at a given h, the instant t_1 at which the pressure drops to zero must satisfy the condition

$$t_1\leq\frac{(1+h)(n+1)}{\sqrt{3}\gamma}\left(\frac{\sqrt{3}p_s}{2kn\left(1-\eta^{\frac{3}{n}}\right)}\right)^{-n} h^{-(n+1)}. \tag{3.25}$$

Under this condition the motion ceases to continue at the instant t_f^0 and the final displacement, not violating the range of infinitesimal strains, is found to be

$$u_{af}^{0}\equiv u_a^0\left(t_f^0\right)=\frac{\gamma a t_1}{\sqrt{3}(1+h)(n+1)}\left(\frac{\sqrt{3}p_s}{2kn\left(1-\eta^{\frac{3}{n}}\right)}\right)^n h^{(n+1)} =\frac{v_a^0(0)t_f^0}{n+1}. \tag{3.26}$$

If, to the contrary, at a given h the instant t_1 does not satisfy the inequality (3.25), then $\varepsilon_{ai}^{0}(t_f^0)>\varepsilon_p$ and the range of validity of the solution

(3.19), (3.20) is bounded by an instant t_ε^0, satisfying the Eq. (3.8),

$$(3.27) \qquad t_\varepsilon^0 = \frac{t_1}{1+h}\left\{h - \left[h^{n+1} - \frac{\varepsilon_p(1+h)(n+1)}{\sqrt{3}\gamma t_1}\left(\frac{\sqrt{3}p_s}{2kn\left(1-\eta^{\frac{3}{n}}\right)}\right)^{-n}\right]^{\frac{1}{n+1}}\right\}$$

$$= t_f^0\left[1-\left(1-\frac{\varepsilon_p}{\varepsilon_{ai}^0\left(t_f^0\right)}\right)^{\frac{1}{n+1}}\right] < t_f^0.$$

To obtain $(3.27)_2$ the relationship (3.24) was used. Substituting (3.27) into (3.19) and (3.20) and making use of (3.22) and (3.24), the final rate

$$(3.28) \qquad v_{a\varepsilon}^0 \equiv v_a^0\left(t_\varepsilon^0\right) = v_a^0(0)\left(1-\frac{\varepsilon_p}{\varepsilon_{ai}^0\left(t_f^0\right)}\right)^{\frac{n}{n+1}},$$

and the final displacement

$$(3.29) \qquad u_{a\varepsilon}^0 \equiv u_a^0\left(t_\varepsilon^0\right) = \frac{v_a^0(0)t_f^0}{n+1}\frac{\varepsilon_p}{\varepsilon_{ai}^0\left(t_f^0\right)} < u_{af}^0$$

are obtained.

In what follows, for the convenience of introducing the dimensionless magnitudes, the larger of the rates v_{ae}^0, v_{an}^0 in the quasi-static process will be denoted by v_{am}^0 and the corresponding time instant by t_m^0; the final instant of the range of validity of the solution will be denoted by t_k^0, and the final displacement – by u_{ak}^0.

4. Nondimensional form of the problem

To express the problem (2.1)–(2.6) in a nondimensional form, the reference magnitudes for the independent variables r, t, for the sought variables v, u, σ_r, σ_φ and for the pressure p should be introduced. Since the problem has already been reduced to the equations (2.13), (2.14) in the stress components and to the initial value problem (2.7)–(2.9), the nondimensional form of only t, v_a, u_a – and, as seen in (2.11) and $(2.12)_1$ – of v, u as well as p must be determined. As follows from Sec. 3, the comparative magnitudes for t, v_a, u_a should be t_k^0, v_{am}^0, u_{am}^0, respectively. The pressure p will be compared with the pressure p_s corresponding to the quasi-static yielding of the considered container. Selection of the remaining magnitudes can be virtually arbitrary. However, as follows from the formulae (2.11)–(2.14), it appears convenient to assume a and $\sqrt{3}\,k$ as comparative magnitudes for the radius a and the stress components σ_r, σ_φ, respectively.

The following nondimensional independent variables

$$r^* = \frac{r}{a}, \qquad t^* = \frac{t}{t_k^0}, \tag{4.1}$$

nondimensional dependent variables

$$v^* = \frac{v}{v_{am}^0}, \qquad u^* = \frac{u}{u_{ak}^0}, \qquad \sigma_r^* = \frac{\sigma_r}{\sqrt{3}\,k}, \qquad \sigma_\varphi^* = \frac{\sigma_\varphi}{\sqrt{3}\,k}, \tag{4.2}$$

and the nondimensional pressure

$$p^*(t^*) = \frac{p\,(t^* t_k^0)}{2\sqrt{3}\,k \ln \eta^{-1}} \tag{4.3}$$

are introduced.

Let us note that from $(4.2)_{1,2}$ we obtain the nondimensional variables

$$v_a^* = \frac{v_a}{v_{am}^0}, \qquad u_a^* = \frac{u_a}{u_{ak}^0}, \tag{4.4}$$

that are necessary to reduce the initial value problem (2.7)–(2.9) to its nondimensional form.

