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THE COMPRESSIVE BARS WITH BOUNDED DISPLACEMENT AS THE TASK OF OPTIMAL CONTROL

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The beam with bounded displacement compressed by the axial force is considered as an optimal control task. The constraints relative to displacement of the bar axis could be active pointwise or in some domain. To solve the task, maximum principle shall be used; necessary conditions reduce the problem of optimal control to a multipoint boundary value problem for ordinary differential equations. The boundary value problems with jump conditions are solved by multiple-shooting techniques.

1. Introduction

The beam with bounded displacement compressed by the axial force is considered as an optimal control task. The shape of the bar axis determined by the function describing the bar curvature is the control variable. We shall consider a typical nonlinear bar compressed by the axial force whose displacement is limited by two parallel lines. The cost function is potential energy of the compressed bar. The constraints relative to displacement of the bar axis could be active pointwise or in some domain. To solve the task, maximum principle shall be used; necessary conditions reduce the problem of optimal control to a multipoint boundary value problem (MPBVP) for ordinary differential equations. The multiple shooting method [1] has been successfully used to solve the resulting multipoint boundary value problem numerically.

2. Task formulation

We consider the nonlinear bar with bounded deflection. The bar is axially compressed by a force P (Fig. 1).

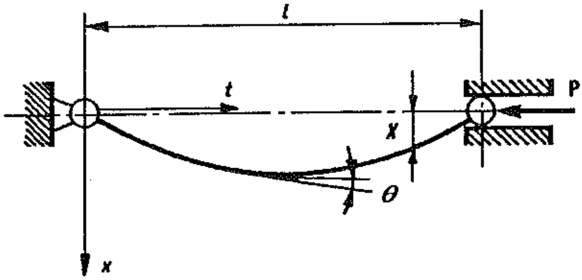

FIG. 1. Column geometry.

In view of the condition of incompressibility we can write

$$\frac{dx}{dt} = \sin\theta\,, \tag{2.1}$$

from which it follows by differentiation that

$$\frac{d^2x}{dt^2} = \cos\theta\frac{d\theta}{dt}\,. \tag{2.2}$$

The true curvature of the deformed curve is

$$\frac{1}{\rho} = \frac{d\theta}{dt}\,, \tag{2.3}$$

so that

$$\frac{1}{\rho} = \frac{\dfrac{d^2x}{dt^2}}{\cos\theta} = \frac{x''}{\sqrt{1-x'^2}}\,. \tag{2.4}$$

For the deflection of the bar we have the following state equations:

$$\begin{aligned} \frac{dx}{dt} &= \sin\theta\,, \\ \frac{d\theta}{dt} &= \frac{x''}{\sqrt{1-x'^2}}\,. \end{aligned} \tag{2.5}$$

The boundary conditions at the simply supported ends are

$$x(0) = 0\,, \qquad x(1) = 0\,. \tag{2.6}$$

At the fixed ends, the boundary conditions are

$$\begin{aligned} x(0) &= 0\,, \qquad & \theta(0) &= 0\,, \\ x(1) &= 0\,, \qquad & \theta(1) &= 0\,. \end{aligned} \tag{2.7}$$

The total potential energy is thus

$$J_\alpha = \frac{1}{2}\int_0^1 \theta'^2\,dt + \alpha\int_0^1 \cos\theta\,dt\,, \tag{2.8}$$

where

$$\alpha = \frac{P}{EJ}\,.$$

In the problem considered the state variables are: x, θ, and the control variable U is defined by the equation $U = \theta'$.

We minimize the potential energy of the bar

$$J_\alpha = \frac{1}{2}\int_0^1 \theta'^2\,dt + \alpha\int_0^1 \cos\theta\,dt \tag{2.9}$$

with constraints

$$x' = \sin\theta\,, \qquad \theta' = U\,, \qquad |x| \le d\,. \tag{2.10}$$

The functional (2.9) is to be minimized subject to the constraints (2.10).

3. Necessary conditions

The bar with bounded deflection is considered as an optimal control problem with bounded state variables. The theory of optimal control provides conditions for the trajectory $x(t)$ and the control function $U(t)$ associated with it.

