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APPLICATION OF PARTIAL MODELS FOR VIBRATION ANALYSIS OF COMBINED DISCRETE-CONTINUOUS SYSTEMS

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A method of analysis of complex dynamic systems by means of partial models is characterized in brief. Some advantages of that method as a means for analysis of combined discrete-continuous systems are pointed out. An iteration procedure for analysing such a system is presented, as well as an example of application of the method discussed for analysing a certain discrete-continuous system by its separation into partial models, for which vibration equations are formulated. The results of computation are presented. They constitute a good illustration of the convergence properties of the iteration procedure used.

1. Introduction

A problem which is often encountered in the domain of machine dynamics is that of reducing the vibration of machines or other devices incorporating structural elements of large surfaces such as housings, screens, covers etc. The mechanism of generation of vibrations is, in such cases, as follows: an (internal or external) excitation source acting on the body or any subassembly of the machine generates also vibrations of other structural elements or assemblies. These vibrations are transferred to the housing, which may become a source of radiation of sonic energy, sometimes of high intensity, both in the audible or subsonic ranges [2]. Those vibrations can be abated by simple engineering methods such as a change in the form or rigidity of the housing or in the excitation parameters, application of vibration isolating pads etc. In more complicated cases it is necesary to analyse the dynamic properties of the machine. This requires the construction of a model of the machine as a vibrating system, identification of the parameters of that model and the excitations acting on it, and the numerical analysis of its properties. The models which are devised in such cases are of the discrete-continuous type, in which elements of the body and subassemblies of the machine are treated as rigid bodies interconnected by vicso-elastic elements (elastic elements with damping) and the housing is modelled by a plate, a system of plates or a complex shell structure, resting on elastic supports or rigidly connected with the solids.

This double character of the elements of machine models is the principal reason for using for vibration analysis the method of partial models.

2. The method for analysing complex dynamic systems by means of partial models

The method to be described is based on the theory, developed by Mandelshtam [5], of weak couplings between partial systems. In a general manner, the method to be submitted can be characterized as follows:

- The complete model is divided into partial (simpler) models on the basis of an analysis of couplings between partial models (all the couplings should be either "weak" or "unidirectionally weak";
- Analysis of the complete model by means of partial models should be conducted by an iteration method, such that the couplings which are considered to be weak are treated as perturbations in consecutive iterations.

This method has been described in paper [6] presenting two principal ways of separating the complete model into partial models and the relevant algorithms of the iteration method. It may have some limitations, for instance, if the natural frequencies of partial models are equal, and the damping in the system is very weak. The limitations of the method are manifested by the iteration procedures being not convergent.

The problem of convergence of iteration procedures has been analysed in [7] (taking into consideration, among other problems, that of influence of damping on the convergence of procedures). In [8] it has been shown that the convergence conditions are weaker than those of Mandelshtam, therefore they are easier to be satisfied.

The use of the method discussed enables us to avoid the inconveniences which may occur with increasing complexity of models, such as some difficulties of numerical nature in the analysis, interpretation of the results or some troubles with identification of the increasing number of parameters of the model.

As regards the applications to the analysis of combined discrete-continuous systems, other advantages of the method proposed become evident. The first of them is the possibility of using models of various types and different methods of analysis used for different partial models. Discrete partial models are described by sets of ordinary differential equations and can be solved in a manner typical for such systems. Continuous partial models, used for housing or screen structures are described by one of the existing difference methods, the method of finite elements or other methods used for the group of models considered.

Such an approach enables us also to apply various integration steps in numerical procedures for solving particular partial models, what reduces the time consumption.

The second advantage of the method is a result of application of a fundamental principle of construction of machines and other devices, that is the principle of symmetry. Moreover, it is known that symmetric structures have always better vibro-acoustic properties, than asymmetric structures [1, 4]. The existing tendency to preserve some elements of symmetry of a structure results in the fact that, in many cases of separation of the complete model into partial models, the proposed method of analysis enables us to select partial models (or groups of models) of identical or very similar structures (after elimination of certain components). In such a case we can use in the computation programs, by judicious choice of the principle of indexing the variables and the parameters of the models, a single procedure (or modulus) for describing them.