With the use of the above quantities the initial-boundary value problem can be written down as

$$\frac{\partial v^*}{\partial r^*} + 2\frac{v^*}{r^*} = 0, \tag{4.5}$$

$$\frac{\partial \sigma_r^*}{\partial r^*} + 2\frac{\sigma_r^* - \sigma_\varphi^*}{r^*} = \frac{2\ln\eta^{-1}}{1-\eta}\beta\frac{\partial v^*}{\partial t^*}, \tag{4.6}$$

$$\sigma_\varphi^* - \sigma_r^* = \Phi^{-1}\left(\delta\frac{v^*}{r^*}\right), \tag{4.7}$$

$$\frac{\partial u^*}{\partial t^*} = \nu v^*, \tag{4.8}$$

$$v^*(r^*,0) = 0, \qquad u^*(r^*,0) = 0, \qquad 1 \le r^* \le \eta^{-1}, \tag{4.9}$$

$$\sigma_r^*(1,t^*) = -2p^*(t^*)\ln\eta^{-1}, \qquad \sigma_r^*\left(\eta^{-1},t^*\right) = 0, \qquad 0 \le t^* \le t_k^*, \tag{4.10}$$

where

$$\beta = \frac{\varrho a\,(1-\eta)\,v^0_{am}}{p_s t^0_k}\,, \tag{4.11}$$

$$\delta = \frac{\sqrt{3}\,v^0_{am}}{\gamma a}\,, \qquad \nu = \frac{v^0_{am}\,t^0_k}{u^0_{ak}}\,, \tag{4.12}$$

are positive-valued nondimensional parameters.

The relationships (2.11), (2.12) take a simple form

$$v^*(r^*,t^*) = \frac{v^*_a(r^*,t^*)}{r^{*2}}\,, \tag{4.13}$$

$$u^*(r^*,t^*) = \frac{u^*_a(r^*,t^*)}{r^{*2}}, \tag{4.14}$$

and the formulae (2.13)–(2.16) can be rewritten in the form

$$\sigma^*_r = -2p^*(t^*)\,\frac{r^{*-1}-\eta}{1-\eta}\ln\eta^{-1} + A(r^*;\eta) + C(r^*;v^*_a;\eta,\delta), \tag{4.15}$$

$$\sigma^*_\varphi = 1 - 2p^*(t^*)\,\frac{r^{*-1}-\eta}{1-\eta}\ln\eta^{-1} + A(r^*;\eta) + C(r^*;v^*_a;\eta,\delta) + \Phi^{-1}\left(\delta\frac{v^*_a}{r^{*3}}\right), \tag{4.16}$$

$$\frac{1}{2}A(r^*;\eta) = \ln r^*\eta + \frac{r^{*-1}-\eta}{1-\eta}\ln\eta^{-1}, \tag{4.17}$$

$$\frac{1}{2}C(r^*;v^*_a;\eta,\delta) = \int_{r^*}^{-1}\Phi^{-1}\left(\delta\frac{v^*_a}{r^{*3}}\right)\frac{dr^*}{r^*} + \frac{r^{*-1}-\eta}{1-\eta}\int_1^{\eta^{-1}}\Phi^{-1}\left(\delta\frac{v^*_a}{r^{*3}}\right)\frac{dr^*}{r^*}\,. \tag{4.18}$$

Finally, the initial value problem (2.7)–(2.9) assumes the form

$$\beta\frac{dv^*_a}{dt^*} + \frac{1}{ln\eta^{-1}}\int_1^{\eta^{-1}}\Phi^{-1}\left(\delta\frac{v^*_a}{r^{*3}}\right)\frac{dr^*}{r^*} = p^*(t^*) - 1, \qquad v^*_a(0) = 0, \tag{4.19}$$

$$\frac{du^*_a}{dt^*} = \nu v^*_a, \qquad u^*_a(0) = 0. \tag{4.20}$$

In the case of a power function of the stress excess $\Phi(*) = (*)^n$, the integrals in (4.18) and (4.19) are equal to

$$\int_{r^*}^{\eta-1} \Phi^{-1}\left(\delta \frac{v_a^*}{r^{*3}}\right) \frac{dr^*}{r^*} = \frac{n}{3}\left(r^{*-\frac{3}{n}} - \eta^{\frac{3}{n}}\right) \delta^{\frac{1}{n}} v_a^{*\frac{1}{n}}, \tag{4.21}$$

$$\int_{1}^{\eta-1} \Phi^{-1}\left(\delta \frac{v_a^*}{r^{*3}}\right) \frac{dr^*}{r^*} = \frac{n}{3}\left(1 - \eta^{\frac{3}{n}}\right) \delta^{\frac{1}{n}} v_a^{*\frac{1}{n}}. \tag{4.22}$$

Denoting

$$\alpha_n = \frac{n}{3} \frac{1 - \eta^{\frac{3}{n}}}{\ln \eta^{-1}} \delta^{\frac{1}{n}}, \tag{4.23}$$

the relationship (4.18) takes the form

$$C(r^*; v_a^*; \eta, \delta) = \alpha_n B(r^*; \eta, n) v_a^{*\frac{1}{n}}, \tag{4.24}$$

in which

$$\frac{1}{2} B(r^*; \eta, n) = \frac{\ln \eta^{-1}}{1 - \eta^{\frac{3}{n}}} \left[r^{*-\frac{3}{n}} - \eta^{\frac{3}{n}} - \frac{\left(r^{*-1} - \eta\right)\left(1 - \eta^{\frac{3}{n}}\right)}{1 - \eta} \right]. \tag{4.25}$$

Consequently, for the power function of the excess the formulae (4.15) and (4.16) for the stress components are given by

$$\sigma_r^* = -2p^*(t^*) \frac{r^{*-1} - \eta}{1 - \eta} \ln \eta^{-1} + A(r^*; \eta) - \alpha_n B(r^*; \eta, n) v_a^{*\frac{1}{n}}, \tag{4.26}$$

$$\sigma_\varphi^* = 1 - 2p^*(t^*) \frac{r^{*-1} - \eta}{1 - \eta} \ln \eta^{-1} + A(r^*; \eta) - \alpha_n \left[B(r^*; \eta, n) - \frac{3}{n} \frac{\ln \eta^{-1}}{1 - \eta^{\frac{3}{n}}} r^{*-\frac{3}{n}} \right] v_a^{*\frac{1}{n}}, \tag{4.27}$$

and the initial value problem (4.19), (4.20) is expressed by

$$\beta \frac{dv_a^*}{dt^*} + \alpha_n v_a^{*\frac{1}{n}} = p^*(t^*) - 1, \qquad v_a^*(0) = 0, \tag{4.28}$$

$$\frac{du_a^*}{dt^*} = \nu v_a^*, \qquad u_a^*(0) = 0. \tag{4.29}$$

On denoting $p_m^* \equiv p^*(t_m^{0*})$, the following two parameters appear:

$$\beta = \frac{\varrho \gamma a^2 (1 - \eta)}{\sqrt{3}\, p_s t_k^0} \left(\frac{3}{n} \frac{\ln \eta^{-1}}{1 - \eta^{\frac{3}{n}}} \right)^n (p_m^* - 1)^n, \tag{4.30}$$

$$\alpha_n = p_m^* - 1, \tag{4.31}$$

which results from (3.15), (4.11), (4.12) and (4.23).