3.1. The task without constraints

In this case the Hamilton function has a form of

$$H = 0.5U^2 + \alpha\cos\theta + \lambda_1\sin\theta + \lambda_2 U\,. \tag{3.1}$$

Here the control variable $U(t)$ is determined by the maximum principle. In the first step, the optimal control variable is eliminated in terms of the state and adjoint variables. In so far U is a function of λ, x. In particular, U is a solution of the equation

$$\frac{\partial H}{\partial U} = 0\,. \tag{3.2}$$

The control function U is obtained on unconstrained subarcs by

$$U = -\lambda_2 \tag{3.3}$$

if the strengthened Legendre–Clebsch condition

$$\frac{\partial^2 H}{\partial U^2} = 1 > 0 \tag{3.4}$$

is valid. In the problem considered the condition (3.4) is always fulfilled. Functions λ_i are the solution of the system of adjoint equations

$$\begin{aligned} \lambda_1' &= 0, \\ \lambda_2' &= \alpha \sin\theta - \lambda_1 \cos\theta. \end{aligned} \tag{3.5}$$

In view of the transversality conditions we can write boundary values for the adjoint functions

$$\lambda_2(0) = 0, \qquad \lambda_2(1) = 0. \tag{3.6}$$

State equations (2.6) and adjoint equations (3.5) together with suitable boundary conditions form two-point boundary value problem (TPBVP) for functions y_1, y_2, λ_1, λ_2; with control defined by the maximum principle (3.2).

In case (A) under discussion, we obtain a system of differential equations:

$$\begin{aligned} x' &= \sin\theta, \\ \theta' &= U, \\ \lambda_1' &= 0, \\ \lambda_2' &= \alpha \sin\theta - \lambda_1 \cos\theta, \\ \alpha' &= 0, \\ U &= -\lambda_2, \end{aligned} \tag{3.7}$$

with the following boundary conditions:

$$\begin{aligned} x(0) &= 0, & \lambda_2(0) &= 0, \\ \theta\left(\frac{1}{2}\right) &= 0, & \lambda_1\left(\frac{1}{2}\right) &= 0, & x\left(\frac{1}{2}\right) &= d_1 \le d. \end{aligned}$$

3.2. The task with constraints

The next degree of complexity is given by optimal control problems with state variable inequality constraints. The constraint (2.10) does not depend on the control. The order of the state constraints has a decisive importance for further considerations. G is defined by

$$G(t) := |x| \le d\,. \tag{3.8}$$

By successive differentiation G with respect to t, we find control U in the explicit form.

Thus, we have

$$\begin{aligned} G^{(0)} &:= |x| \le d\,, \\ G^{(1)} &:= \pm \sin\theta\,, \\ G^{(2)} &:= \pm U\cos\theta\,. \end{aligned} \tag{3.9}$$

The expression $G^{(2)}$ includes control U in explicit form so that the order of the state constraints is equal $q = 2$. This constraint can be active pointwise or in some domain [4].

From a practical point of view the transition from the unconstrained to the constrained solution comprises two stages: (1) constrained solution which touches the boundary only; (2) constrained solution containing boundary arcs.

If the constraint (3.8) is active, then control $U = 0$, moreover $x(t) = \pm d$, and $\theta(t) = 0$.

By using the necessary conditions from the calculus of variations this problem can be transformed into a multipoint boundary value problem for the state vector $\mathbf{x}$ and adjoint variables λ, which can be solved by the multiple shooting method (see, Bulirsch [1]).

If the constraint becomes active, it first happens in point $t_1 = 0.5$.

This point is subject to the boundary conditions

$$x(t_1) = d, \qquad \theta(t_1) = 0, \tag{3.10}$$

and the jump condition

$$\begin{aligned} \lambda_1(\xi_1^+) &= \lambda_1(\xi_1^-) - \nu\,, \\ \lambda_2(\xi_1^+) &= \lambda_2(\xi_1^-)\,. \end{aligned} \tag{3.11}$$

In contradistinction to the problem without constraints, factor $\lambda_1 \neq 0$.