The third advantage is the possibility of solving complicated problems, if poor computation facilities are available (a computer of the IBM PC class, for instance). In many cases the necessity of taking into consideration the complicated form of some machine elements (bodies, housing or certain assemblies) leads to very complicated dynamic models, with large numbers of degrees of freedom, e.g. if we want to treat several structural elements as continuous systems, and to analyse them by the method of finite elements. Decomposition of the complete model into partial models makes it possible to use several programs (separate programs for particularly complicated partial systems), which are started consecutively and, therefore, it creates a possibility to solve the problem by means of a computer of lower computing power.

A diagrammatic illustration of the analysis of a certain discrete-continuous system of partial models is shown in Fig. 1.

The complete model has been decomposed into five partial models. The models I to IV are of the discrete type and the model V is continuous. The models II, III and IV are of identical structure, their analysis thus requiring a single description procedure. Owing to this fact, there are only three procedures which are required for the computation program (despite the fact that five partial models have been separated) to describe the models I, V, and the common procedure for models II, III and IV.

Table 1 illustrates the iteration procedure of analysis of the complete

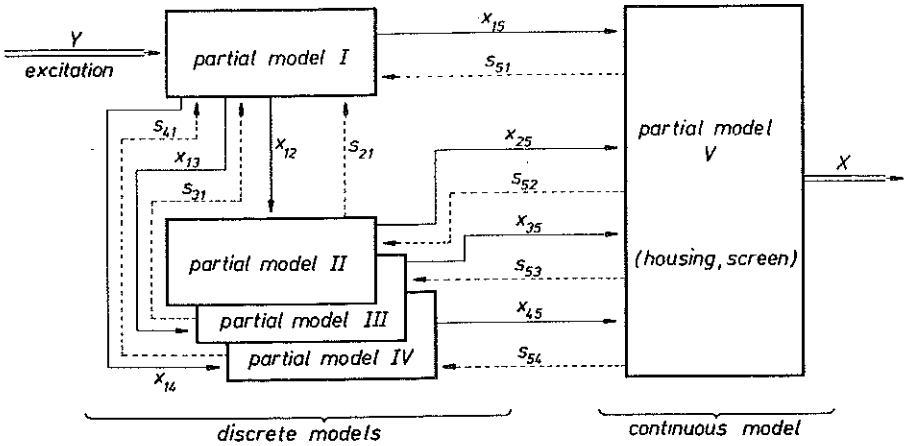

FIG. 1. Diagrammatic representation of the dynamic analysis of a system by means of partial models.

model (letter s in Fig. 1 and in the table marks the signals having a character of weak couplings). The computation should be continued until the required agreement is obtained between two consecutive iterations of a definite signal. In most cases the agreement is already satisfactory after a few iterations.

Table 1.

	Inlet	Model	Outlet
Iteration 0			
step 1	Y	Partial model I	$x_{12}^{(0)}, x_{13}^{(0)}, x_{14}^{(0)}, x_{15}^{(0)}$
step 2	$x_{12}^{(0)}$	Partial model II	$x_{25}^{(0)}, s_{21}^{(0)}$
	$x_{13}^{(0)}$	Partial model III	$x_{35}^{(0)}, s_{31}^{(0)}$
	$x_{14}^{(0)}$	Partial model IV	$x_4^{(0)}5, s_{41}^{(0)}$
step 3	$x_{15}^{(0)}, x_{25}^{(0)}, x_{35}^{(0)}, x_{45}^{(0)}$	Partial model V	$X^{(0)}, s_{51}^{(0)}, s_{52}^{(0)}, s_{53}^{(0)}, s_{54}^{(0)}$
Iteration i			
step 1	$Y, s_{21}^{(i-1)}, s_{31}^{(i-1)}, s_{41}^{(i-1)}, s_{51}^{(i-1)}$	Partial model I	$x_{12}^{(i)}, x_{13}^{(i)}, x_{14}^{(i)}, x_{15}^{(i)}$
step 2	$x_{12}^{(i)}, s_{52}^{(i-1)}$	Partial model II	$x_{25}^{(i)}, s_{21}^{(i)}$
	$x_{13}^{(i)}, s_{53}^{(i-1)}$	Partial model III	$x_{35}^{(i)}, s_{31}^{(i)}$
	$x_{14}^{(i)}, s_{54}^{(i-1)}$	Partial model IV	$x_{45}^{(i)}, s_{41}^{(i)}$
step 3	$x_{15}^{(i)}, x_{25}^{(i)}, x_{35}^{(i)}, x_{45}^{(i)}$	Partial model V	$X^{(i)}, s_{51}^{(i)}, s_{52}^{(i)}, s_{53}^{(i)}, s_{54}^{(i)}$