5. Perturbation solution

Let us observe that, on the basis of the relations (4.11) or (4.30), when $\varrho \to 0$ the dimensionless parameter $\beta \to 0$. With the assumed values of the remaining magnitudes, such a ϱ can always be found that makes the parameter β sufficiently small.

Assume that in Eqs. $(4.28)_1$, $(4.29)_1$, the parameter β is small and the remaining parameters α_n, ν are not. As stated in Sec.3, the inital perturbation problem (4.28), (4.29) with respect to β becomes irregular and in order to solve it one of the singular perturbation theory techniques must be employed (see, e.g. [3] or [5]).

In this paper the method of matched asymptotic expansions will be employed. It consists of four stages: solution of an outer problem, i.e. construction of an expansion valid after a certain short lapse of time measured from an initial time, solution of an inner problem, i.e. the determination of an expansion in the neighbourhood of the initial time, making these two expansions matched and, finally, composition of these two solutions into one, the so-called uniformly valid asymptotic expansion in the whole time interval under consideration, see e.g. [2] or[4].

5.1. Outer problem

Outer problem is obtained from the initial value problem (4.28), (4.29) as a limiting case for

$$\beta \to 0, \qquad t^* - \text{steady time}, \tag{5.1}$$

with simultaneous rejection of the initial conditions $(4.28)_2$ and $(4.29)_2$. The asymptotic expansions with respect to the small parameter β for the outer solutions v_a^{*0}, u_a^{*0} are assumed to have the form

$$v_a^{*0}(t^*;\beta) = v_{a0}^{*0}(t^*) + \beta v_{a1}^{*0}(t^*) + \cdots, \tag{5.2}$$

$$u_a^{*0}(t^*;\beta) = u_{a0}^{*0}(t^*) + \beta u_{a1}^{*0}(t^*) + \cdots. \tag{5.3}$$

Substituting the expansions (5.2), (5.3) into the Eqs. $(4.28)_1$ and $(4.29)_1$ and equating the expressions of the same order with respect to β we obtain the relations

$$\alpha_n v_{a0}^{*0\frac{1}{n}} = p^*(t^*) - 1, \tag{5.4}$$

$$\frac{du_{a0}^{*0}}{dt^*} = \nu v_{a0}^{*0}(t^*), \tag{5.5}$$

for the zero perturbations v_{a0}^{*0}, u_{a0}^{*0}, and the relations

$$\frac{\alpha_n}{n} v_{a0}^{*0\,\frac{1}{n}-1} v_{a1}^{*0} = -\frac{dv_{a0}^{*0}}{dt^*}, \tag{5.6}$$

$$\frac{du_{a1}^{*0}}{dt^*} = \nu v_{a1}^{*0}(t^*), \tag{5.7}$$

for the first perturbations v_{a1}^{*0}, u_{a1}^{*0}.

From the relations (5.4)–(5.7) the general solution follows,

$$v_{a0}^{*0} = \frac{1}{\alpha_n^n}\left[p^*(t^*) - 1\right]^n, \tag{5.8}$$

$$u_{a0}^{*0} = D_0 + \frac{\nu}{\alpha_n^n}\int_0^{t^*}\left[p^*(\xi) - 1\right]^n d\xi, \tag{5.9}$$

$$v_{a1}^{*0} = -\frac{n^2}{\alpha_n^{2n}}\left[p^*(t^*) - 1\right]^{2(n-1)}\frac{dp^*}{dt^*}, \tag{5.10}$$

$$u_{a1}^{*0} = D_1 - \frac{n^2}{2n-1}\frac{\nu}{\alpha_n^{2n}}\left[(p^*(t^*) - 1)^{2n-1} - (p^*(0) - 1)^{2n-1}\right], \tag{5.11}$$

in which two constants D_0 and D_1 appear to be determined later on by matching the expansions.

5.2. *Inner problem*

In order to determine an expansion for a limiting layer, we must first make it larger by extensional transformation of the time t^* in the initial problem,

$$\tau = \frac{t^*}{\beta}, \tag{5.12}$$

that follows from the analysis of the orders of smallness of the terms entering the Eqs. $(4.28)_1$, $(4.29)_1$. After this transformation the inner problem takes the form

$$\beta\frac{dv_a^{*i}}{d\tau} + \alpha_n v_a^{*i\,\frac{1}{n}} = p^*(\beta\tau) - 1, \qquad v_a^{*i}(0) = 0, \tag{5.13}$$

$$\frac{du_a^{*i}}{d\tau} = \beta\nu v_a^{*i}, \qquad u_a^{*i}(0) = 0, \tag{5.14}$$

where new notation for the sought functions is introduced, namely $v_a^{*i} \equiv v_a^*(\beta\tau)$, $u_a^{*i} \equiv u_a^*(\beta\tau)$. Similarly as in the case of an outer problem, the asymptotic expansion with respect to the small parameter β has a power series form

$$v_a^{*i}(\tau;\beta) = v_{a0}^{*i}(\tau) + \beta v_{a1}^{*i}(\tau) + \cdots, \tag{5.15}$$

$$u_a^{*i}(\tau;\beta) = u_{a0}^{*i}(\tau) + \beta u_{a1}^{*i}(\tau) + \cdots. \tag{5.16}$$