In this case (B) state equations have a form

$$
\begin{aligned}
x' &= \sin\theta\,,\\
\theta' &= U,\\
\lambda_1' &= 0,\\
\lambda_2' &= \alpha\sin\theta - \lambda_1\cos\theta\,,\\
\alpha' &= 0,\\
\nu' &= 0,\\
U &= -\lambda_2\,.
\end{aligned}
\tag{3.12}
$$

with the following boundary conditions:

$$
\begin{aligned}
x(0) &= 0, \qquad \lambda_2(0) = 0,\\
\theta\left(\frac{1}{2}\right) &= 0, \quad \lambda_1\left(\frac{1}{2}\right) = c, \quad x\left(\frac{1}{2}\right) = d,\\
\lambda_1\left(\frac{1^+}{2}\right) &= \lambda_1\left(\frac{1^-}{2}\right) - \nu\left(\frac{1}{2}\right).
\end{aligned}
$$

The system of Eqs. (3.12) has a solution if $\alpha \in (39.601078, 157.316212)$, (Figs. 7, 8).

When the constraint (3.8) is active in some point $t_1 \neq 0.5$, we can calculate this point using the condition

$$
\begin{aligned}
x(t_1) &= d,\\
\theta(t_1) &= 0,\\
\lambda_2(t_1) &= 0.
\end{aligned}
\tag{3.13}
$$

This case (C) is described by equations

$$
\begin{aligned}
x' &= \sin\theta\,,\\
\theta' &= -\lambda_2\,,\\
\lambda_1' &= 0\,,\\
\lambda_2' &= \alpha\sin\theta - \lambda_1\cos\theta\,,\\
\alpha' &= 0\,,\\
\nu' &= 0\,,
\end{aligned}
\tag{3.14}
$$

with the following boundary and jump conditions

$$\begin{aligned} x(0) &= 0, & \lambda_2(0) &= 0, \\ \theta\left(\frac{1}{2}\right) &= 0, & x\left(\frac{1}{2}\right) &= d, \\ x(t_1) &= d, & \theta(t_1) &= 0, & \lambda_2(t_1) &= 0, \\ \lambda_1(t_1^+) &= \lambda_1(t_1^-) - \nu(t_1). \end{aligned}$$

The solution of Eqs. (3.14) with conditions given above is obtained for $157.316212 \le \alpha \le 223.123747$, and point t_1 of control change can be found in the interval $t_1 \in (0.4195, 0.5)$, (Fig. 9).

Hence, considering additionally constraint $x \ge -d$ we obtain the following system (case D):

$$\begin{aligned} x' &= \sin\theta\,, \\ \theta' &= -\lambda_2\,, \\ \lambda_1' &= 0\,, \\ \lambda_2' &= \alpha\sin\theta - \lambda_1\cos\theta\,, \\ \alpha' &= 0\,, \\ \nu' &= 0\,, \end{aligned} \tag{3.15}$$

with conditions:

$$\begin{aligned} x(0) &= 0, & \lambda_2(0) &= 0, \\ \theta\left(\frac{1}{2}\right) &= 0, & x\left(\frac{1}{2}\right) &= -d, \\ x(t_1^*) &= d, & \theta(t_1^*) &= 0, & \lambda_2(t_1^*) &= c, \\ \lambda_1(t_1^{*+}) &= \lambda_1(t_1^{*-}) - \nu(t_1^*). \end{aligned}$$

The system (3.15) describing case D concerns constraints shown in Fig. 5. The results are obtained when $\alpha \in (91.469901, 342.253)$; point t_1^* does not depend on α and has state value equal $t_1^* = 0.1666$.

If $\alpha > 338.667$ constraints will be active as shown in Fig. 6.

