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FREE VIBRATIONS OF TIMOSHENKO BEAMS ON RIGID BLOCKS

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The aim of the paper is to analyze the vibration of Timoshenko beam resting on the most arbitrary elastically flexible supports. The boundary conditions are defined by means of a 2×2 flexibility matrix, and the structure is discretized according to the so-called cell procedure. The Lagrangian coordinates are selected to be the vertical displacements of the end points of the rigid bars, plus the four possible displacements of the external constraints. A numerical example is worked out in which the obtained results are compared with some known results.

1. Introduction

Free vibration analysis of beams structures is the basic model of a large number of structural problems. Sometimes, the classical Euler – Bernoulli hypothesis is unsatisfactory because shear flexibility and rotatory inertia are considered to be negligible. On the other hand, towers, antennae, shear walls, movable arms, and deep beams are strongly influenced by both these factors and therefore must be analysed by adopting the more refined Timoshenko theory. According to this theory, some authors studied the beam model and useful results are now available for different boundary conditions, so that all the major practical problems can be considered as solved.

In particular, all the authors uncouple the translational flexibilities from the angular flexibility when analysing beams on flexible supports.

The aim of this paper is to use the so-called cells discretization method in order to allow for a more general case, in which the boundary flexibilities are of the most arbitrary type. In this way it is possible to simulate the real behaviour of beams supported on rigid blocks, which in turn are supposed to be immersed in an elastic Winkler foundation. Each block is defined by a full 2×2 flexibility matrix, and all the classical boundary conditions (clamped beam, simply supported beam, cantilever beam, etc) can be easily specified as limiting cases.

The structure will be discretized into t rigid bars, linked together by elastic constrains (henceforth cells), according to a discretization method which was already adopted to investigate the static and dynamic behaviour of various structural systems [1–3]. In this paper we take into account the shear deformations and the rotatory inertia, so that the structure becomes a $(2t+4)$-degrees-of-freedom system. It is convenient to assume as Lagrangian coordinates the vertical displacements of the end points of each rigid bar [4], plus the 4 displacements of the blocks. The paper ends with a numerical example in which a beam with various boundary conditions is examined and some results from the literature can be compared. Moreover, a cantilever beam with a concentrated mass at its free end is closely examined, because this system seems to be of paramount importance in the analysis of towers, antennae and flexible arms [5–6].

2. Discretization method

Let us consider a system shown in Fig. 1, which has been obtained by dividing the beam into t rigid bars, resting on rigid blocks, resting on rigid blocks, which in turn are immersed in a Winkler foundation. The bars are connected together by means of elastic cells, in which the bending strain energy and the shearing strain energy are concentrated. Consequently, the possible deformed shapes of the structure are univocally defined by the values of $2t$ vertical displacements of the end points of the rigid bars and by 4 possible displacements of the blocks. The distributed mass of each bar is lumped in the most natural way by placing half of the total mass at each end. The vector of the Lagrangian coordinates is therefore given by

$$\mathbf{c}^T = [v_1, v_2, v_3 \ldots, v_{2t}, r_A, r_B, v_A, v_B], \tag{2.1}$$

where v_i are the vertical displacements of the end points of each rigid bar, r_A, r_B are rotations of the blocks and v_A, v_B are the vertical displacements of the blocks. The equation of motion will be formulated by means of the Lagrange equations, and consequently it is necessary to express the total potential energy and the kinetic energy as functions of the Lagrangian coordinates c.