These expansions are substituted into the relations (5.13), (5.14) and the process of an inner limiting transition follows,

$$\beta \to 0, \qquad \tau - \text{steady time}. \tag{5.17}$$

For the power expansions this process consists in the procedure of equating the coefficients of the same powers of small parameter at both sides of the obtained equations and the initial conditions. A nonlinear initial problem is thus arrived at,

$$\frac{dv_{a0}^{*i}}{d\tau} + \alpha_n {v_{a0}^{*i}}^{\frac{1}{n}} = p^*(0) - 1, \qquad v_{a0}^{*i}(0) = 0, \tag{5.18}$$

$$\frac{du_{a0}^{*i}}{d\tau} = 0, \qquad u_{a0}^{*i}(0) = 0, \tag{5.19}$$

for the zeroth perturbations v_{a0}^{*i}, u_{a0}^{*i}, and a linear one

$$\frac{dv_{a1}^{*i}}{d\tau} + \frac{\alpha_n}{n} {v_{a0}^{*i}}^{\frac{1}{n}-1} v_{a1}^{*i} = p^{*'}(0)\tau, \qquad v_{a1}^{*i}(0) = 0, \tag{5.20}$$

$$\frac{du_{a1}^{*i}}{d\tau} = \nu v_{a0}^{*i}, \qquad u_{a1}^{*i}(0) = 0, \tag{5.21}$$

fot the first perturbations v_{a1}^{*i}, u_{a1}^{*i}, in which

$$p^{*'}(0) = \left.\frac{dp^*}{dt^*}\right|_{t^*=0}. \tag{5.22}$$

The inital value problem can be integrated by separation of variables. However, the solution $\tau = \tau(v_{a0}^{*i})$ is obtained in the form of a transcendental function which cannot be inverted in an explicit manner in order to solve the problem (5.20), (5.21) for the first perturbations. That is why the solution to the problem (5.18), (5.19) has to be expressed in a parametric form

$$\bar{v}_{a0}^{*i}(\zeta;\beta) = \left(\frac{p^*(0)-1}{\alpha_n}\zeta\right)^n, \qquad 0 \le \zeta < 1, \tag{5.23}$$

(5.24)
$$\tau = \frac{n}{\alpha_n^n}(p^*(0)-1)^{n-1}\int_0^\zeta \frac{\xi^{n-1}}{1-\xi}\,d\xi$$
$$= -\frac{n}{\alpha_n^n}(p^*(0)-1)^{n-1}\left[\sum_{i=1}^{n-1}\frac{1}{n-i}\zeta^{n-i}+\ln(1-\zeta)\right],$$

(5.25)
$$u_{a0}^{*i}(\tau)\equiv 0,$$

where the notation $\bar{v}_{a0}^{*i}(\zeta)\equiv v_{a0}^{*i}(\tau(\zeta))$ is introduced and the transformation (5.24) in (5.20), (5.21) has also to be performed. On simultaneous using the solution (5.23), the problem emerges

(5.26)
$$\frac{d\bar{v}_{a1}^{*i}}{d\zeta}+\frac{1}{1-\zeta}\bar{v}_{a1}^{*i}=p^{*\prime}(0)\tau\frac{d\tau}{d\zeta}, \qquad \bar{v}_{a1}^{*i}(0)=0,$$

(5.27)
$$\frac{d\bar{u}_{a1}^{*i}}{d\zeta}=\nu\left(\frac{p^*(0)-1}{\alpha_n}\zeta\right)^n\frac{d\tau}{d\zeta}, \qquad \bar{u}_{a1}^{*i}(0)=0,$$

for the first perturbations $\bar{v}_{a1}^{*i}(\zeta)\equiv v_{a1}^{*i}(\tau(\zeta))$, and $\bar{u}_{a1}^{*i}(\zeta)\equiv u_{a1}^{*i}(\tau(\zeta))$.

On substituting into (5.26), (5.27) the derivative $\dfrac{d\tau}{d\zeta}$ determined from (5.24), we can tackle the above problem and obtain the relations

(5.28)
$$\bar{v}_{a1}^{*i}=p^{*\prime}(0)\left[\frac{n}{\alpha_n^n}(p^*(0)-1)^{n-1}\right]^2(1-\zeta)\int_0^\zeta\frac{\xi^{n-1}}{(1-\xi)^2}\int_0^\xi\frac{\chi^{n-1}}{1-\chi}\,d\xi\,d\chi,$$

(5.29)
$$\bar{u}_{a1}^{*i}=\nu\frac{n}{\alpha_n^{2n}}(p^*(0)-1)^{2n-1}\int_0^\zeta\frac{\xi^{2n-1}}{1-\xi}\,d\xi$$
$$=-\nu\frac{n}{\alpha_n^{2n}}(p^*(0)-1)^{2n-1}\left[\sum_{i=1}^{2n-1}\frac{1}{2n-i}\zeta^{2n-i}+\ln(1-\zeta)\right].$$

5.3. Matching of expansions

To match the zeroth perturbations let us use Prandtl's criterion which requires the following limits to be equal:

(5.30)
$$\lim_{t^*\to 0}v_{a0}^{*0}=\lim_{\tau\to\infty}v_{a0}^{*i}, \qquad \lim_{t^*\to 0}u_{a0}^{*0}=\lim_{\tau\to\infty}u_{a0}^{*i}.$$

Let us at the same time observe that the integral in the formula (5.24) tends to infinity when its upper limit tends to unity. Thus we have

(5.31)
$$\lim_{\tau\to\infty}v_{a0}^{*i}=\lim_{\zeta\to 1}\bar{v}_{a0}^{*i}.$$

The criterion $(5.30)_1$ is reduced to the condition

$$\lim_{t^* \to 0} v_{a0}^{*0} = \lim_{\zeta \to 1} \bar{v}_{a0}^{*i}, \tag{5.32}$$

which – as can be readily verified by using (5.8), (5.23) – is already fulfilled.