For this case we obtain the following system of equations and boundary conditions (case E):

$$\begin{aligned} x' &= \sin\theta\,, \\ \theta' &= -\lambda_2\,, \\ \lambda_1' &= 0, \\ \lambda_2' &= \alpha\sin\theta - \lambda_1\cos\theta\,, \end{aligned} \tag{3.16}$$

(3.16) [cont.]

$$\alpha' = 0,$$
$$\nu' = 0,$$
$$\gamma' = 0.$$

$$\begin{aligned}
x(0) &= 0, & \lambda_2(0) &= 0, \\
\theta\left(\frac{1}{2}\right) &= 0, & x\left(\frac{1}{2}\right) &= -d, \\
x(t_1^*) &= d, & \theta(t_1^*) &= 0, \\
x(t_2) &= -d, & \theta(t_2) &= 0, \quad \lambda_2(t_2) = 0, \\
\lambda_1(t_1^{*+}) &= \lambda_1(t_1^{*-}) - \nu(t_1^*). \\
\lambda_1(t_2^{+}) &= \lambda_1(t_2^{-}) - \gamma(t_2).
\end{aligned}$$

State equations (A,B,C,D,E) with suitable boundary conditions and with jump condition have been numerically solved by means of the multiple shooting method [1].

4. Comparison of the obtained results

The obtained numerical results are presented in Tables 1–4, state variables x, θ, λ_1, λ_2 are shown in Figs. (2–11), depending on the cases and boundary conditions.

Table 1. Cases A and B.

α	$\lambda_2(\frac{1}{2})$	Fig.	α	$\lambda_2(\frac{1}{2})$	Fig.
9.900269	0.495013	2	39.601078	0.990026	7
27.262693	0.25	–	140.220688	0.20	–
39.329083	0.0	3	157.316212	0.0	8

Table 2. Case C.

α	t_1	Fig.	α	t_1	Fig.
62.162184	0.397260	–	163.733296	0.490065	–
83.196326	0.343026	–	187.731343	0.457537	–
101.682056	0.309991	–	189.402190	0.455505	–
162.096647	0.244740	4	202.947450	0.439968	–
304.756492	0.177037	–	223.123747	0.419500	9

Table 3. Case D.

α	t_1^*	Fig.	α	t_1^*	Fig.
91.469901	0.166666	–	599.886312	0.252960	–
156.673502	0.166666	5	587.315184	0.257386	–
281.100900	0.166666	–	603.830773	0.253816	10
338.667292	0.166666	–	603.460747	0.253839	–

Table 4. Case E.

α	t_1	t_2	Fig.	α	t_1	t_2	Fig.
345.927928	0.1657	0.4972	–	737.5331	0.2260	0.4425	11
373.765114	0.1591	0.4774	6	773.2004	0.2237	0.4375	–

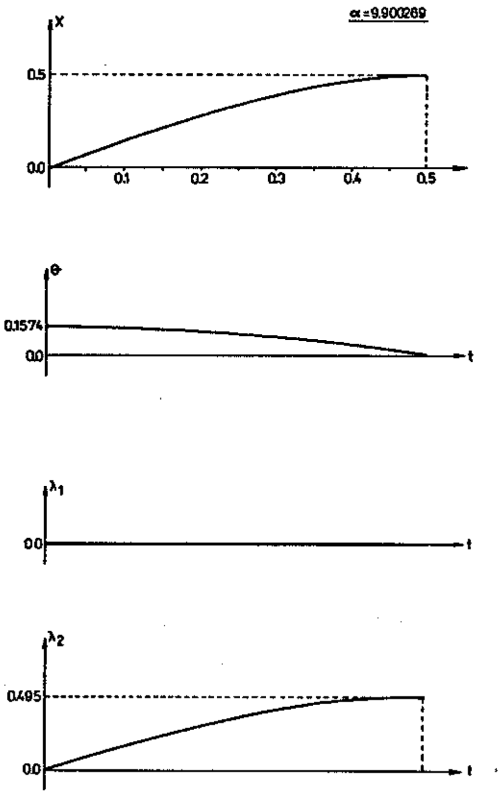

FIG. 2. Activity of constraints in point $t_1 = 0.5$.