A condition of correctness of the analysis is that of correctness of the way in which the complete model has been decomposed into partial models [6, 7], and the correctness of definition of the weak couplings between particular partial models.

3. Example of application

As an example of application of the method of partial models to the analysis of vibrations of discrete-continuous systems, we shall discuss the following analysis of the model represented in Fig. 2. This model corresponds to the scheme shown in Fig. 1, except that the partial model I has the same structure as the models II to IV, and no direct couplings exist between the models I, II, III and IV.

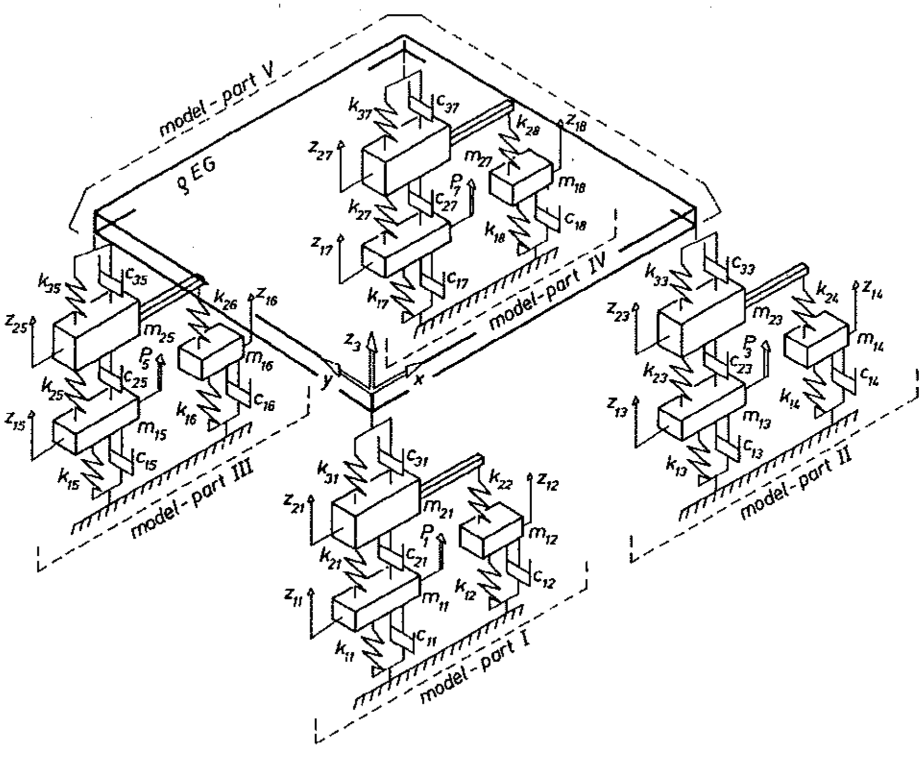

FIG. 2. The dynamic model analysed in the paper.