Let us now proceed to satisfy the criterion $(5.30)_2$. To this end let us notice that from the relation (5.9) it follows that its left-hand side is equal to the constant D_0. From the relation (5.25) we conclude that its right-hand side vanishes. To satisfy the discussed criterion we must assume that $D_0 = 0$ which reduces (5.9) to the formula

$$u_{a0}^{*0} = \frac{\nu}{\alpha_n^n} \int_0^{t^*} [p^*(\xi) - 1]^n \, d\xi. \tag{5.33}$$

To determine the constant D_1, that appears in the first perturbation of the expansion for the displacement, we must match two-term expansions. Experience shows that, in general, matching of multi-term expansions is much more difficult than the matching of single-term expansions. Van Dyke's [4] principle is here widely used as a relatively simple one. It is postulated that a p-term inner expansion of an r-term outer expansion should be equal to an r-term outer expansion of a p-term inner expansion.

However, to employ this method of matching, as well as each of other matchings proposed by a number of authors, it is necessary to make inner expansions to be the functions of the time τ. In our case the expansions are functions of a time-like parameter ζ. To make use of the quoted principle, a function $\tau = \tau(\zeta)$ should be found, inverse to the function (5.24), and inserted into the relations (5.23), (5.28), (5.29). However, except for $n = 1$ the function (5.24) is transcendental and its exact inversion is clearly impossible. The case $n = 1$ is much less interesting since the initial value problem (4.28), (4.29) reduces to a linear one having a closed-form solution. Nevertheless, the case $n = 1$ can play an instructive role and will be dealt with below.

Thus, to determine D_1 we make use of the matching rule with $p = r = 2$. At $n = 1$ the relation (5.24) yields an inversion

$$\zeta = 1 - \exp(-\alpha_1 \tau), \tag{5.34}$$

which, after substituting into the perturbation (5.33), (5.11), (5.29), enables us – in the light of the relation (5.25) – to express the displacement as follows:

- two-term outer expansion with $t^* = \beta\tau$

$$u_a^{*0} = \frac{\nu}{\alpha_1 n}\int_0^{\beta\tau} [p^*(\xi) - 1]\, d\xi + \beta\left[D_1 - \frac{\nu}{\alpha_1^2}\left(p^*(\beta\tau) - p^*(0)\right)\right],$$

- two-term inner expansion with $\tau = \dfrac{t^*}{\beta}$

$$u_a^{*i} = -\beta\frac{\nu}{\alpha_1^2}\left(p^*(0) - 1\right)\left(1 - \exp\left(-\alpha_1\frac{t^*}{\beta}\right) - \alpha_1\frac{t^*}{\beta}\right),$$

- two-term inner expansion of the two-term outer expansion

$$\left(u_a^{*0}\right)^i = \beta\left[\frac{\nu}{\alpha_1}(p^*(0) - 1)\tau + D_1\right],$$

- two-term outer expansion of the two-term inner expansion

$$\left(u_a^{*i}\right)^0 = \frac{\nu}{\alpha_1}(p^*(0) - 1)t^* - \beta\frac{\nu}{\alpha_1^2}\left(p^*(0) - 1\right).$$

- Finally, on comparing the last two expansions we get the relation

$$D_1 = -\beta\frac{\nu}{\alpha_1^2}\left(p^*(0) - 1\right), \qquad n = 1.$$

In the next section it will be shown that in our problem the lack of knowledge of D_1 for $n \neq 1$ does not make it impossible to construct – both for the displacement and for its rate – a two-term combined asymptotic expansion uniformly valid in the whole range of the solution sought.

5.4. Combination of expansions

As stated before, the outer expansion is not valid in the vicinity of an initial instant and outside this interval it is the inner expansion that is not valid. To obtain such an expansion that remains valid in the whole time interval under consideration, let us create the following compositions [4]:

for the displacement rate

$$v_a^{*c} = v_a^{*0} + v_a^{*i} - \left(v_a^{*0}\right)^i = v_a^{*0} + v_a^{*i} - \left(v_a^{*i}\right)^0, \tag{5.35}$$

and for the displacement

$$u_a^{*c} = u_a^{*0} + u_a^{*i} - \left(u_a^{*0}\right)^i = u_a^{*0} + u_a^{*i} - \left(u_a^{*i}\right)^0. \tag{5.36}$$

Due to the equality

$$(5.37)\qquad \left(u_a^{*0}\right)^i = \left(u_a^{*i}\right)^0,$$

and the matching procedure for $p = r$, both methods of creating composed expansions are equivalent.

Let us first find, with the use of $(5.35)_1$, a one-term composed expansion for the rates. To this end, replace in (5.8) t^* by $\beta\tau$ and expand the result with respect to the parameter β. Preserving the first term only, we get

$$(5.38)\qquad \left(u_{a0}^{*0}\right)^i = \left(\frac{p^*(0)-1}{\alpha_n}\right)^n,$$

which, in the light of the perturbation (5.8) and (5.23), yields an expansion

$$(5.39)\qquad v_a^{*c} = \left(\frac{p^*(t^*)-1}{\alpha_n}\right)^n + \left(\frac{p^*(0)-1}{\alpha_n}\zeta\right)^n - \left(\frac{p^*(0)-1}{\alpha_n}\right)^n.$$

As to the displacement, the relation $(5.36)_2$ appears more convenient to be used since, as seen from (5.25), a one-term outer expansion of the zeroth displacement perturbation is equal to zero. As a consequence, in the presence of (5.33) the conclusion is that

$$(5.40)\qquad u_a^{*c} = u_{a0}^{*0} = \frac{\nu}{\alpha_n^n}\int_0^{t^*}\left[p^*(\xi)-1\right]^n d\xi.$$

The formulae (5.12), (5.24), (5.39) and (5.40) constitute a very simple parametric form of the asymptotic solution to the initial value problem (4.28), (4.29) when $\beta \to 0$.