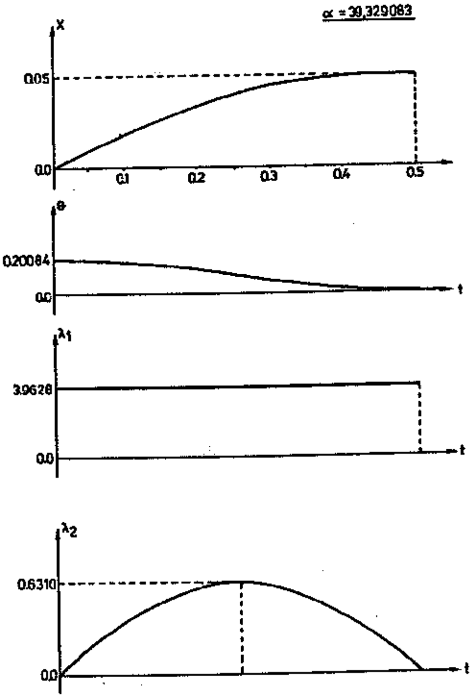

FIG. 3. Activity of constraints in point $t_1 = 0.5$.

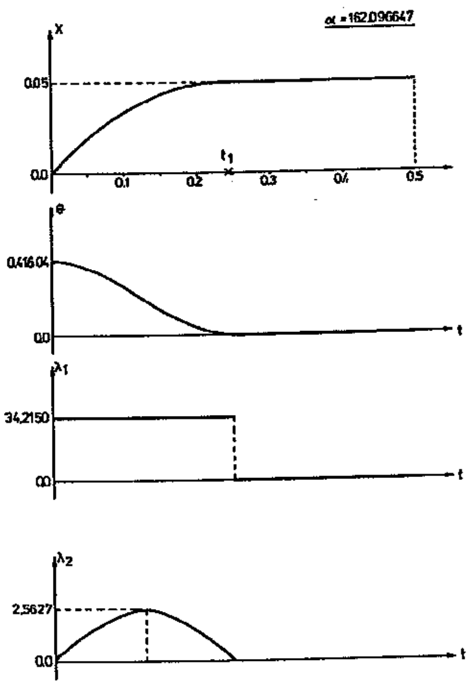

FIG. 4. Activity of constraints in point $t_1 \neq 0.5$.

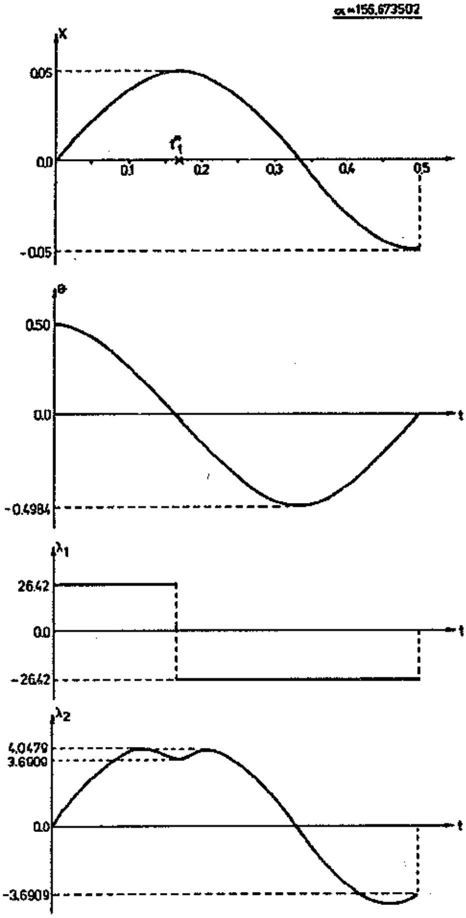

FIG. 5. Activity of constraints in point t_1^*.