Let us assume that l is the span of the beam, $I(z)$ is the second moment of area, E is Young's modulus, G is the shear modulus, $A(z)$ is the

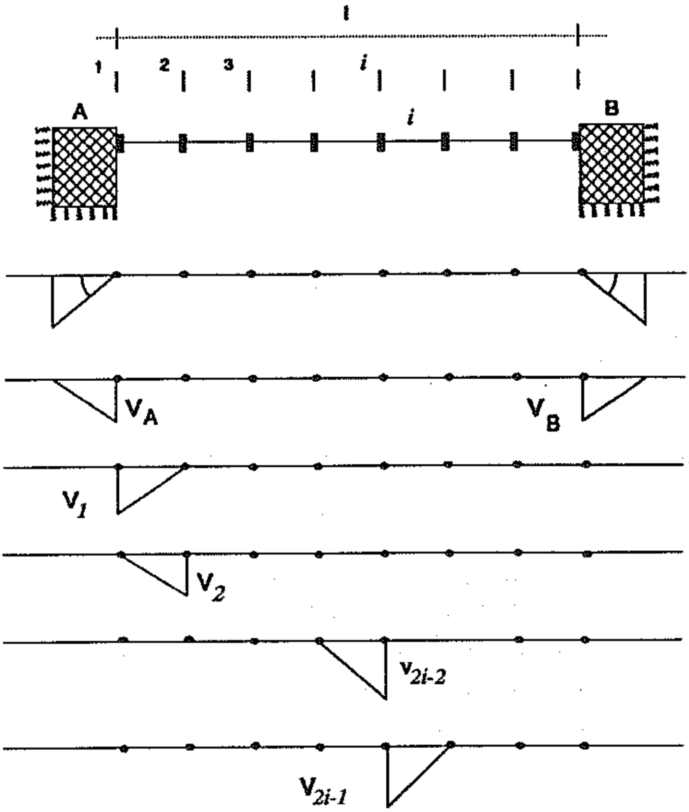

FIG. 1. Lagrangian coordinates.

cross-sectional area and χ is the shear factor. The strain energy is given by the sum of the beam strain energy and of the foundation strain energy, where the rigid blocks are supposed to be immersed.

The strain energy of the beam can in turn be expressed as a sum of the bending strain energy and the shearing strain energy. The bending strain energy will be given by

$$L_{fi} = \frac{1}{2} M_i \Delta\varphi_i \,, \tag{2.2}$$

where $\Delta\varphi_i$ is a relative rotation at the i-th cells abscissa.

The following relationship holds:

$$M_i = \frac{EI}{l_i} \Delta\varphi_i = k_{fi} \Delta\varphi_i \tag{2.3}$$

so that it is possible to write:

$$L_{fi} = \frac{1}{2} k_{fi} \Delta\varphi_i^2 \,. \tag{2.4}$$

In the same way it is possible to derive the shearing strain energy

$$L_{si} = \frac{1}{2} T_i \Delta v_i \,, \tag{2.5}$$

where Δv_i is the relative displacement corresponding to the i-th cell abscissa.

On account of

$$T_i = \frac{GA}{\chi l_i} \Delta v_i \tag{2.6}$$

the shearing strain energy becomes

$$L_{si} = \frac{1}{2} k_{si} \Delta v_i^2 \,. \tag{2.7}$$

The strain energy of the whole system can be written as

$$L = L_f + L_s + L_t, \tag{2.8}$$

where L_t is the strain energy of the foundation beneath the blocks.

The kinetic energy is given by the translational interia of the lumped masses at the ends of each rigid bar, as given in [3]. In addition, the presence of the masses of the rigid blocks has to be taken into account together with the rotatory inertia, expressed as a function of Lagrangian coordinates.

3. The equation of motion

All the terms in Eq. (2.8) can be expressed as functions of the Lagrangian coordinates. In fact, if $\Delta \Phi$ and Δv are the column vectors of relative rotations and displacements, respectively, corresponding to the cells abscissae, then it will be (Fig. 2):

$$\begin{aligned} \Delta \Phi_1 &= -\frac{v_2 - v_1}{l_1} - \varphi_A + \frac{v_A}{l_A} \,, \\ \Delta \Phi_i &= -\frac{v_{2i} - v_{2i-1}}{l_i} + \frac{v_{2i-2} - v_{2i-3}}{l_{i-1}} \,, \\ \Delta \Phi_{t+1} &= \frac{v_{2t} - v_{2t-1}}{l_t} + \frac{v_B}{l_B} + \varphi_B \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \Delta v_1 &= v_1 - v_A \,, \\ \Delta v_i &= v_{2i-1} - v_{2i-2} \,, \\ \Delta v_{t+1} &= v_B - v_{2t} \,. \end{aligned} \tag{3.2}$$