Let us now compose the two-term expansions. For rates, from the formulae (5.8) and (5.10) we create a two-term expansion (5.2) with simultaneous substitution of $\beta\tau$ in place of the time t^*. Next, we expand the resulting expression with respect to β confining ourselves to two terms only. We get

$$(5.41)\qquad \left(v_a^{*0}\right)^i = \left(\frac{p^*(0)-1}{\alpha_n}\right)^n + \beta\frac{n}{\alpha_n^n}(p^*(0)-1)^{n-1}p^{*\prime}(0)\left[\tau - \frac{n}{\alpha_n^n}(p^*(0)-1)^{n-1}\right].$$

Making use of the formulae (5.23) and (5.29) we create a two-term inner expansion that, together with the two-term outer expansion and the recent result (5.41), is inserted into $(5.35)_1$. As a result, a two-term composed rate expansion is arrived at in the form

$$v_a^{*c} = \left(\frac{p^*(t^*)-1}{\alpha_n}\right)^n + \left(\frac{p^*(0)-1}{\alpha_n}\zeta\right)^n - \left(\frac{p^*(0)-1}{\alpha_n}\right)^n + \beta\left(-\frac{n^2}{\alpha_n^{2n}}(p^*(t^*)-1)^{2(n-1)}p^{*\prime}(t^*) - \frac{n}{\alpha_n^n}(p^*(0)-1)^{n-1}p^{*\prime}(0)\left[\tau - \frac{n}{\alpha_n^n}(p^*(0)-1)^{n-1}\right] + p^{*\prime}(0)\left[\frac{n}{\alpha_n^n}(p^*(0)-1)^{n-1}\right]^2(1-\zeta)\int_0^\zeta \frac{\xi^{n-1}}{(1-\xi)^2}\int_0^\xi \frac{\chi^{n-1}}{1-\chi}\,d\xi\,d\chi\right). \tag{5.42}$$

The same procedure is used in the case of the displacements. Two-term outer expansion is made (cf. Sec. 5.3) by substituting the perturbations (5.33) and (5.11) and replacing the time t^* by the product $\beta\tau$. Next, the result is expanded with respect to the parameter β which yields a two-term inner expansion of the two-term outer expansion in the form

$$\left(v_a^{*0}\right)^i = \beta\left[D_1 + \frac{\nu}{\alpha_n^n}(p^*(0)-1)^n\tau\right]. \tag{5.43}$$

Further on, by substituting the formulae (5.25), (5.29) into (5.16), a two-term inner expansion is obtained. Finally, we substitute the last result together with the created two-term expansions into (5.36) and obtain the composed expansion sought

$$u_a^{*c} = \frac{\nu}{\alpha_n^n}\int_0^{t^*}(p^*(\xi)-1)^n\,d\xi + \beta\left(-D_1 - \frac{\nu}{\alpha_n^n}(p^*(0)-1)^n\tau + D_1 - \frac{n^2}{2n-1}\frac{\nu}{\alpha_n^{2n}}\left[(p^*(t^*)-1)^{2n-1} - (p^*(0)-1)^{2n-1}\right] - \nu\frac{n}{\alpha_n^{2n}}(p^*(0)-1)^{2n-1}\left[\sum_{i=1}^{2n-1}\frac{1}{2n-i}\zeta^{2n-i} + \ln(1-\zeta)\right]\right). \tag{5.44}$$

To emphasize that (5.44) is independent of D_1, this constant is purposefully left to make us see its cancelling. The expansions (5.42) and (5.44) are the first improvement on the asymptotic solution (5.39) and (5.40) for small values of β.

6. Example

To demonstrate the practical usefulness of the obtained solution, let us assume that the container is made of mild steel with $k = 147.15\,\mathrm{MPa}$, $\rho = 7800\,\mathrm{kgm^{-3}}$, $n = 5$ and $\gamma = 40.4$. In addition, let $a = 0.1\,\mathrm{m}$ and the diameter ratio be $\eta = 0.6$. Assume the pressure to decrease linearly according to the formula (3.18) and to vanish at $t_1 = 0.0008\,\mathrm{s}$. Under such type of pressure the basic comparative magnitudes v^0_{am}, u^0_{ak}, t^0_k can be reasonably assumed as constants v^0_{an}, u^0_{af}, t^0_f, defined by the formulae (3.22), (3.26) and (3.27), respectively. Thus, in the formulae $(4.1)_2$, $(4.2)_{1,2}$ and (4.4) we put

$$v^0_{am} = v^0_{an}, \qquad u^0_{ak} = u^0_{af}, \qquad t^0_k = t^0_f, \tag{6.1}$$

while a manner of introduction of the remaining nondimensional variables remains unchanged.

As follows from (3.18) and (4.3), the pressure function assumes now the form

$$p^*(t) = 1 + h(1 - t^*), \tag{6.2}$$

which, in view of (4.30) and (4.31), yields the expressions for the dimensionless parameters

$$\beta = \frac{\varrho\gamma a^2(1-\eta)(1+h)}{6kht_1 \ln \eta^{-1}} \left(\frac{3}{n} \frac{h \ln \eta^{-1}}{1 - \eta^{\frac{3}{n}}} \right)^n, \tag{6.3}$$

$$\alpha_n = h. \tag{6.4}$$

Finally, using (3.22), (3.23) and (3.24), we can – for a parameter ν defined by $(4.12)_2$ – find a simple solution

$$\nu = n + 1. \tag{6.5}$$

The expressions (6.3)–(6.5) represent a set of parameters that appears in the initial value problems (4.28) and (4.29).