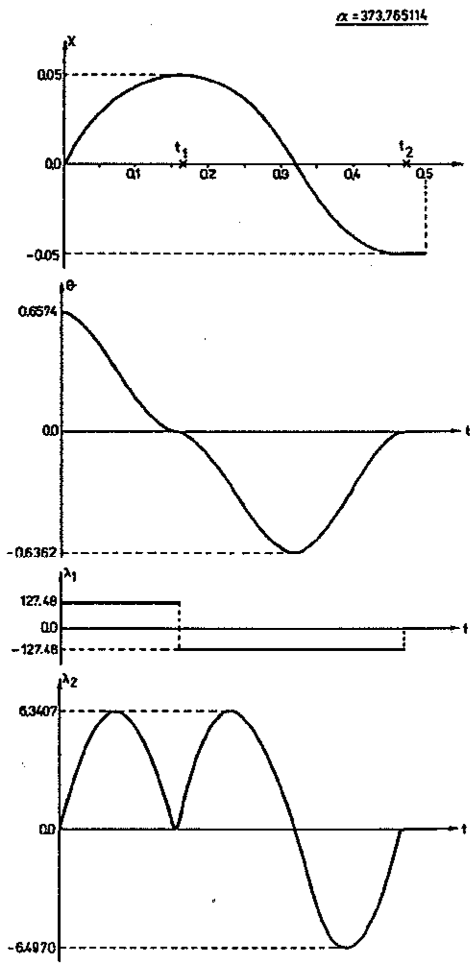

FIG. 6. Activity of constraints in points t_1, t_2.

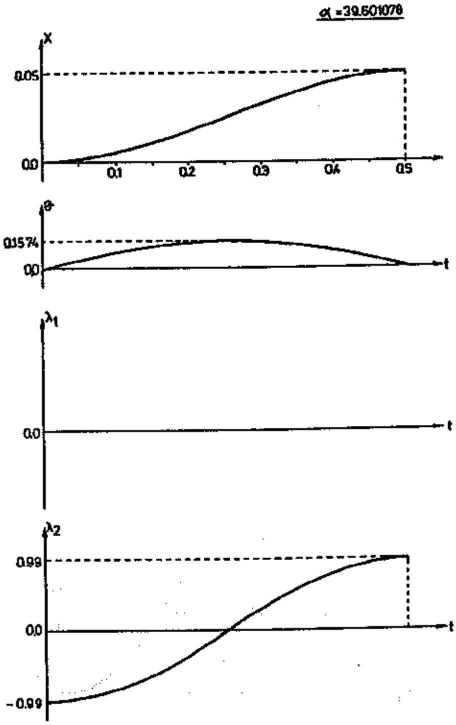

FIG. 7. Activity of constraints in point $t_1 = 0.5$ (fixed ends).

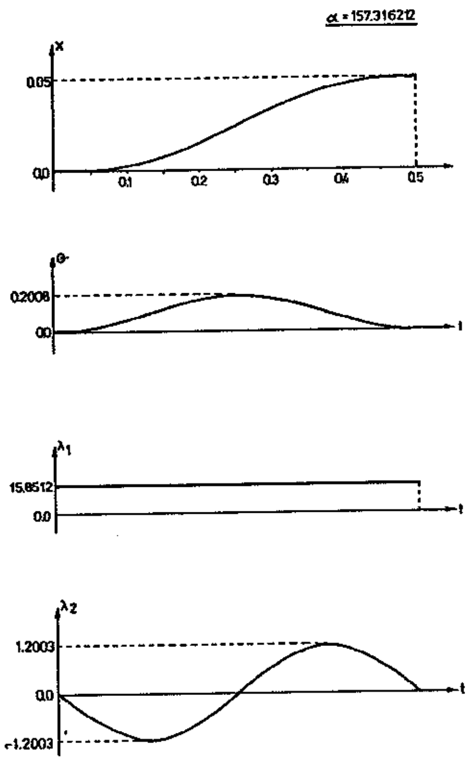

FIG. 8. Activity of constraints in point $t_1 = 0.5$ (fixed ends).

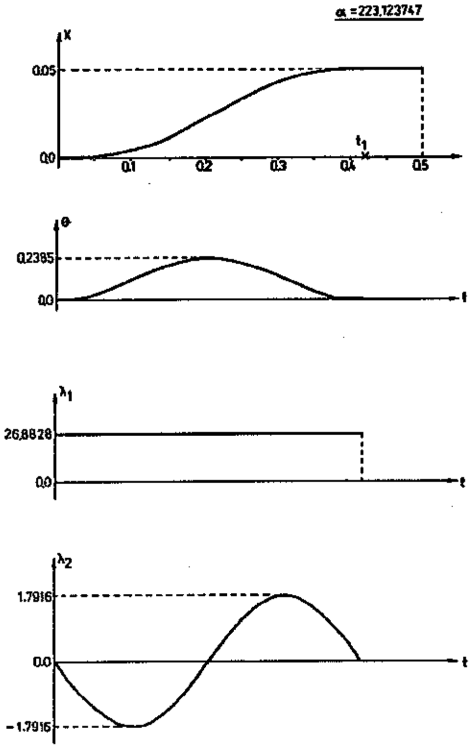

FIG. 9. Activity of constraints in point $t_1 \neq 0.5$ (fixed ends).