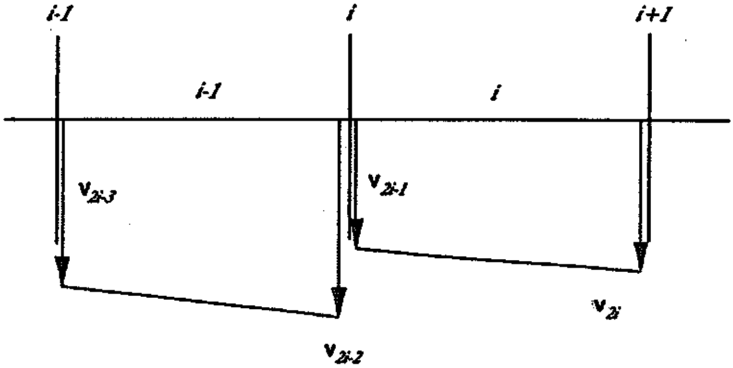

FIG. 2. Displacements and rotations.

In matrix terms

$$\Delta\Phi = \mathbf{A}\,\mathbf{c}\,, \qquad \Delta v = \mathbf{B}\,\mathbf{c}\,,$$

where $\mathbf{A}$ and $\mathbf{B}$ are the matrices with $t+1$ rows and $2t+4$ columns.

The bending strain energy can be calculated from Eq. (3.1) as:

$$L_f = \frac{1}{2}\,\mathbf{c}^T\mathbf{k}_f\,\mathbf{c}\,, \tag{3.3}$$

where

$$\mathbf{K}_f = \mathbf{A}^T\mathbf{D}_f\mathbf{A}$$

and $\mathbf{D}_f$ is the diagonal $(t+1, t+1)$ matrix of the stiffness coefficients $\mathbf{K}_{f_i}$.

Quite in the same way the shearing strain energy can be expressed as

$$L_s = \frac{1}{2}\,\mathbf{c}^T\mathbf{K}_s\,\mathbf{c}\,, \tag{3.4}$$

where

$$\mathbf{K}_s = \mathbf{B}^T\mathbf{D}_s\mathbf{B}$$

and $\mathbf{D}_s$ is the diagonal $(t+1, t+1)$ matrix of the shearing stiffness.

The strain energy of the foundation can be conveniently expressed as a function of the stiffness matrices of the blocks. Let us consider the left block. It is:

$$L_{tA} = \frac{1}{2}\mathbf{s}_A^T\mathbf{K}_A\mathbf{s}_A\,, \tag{3.5}$$

where

$$\mathbf{s}_A = \begin{vmatrix} r_A \\ v_A \end{vmatrix}, \qquad \mathbf{f}_A = \begin{vmatrix} \mathcal{M}_A \\ V_A \end{vmatrix} \tag{3.6}$$

and $\mathbf{s}_A = \mathbf{C}_A\mathbf{f}_A$.

The $\mathbf{C}_A$ is the flexibility matrix of the left block and can be considered as a function of the foundation properties. If a Winkler foundations is assumed, then it is a function of the modulus of subgrade reaction.

In the same way, it is possible to write, for the right-hand side block, $\mathbf{s}_B = \mathbf{C}_B \mathbf{f}_B$.