The values of the small parameter β as dependent on the inner radius a for the pressure excesses $h = (1, 0.5)$ and the radii ratios $\eta = (0.2, 0.4, 0.6, 0.8)$ are shown in Fig. 1. A conclusion may be drawn that in the considered range of η, for $a = 0.1\,\mathrm{m}$ and the value $h = 1$, the values β are significantly lower, $\beta < 0.04$. Similar conclusion remains true for $h = 0.5$, when $\beta < 0.0025$. In particular, as it directly follows from (6.3), for the value $h = (0.5, 0.8)$

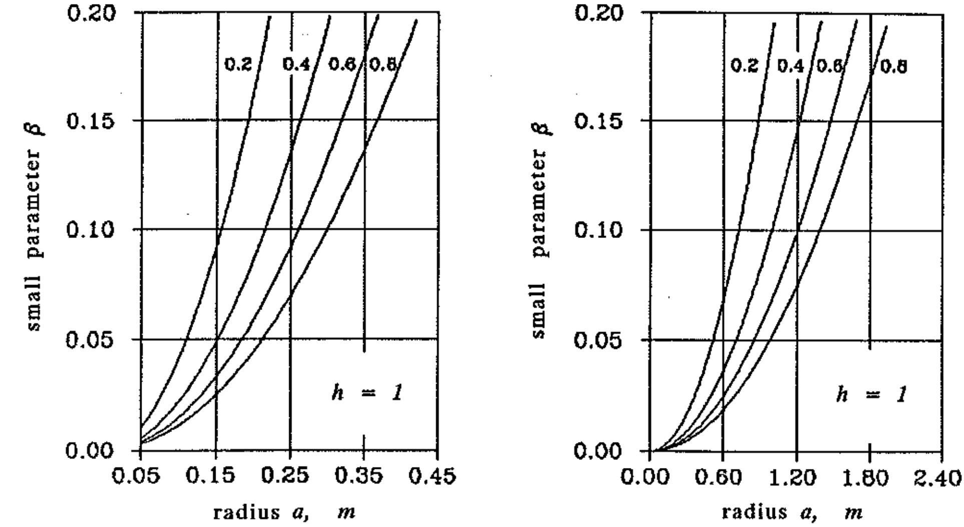

FIG. 1. Small parameter as a function of the inner radius for selected radii ratios and pressure excesses.

we have, respectively, $\beta = (0.0007, 0.0054)$. The ratios β/h are definitely small, $\beta/h = (0.0014, 0.0067)$.

Making use of the values from (6.2)–(6.5) in the Eqs. (4.28), (4.29) and remembering the assumed data, the initial value problem is formulated as

$$\beta\frac{dv_a^*}{dt^*} + hv_a^{*\frac{1}{5}} = h(1-t^*), \qquad v_a^*(0) = 0, \tag{6.6}$$

$$\frac{du_a^*}{dt^*} = 6v_a^*, \qquad u_a^*(0) = 0, \tag{6.7}$$

with the parameter β significantly smaller than the excess h. Under these conditions the presented perturbation solution can be successfully used. From the practical point of view, the solution will be limited to an asymptotic one (5.39), (5.40) and (5.24). Surprisingly simple relationships are found to describe the motion:

$$v_{a0}^{*c} = (1-t^*)^5 + \xi^5 - 1, \tag{6.8}$$

$$u_{a0}^{*c} = 1 - (1-t^*)^6, \tag{6.9}$$

$$t^* = -\beta\frac{5}{h}\left[\frac{1}{4}\zeta^4 + \frac{1}{3}\zeta^3 + \frac{1}{2}\zeta^2 + \zeta + \ln(1-\zeta)\right]. \tag{6.10}$$

To test their applicability, let us compare the results with the numerical solution of the problem (6.6), (6.7).

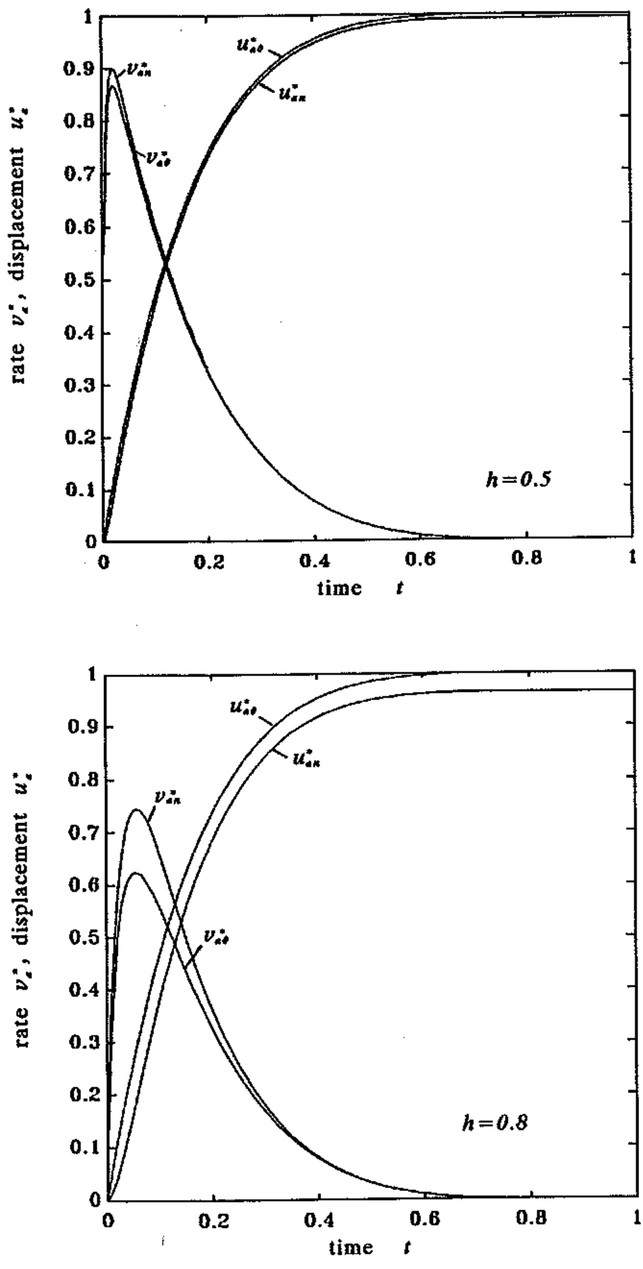

FIG. 2. Displacement rates and displacements for two excesses of pressure.