The vibrations of the complete model are described by the following set of equations:

$$(3.1)\qquad \begin{aligned} &m_{11}\,\ddot{z}_{11} + \dot{z}_{11}\,c_{11} - (\dot{z}_{21} - \dot{z}_{11})c_{21} + z_{11}k_{11} - (z_{21} - z_{11})k_{21} = P_1,\\ &m_{12}\,\ddot{z}_{12} + \dot{z}_{12}\,c_{12} + z_{12}k_{12} - (z_{21} - z_{12})k_{22} = 0, \end{aligned}$$

(3.1) [cont.]

$$m_{21}\ddot{z}_{21}+(\dot{z}_{21}-\dot{z}_{11})c_{21}+(z_{21}-z_{11})k_{21}+(z_{21}-z_{12})k_{22}+\dot{z}_{21}c_{31}+z_{21}k_{31}\underline{-\dot{z}_{31}c_{31}-z_{31}k_{31}}=0,$$

$$m_{13}\ddot{z}_{13}+\dot{z}_{13}c_{13}-(\dot{z}_{23}-\dot{z}_{13})c_{23}+z_{13}k_{13}-(z_{23}-z_{13})k_{23}=P_3,$$

$$m_{14}\ddot{z}_{14}+\dot{z}_{14}c_{14}+z_{14}k_{14}-(z_{23}-z_{14})k_{24}=0,$$

$$m_{23}\ddot{z}_{23}+(\dot{z}_{23}-\dot{z}_{13})c_{23}+(z_{23}-z_{13})k_{23}+(z_{23}-z_{14})k_{24}+\dot{z}_{23}c_{33}+z_{23}k_{33}\underline{-\dot{z}_{33}c_{33}-z_{33}k_{33}}=0,$$

$$m_{15}\ddot{z}_{15}+\dot{z}_{15}c_{15}-(\dot{z}_{25}-\dot{z}_{15})c_{25}+z_{15}k_{15}-(z_{25}-z_{15})k_{25}=P_5,$$

$$m_{16}\ddot{z}_{16}+\dot{z}_{16}c_{16}+z_{16}k_{16}-(z_{25}-z_{16})k_{26}=0,$$

$$m_{25}\ddot{z}_{25}+(\dot{z}_{25}-\dot{z}_{15})c_{25}+(z_{25}-z_{15})k_{25}+(z_{25}-z_{16})k_{26}+\dot{z}_{25}c_{35}+z_{25}k_{35}\underline{-\dot{z}_{35}c_{35}-z_{35}k_{35}}=0,$$

$$m_{17}\ddot{z}_{17}+\dot{z}_{17}c_{17}-(\dot{z}_{27}-\dot{z}_{17})c_{27}+z_{17}k_{17}-(z_{27}-z_{17})k_{27}=P_7,$$

$$m_{18}\ddot{z}_{18}+\dot{z}_{18}c_{18}+z_{18}k_{18}-(z_{27}-z_{18})k_{28}=0,$$

$$m_{27}\ddot{z}_{27}+(\dot{z}_{27}-\dot{z}_{17})c_{27}+(z_{27}-z_{17})k_{27}+(z_{27}-z_{18})k_{28}+\dot{z}_{27}c_{37}+z_{27}k_{37}\underline{-\dot{z}_{37}c_{37}-z_{37}k_{37}}=0,$$

$$D\left(\frac{\partial^4 z_3}{\partial x^4}+\frac{2\partial^4 z_3}{\partial x^2\partial y^2}+\frac{\partial^4 z_3}{\partial y^4}\right)+\rho h\frac{\partial^2 z_3}{\partial t^2}$$
$$=c_{31}(\dot{z}_3-\dot{z}_{21})\delta(x)\delta(y)+c_{33}(\dot{z}_3-\dot{z}_{23})\delta(x-1)\delta(y)$$
$$+c_{35}(\dot{z}_3-\dot{z}_{25})\delta(x)\delta(y-b)+c_{37}(\dot{z}_3-\dot{z}_{27})\delta(x-1)\delta(y-b)$$
$$+k_{31}(z_3-z_{21})\delta(x)\delta(y)+k_{33}(z_3-z_{23})\delta(x-1)\delta(y)$$
$$+k_{35}(z_3-z_{25})\delta(x)\delta(y-b)+k_{37}(z_3-z_{27})\delta(x-1)\delta(y-b),$$

where D is the cylindrical bending rigidity of the plate

$$D=\frac{Eh^3}{12(1-\nu)^2},\tag{3.2}$$

and E is Young's modulus, h – plate thickness and ν – Poisson's ratio. The symbols z_{31}, z_{33}, z_{35} and z_{37} denote the vertical displacements of the respective corners of the plate.