Finnally, the strain energy of the foundation is given by

$$L_t = \frac{1}{2} c^T \mathbf{K}_t c \,, \tag{3.7}$$

where the matrix $\mathbf{K}_t$ is given by [2]:

$$\mathbf{K}_t = \begin{array}{c|c|cccc|c}
 & 1,2.. \quad ..2t & r_A & r_B & v_A & v_B & \\ \hline
1 \\ \vdots & 0 & & 0 & & & \\ 2t & & & & & & \\ \hline
 & & K_{A11} & 0 & K_{A12} & 0 & r_A \\
 & & & K_{B11} & 0 & K_{B12} & r_B \\
 & 0 & & & K_{A22} & 0 & v_A \\
 & & & \text{SYM} & & & \\
 & & & & & K_{B22} & v_B \\ \hline
\end{array}$$

It is worth noting that symmetry considerations imply:

$$K_{A11} = K_{B11}, \qquad K_{A12} = -K_{B12}, \qquad K_{A22} = K_{B22} \,. \tag{3.8}$$

The strain energy of the whole structure, as a function of the Lagrangian coordinates, is given by

$$L = \frac{1}{2} \mathbf{c}^T \mathbf{K} \mathbf{c} \,, \tag{3.9}$$

where

$$\mathbf{K} = \mathbf{K}_f + \mathbf{K}_s + \mathbf{K}_t \tag{3.10}$$

is the symmetric, positive definite global stiffness matrix.

4. Kinetic energy

In order to calculate the kinetic energy of the system it is necessary to calculate the vertical displacements of the masses at the ends of the rigid bars, the rotations of the masses at the ends of the rigid bars and the same quantities referred to the foundations blocks.

It is, therefore:

$$T = \frac{1}{2}\dot{\mathbf{c}}^T \mathbf{M}_V \dot{\mathbf{c}} + \frac{1}{2}\dot{\mathbf{r}}^T \hat{\mathbf{M}}_I \dot{\mathbf{r}}, \tag{4.1}$$

where $\mathbf{M}_V$ represents the $(2t+4,\, 2t+4)$ mass matrix, $\mathbf{r}$ is the column vector of the absolute rotations, and $\hat{\mathbf{M}}_I$ is the $(t+2,\, t+2)$ matrix of the rotary inertia.

The diagonal terms of the mass matrix $\mathbf{M}_V$ are given by

$$\begin{aligned} M_i &= \frac{m_i l_i}{2}, & i &= 1, 2, \ldots, 2t \\ M_{2t+1} &= I_{mA}, & M_{2t+2} &= I_{mB}, \\ M_{2t+3} &= m_A, & M_{2t+4} &= m_B, \end{aligned} \tag{4.2}$$

where I_{mA} and I_{mB} are the polar moments of inertia of the masses m_A and m_B with respect to A and B, respectively.

The rotations of the rigid bars of the beam are given by

$$r_i = -\frac{v_{2i} - v_{2i-1}}{l_i}, \qquad i = 1, 2, \ldots, t \tag{4.3}$$

which can be written as

$$\mathbf{r} = \mathbf{V}\mathbf{c}, \tag{4.4}$$

where $\mathbf{V}$ is a rectangular $(t+2,\, 2t+4)$ "transfer" matrix. Finally, Eq. (4.1) can be expressed in terms of the Lagrangian coordinates as

$$T = \frac{1}{2}\dot{\mathbf{c}}^T \mathbf{M}_V \dot{\mathbf{c}} + \frac{1}{2}\dot{\mathbf{c}}^T \mathbf{M}_I \dot{\mathbf{c}}, \tag{4.5}$$

where

$$\mathbf{M}_I = \mathbf{V}^T \hat{\mathbf{M}}_I \mathbf{V}. \tag{4.6}$$

Each term of the matrix $\hat{\mathbf{M}}_I$ is given by the product of the mass of the beam and the moment of inertia of the cross-section [3].