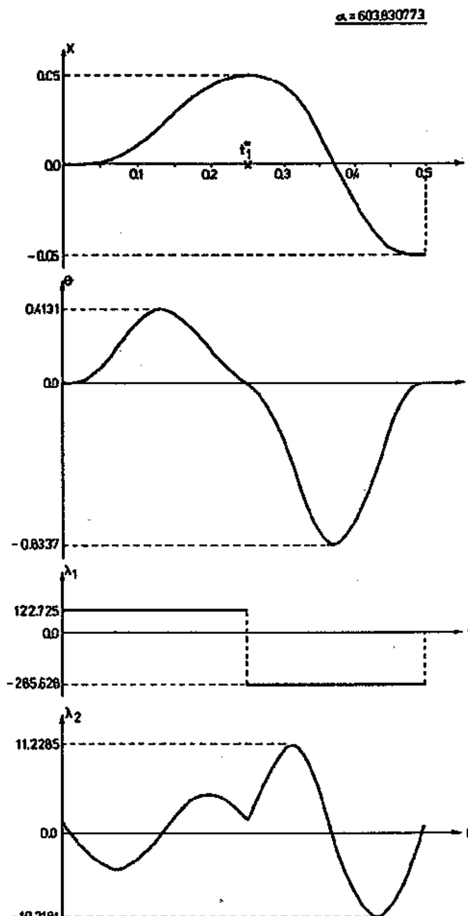

FIG. 10. Activity of constraints in point t_1^* (fixed ends).

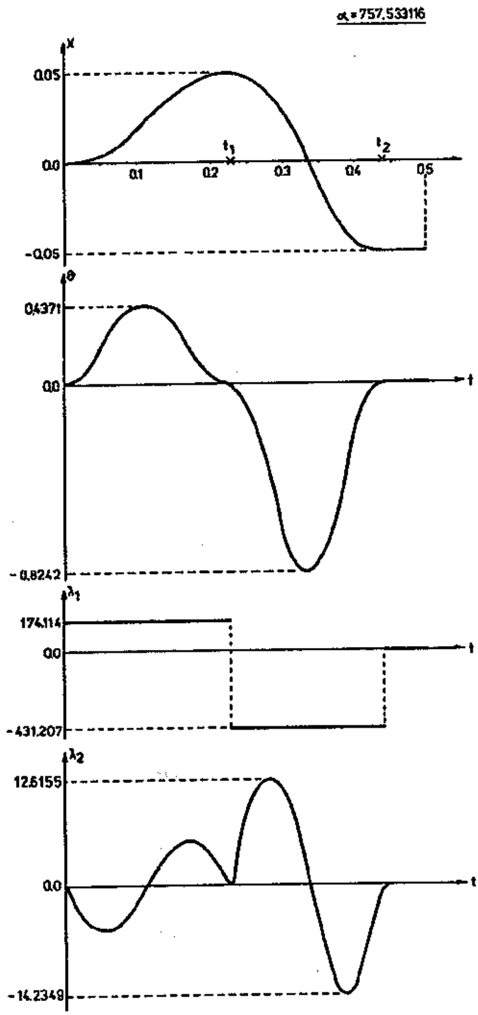

FIG. 11. Activity of constraints in points t_1, t_2 (fixed ends).

5. FINAL REMARKS

The constrained bar is considered as an optimal control problem with bounded state variables. The underlying boundary value problems are solved with multiple-shooting methods which give numerical results of high accuracy. The computed examples confirm the efficiency of the methods of the optimal control which, in connection with multiple shooting method, are effective in solving the problems of optimal elastic bars design with constraints of the state variables.

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