If the model considered satisfies the condition

$$k_{3i}\ll k_{2i}+k_{2,i+1},\qquad i=1;3;5;7;\tag{3.3}$$

then the underlined terms of the Eqs. (3.1) have the character of weak couplings. Thus, the complete model can be decomposed into five partial models (as shown in Fig. 2).

Because the partial models I to IV have the same structure, a single common procedure has been used for their analysis, involving the following set of three differential equations:

$$
\begin{aligned}
&m_{1,1+r}\,\ddot{z}^{(i)}_{1,1+r} + \dot{z}^{(i)}_{1,1+r}\,c_{1,1+r} - (\dot{z}^{(i)}_{2,1+r} - \dot{z}^{(i)}_{1,1+r})c_{2,1+r} \\
&\qquad + z^{(i)}_{1,1+r}k_{1,1+r} - (z^{(i)}_{2,1+r} - z^{(i)}_{1,1+r})k_{2,1+r} = P_{1+r}, \\
&m_{1,2+r}\,\ddot{z}^{(i)}_{1,2+r} + \dot{z}^{(i)}_{1,2+r}\,c_{1,2+r} + z^{(i)}_{1,2+r}k_{1,2+r} \\
&\qquad - (z^{(i)}_{2,1+r} - z^{(i)}_{1,2+r})k_{2,2+r} = 0, \\
&m_{2,1+r}\,\ddot{z}^{(i)}_{2,1+r} + (\dot{z}^{(i)}_{2,1+r} - \dot{z}^{(i)}_{1,1+r})c_{2,1+r} + \dot{z}^{(i)}_{2,1+r}\,c_{3,1+r} \\
&\qquad + (z^{(i)}_{2,1+r} - z^{(i)}_{1,1+r})k_{2,1+r} + (z^{(i)}_{2,1+r} - z^{(i)}_{1,2+r})k_{2,2+r} + z^{(i)}_{2,1+r}k_{3,1+r} \\
&\qquad = \dot{z}^{(i-1)}_{3,1+r}\,c_{3,1+r} + z^{(i-1)}_{3,1+r}k_{3,1+r},
\end{aligned}
\tag{3.4}
$$

where $i = 0, 1, 2, \ldots$ is the number of iteration, $r = 0, 2, 4, 6$ for the partial models I, II, III, IV, respectively, symbols $z^{(i-1)}_{31}$, $z^{(i-1)}_{33}$, $z^{(i-1)}_{35}$ and $z^{(i-1)}_{37}$ denote the vertical displacements of the plate corners; they appear in expressions representing the weak couplings acting on the partial models I to IV. They are assumed to be zero in the zero iteration, and they are taken into account as non-zero quantities beginning from the first iteration.

The method of rigid finite elements has been used for the analysis of vibrations of the partial model V (the plate). According to this method, the plate was initially divided into $n_x \times n_y$ rigid finite elements (RFE). By proceeding in a manner as that described in [3], $(n_x + 1) \times (n_y + 1)$ such elements were obtained for a plate with free or simply supported edges. Each RFE has three degrees of freedom. Its motion is described by the coordinates z_{3ij}, φ_{xij} and φ_{yij}. Each RFE (except those located at the edges) is connected with other RFE elements by means of 8 elastic elements with damping (EED). All the EED are characterized by their ability to move in the direction of three coordinate axes.

The way of indexing the RFE and the way of numbering the EED co-operating with the RFE having indices i, j are shown in Fig. 3.