The equation of motion of the system can be immediatelly derived from the Lagrange equation

$$\mathbf{M}\ddot{\mathbf{c}} + \mathbf{K}\mathbf{c} = \mathbf{0}, \tag{4.7}$$

with the corresponding eigenvalue problem

$$(-\omega^2\mathbf{M} + \mathbf{K})\mathbf{c} = \mathbf{0}. \tag{4.8}$$

5. Numerical examples

As a first example, let us consider a uniform beam with span l, ratio of the elasticity moduli $E/G = 2.6$, shear factor $\chi = 1/0.85$, radius of gyration $r_g = 0.08\,l$, and let us calculate the first three free vibration frequencies. More particularly, the nondimensional parameter

$$\lambda_i^2 = \omega_i \sqrt{\frac{\rho A l^4}{EI}} \,,$$

has been introduced where ρ is the mass density and I is the cross-sectional inertia. In Table 1 the λ_i^2 values are compared with the nondimensional frequencies of simply supported beam, as given in [3]. It is interesting to observe that the discretization method used leads to an overly stiffened structure, and consequently the nondimensional frequencies are slightly underestimated with respect to other discretization methods.

Table 1.

	Author	[3]	exact value
λ_1^2	8.82	8.80	8.84
λ_2^2	28.29	28.15	28.46
λ_3^2	50.96	–	–

In Table 2 the nondimensional frequency λ_i^2 is given as a function of the ratio $\alpha = r_g/l$.

Table 2.

r_g/l	λ_1^2	λ_2^2	λ_3^2
0.01	9.77	38.67	84.90
0.02	9.66	37.24	78.06
0.04	9.54	35.02	70.49
0.06	9.21	31.62	59.82
0.08	8.82	28.29	50.96

r_g/l	λ_1^2	λ_2^2	λ_3^2
0.10	8.38	25.33	43.95
0.20	6.36	14.12	15.81
0.25	5.57	9.07	13.14
0.30	4.93	6.39	11.81
0.35	4.40	4.69	9.75

As $\alpha \to 0$ the classical Euler – Bernoulli results are clearly recovered, as illustrated in Table 3.

Table 3.

	Author	[3]	exact value
λ_1^2	9.84	9.81	9.87
λ_2^2	38.91	38.58	39.48
λ_3^2	85.94	–	–

It is sometimes useful to investigate cantilever beam with a tip mass, for example in the analysis of tower vibrations.

The problem was first approached by BRUCK and MITCHELL [5] and then by ABRAMOVICH and HAMBURGER [6]. They studied the free vibration frequencies of a cantilever beam with a tip mass, taking into account shear deformations and rotary inertia. Moreover, in [6] an influence of the rotary inertia of the concentrated mass at the tip is elucidated by showing that the nondimensional frequency decreases. In order to perform a comparison, in Table 4 the first four nondimensional frequencies λ_i^2 are shown as given by the used discretization method in [5] and [6]. The discrepancies become noticeable for higher modes because of the stiffening of the structure. In the table it is assumed that $1/\chi = 2/3$, $E/G = 8/3$, $\alpha = 0.02$ and a rotary inertia of the tip mass $J = \mathbf{M} \times \mathbf{K}^2$.

Table 4.

	[5]	[6]	author	λ^2
$Y = 0$, $z = 0$	3.50	3.50	3.485	1
	21.35	21.35	20.77	2
	57.47	57.42	54.58	3
	106.93	106.58	98.95	4
$Y = 1$, $z = .5$	1.40	1.27	1.27	1
	5.73	4.53	4.51	2
	23.64	23.32	22.99	3
	58.41	58.24	56.60	4
$Y = M/(mL)$, $z = K/L$				

Finally, the results have been obtained by dividing the structure into 12 rigid bars leading to a discretized system with 28 Lagrangian coordinates.

6. Conclusion

It is well known that the Timoshenko theory allows us to take into account the shear deformation and rotary inertia of deep structures. This theory is very useful even for tall buildings and shear walls and it is usual to reduce a tall building to a cantilever Timoshenko beam with a flexible constraint at the base. In this case, the constraint is viewed as a couple of independent bending and axial springs, according to the classical Winkler hypothesis.

In this paper a more refined model has been adopted, in which the constraints are reduced to rigid blocks on Winkler foundation, defined by a full 2×2 flexibility matrix. This model leads to somewhat different results than the classical uncoupled case would provide.

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