The diagrams of the rates v_a^* and displacements u_a^* for two typical pressure excesses $h = (0.5, 0.8)$ are drawn in Fig. 2. The curves following from the numerical solution are denoted by v_{an}^*, u_{an}^*, while the perturbation method provides the curves v_{a0}^*, u_{a0}^*. It is readily seen that the compared curves are similar and show good quantitative agreement. Moreover, the rate values obtained from the asymptotic solution, initially lower than the numerical ones, starting from a certain instant begin to coincide (within the scale of the figure). In the case of displacements the situation is reversed: asymptotic values are larger than the numerical ones over the whole range.

Let us finally notice that the largest differences in rates occur near the maximum values, and the largest differences in displacements take place near the terminal values. These differences grow as the pressure excess increases from 0.5 to 0.8.

The curves showing the maximum rates and final displacements as functions of the pressure excess are shown in Fig. 3. The numerical results are denoted by v_{am}^* and u_{ak}^*, the asymptotic results are designated by v_{aom}^* and u_{aok}^*. Increase in the differences of the maximum rates is rather sensitive to the increase in h while the sensitivity of the final differences in displacements is remarkably less pronounced. It is worth emphasizing that the asymptotic value of the rate constitutes a lower bound on the numerically obtained rate and that the asymptotic value of the displacement is the upper bound of the numerical one.

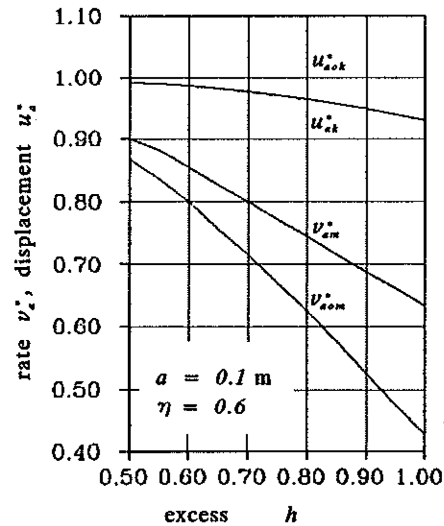

FIG. 3. Maximum rate and final displacement.

Figures 4 and 5 are complementary to the previous curves; they make it possible to compare the times t^*_{vmn}, t^*_{vmo} at which the maximum rate occurs and the times t^*_{kn}, t^*_{ko} of the termination of motion as a function of the excess h. The duration of maximum rate at the increase of the excess grows significantly but the increase of the final time of motion is negligible. The differences of the corresponding instants depend on the excess h in a weak manner. Lastly, the times obtained from the asymptotic formulae are lower bounds on the times derived from the numerical solutions.

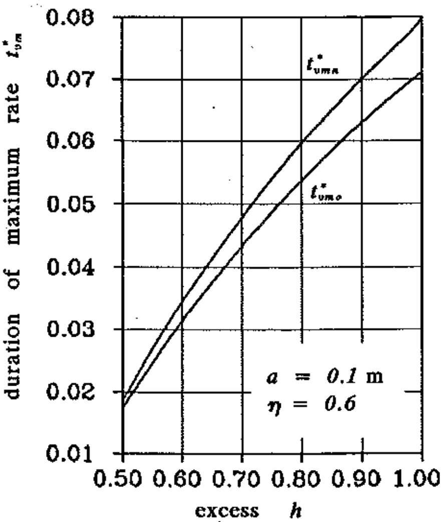

FIG. 4. Duration of the maximum rate.

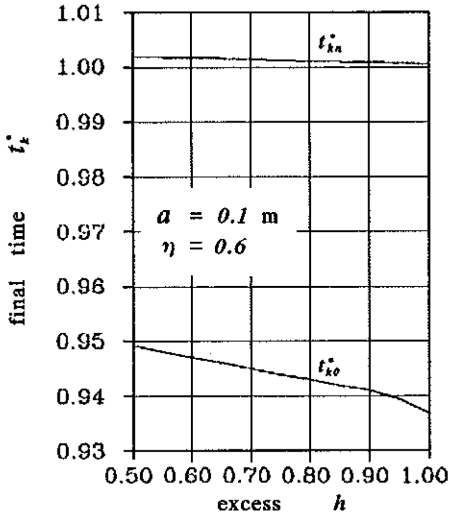

FIG. 5. Final time of the motion.

7. Conclusions

The comparison of results presented above allows us to expect that, with $\rho \to 0$ the obtained asymptotic solution and the perturbation solution, of the dynamic problem for a thick-walled rigid-viscoplastic spherical container made from steel described by Perzyna's constitutive equation, can also be applicable to other metals.

The asymptotic solution not only allowed us to describe approximately the motion, but also to assess such its properties as the maximum rate, final displacement and duration of the process. However, the fact should not be overlooked that the simplicity of the solution reduces its practical application to small inner radii and the pressure excesses not larger than one.

The results of the paper suggest that further investigations should be continued to assess the magnitudes that characterize the dynamic expansion of a thick-walled sphere with small mass density.

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