The values of the coefficients of stiffness for the EED numbered 1, 2, 3 and 4 are determined by the relations

$$
k_{zh} = \frac{Ghl_x}{2.4l_y}, \qquad k_{fxh} = \frac{Gh^3l_x}{12l_y}, \qquad k_{fyh} = \frac{Dl_x}{2l_y},
\tag{3.5}
$$

and for 5, 6, 7 and 8 – by the relations

$$
k_{zw} = \frac{Ghl_y}{2.4l_x}, \qquad k_{fxw} = \frac{Dl_y}{2l_x}, \qquad k_{fyw} = \frac{Gh^3l_y}{12l_x}.
\tag{3.6}
$$

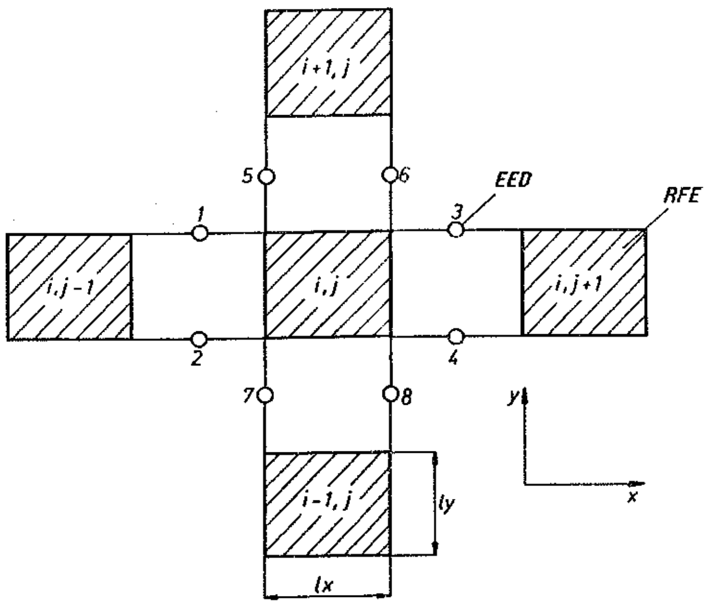

FIG. 3. Principle of indexing the RFE (rigid finite elements) and EED (elastic elements with damping).

The damping coefficients are defined as for an isotropic material, the properties of which are those of the Kelvin - Voigt model [3], in agreement with the relation

$$c_{\dots} = \frac{Q^{-1}}{\omega} k_{\dots} \,, \tag{3.7}$$

where $k_{\dots}$ – arbitrary stiffness coefficient determined by the relations (3.5) and (3.6), $c_{\dots}$ – damping coefficient corresponding to any given $k_{\dots}$, Q^{-1} – coefficient of losses, ω – angular frequency of vibration.

In all the relations given below, the way of indexing the variables and coefficients will be simplified as follows:

- the indices i, j of the variables and coefficients will be disregarded. Thus, for instance, $z_{i,j} \to z$.
- The remaining modifications of indices are

$$i,\ j+1 \to j+1,$$
$$i,\ j-1 \to j-1,$$
$$i+1,\ j \to i+1,$$
$$i-1,\ j \to i-1.$$

Let us substitute

$$
\begin{aligned}
B_1 &= c_{zh}(\dot z + \dot\varphi_x Y2 + \dot\varphi_y X1 - \dot z_{j-1} - \dot\varphi_{xj-1} Y2_{j-1} - \dot\varphi_{yj-1} X2_{j-1})\\
&\quad + k_{zh}(z + \varphi_x Y2 + \varphi_y X1 - z_{j-1} - \varphi_{xj-1} Y2_{j-1} - \varphi_{yj-1} X2_{j-1}),\\
B_2 &= c_{zh}(\dot z + \dot\varphi_x Y1 + \dot\varphi_y X1 - \dot z_{j-1} - \dot\varphi_{xj-1} Y1_{j-1} - \dot\varphi_{yj-1} X2_{j-1})\\
&\quad + k_{zh}(z + \varphi_x Y1 + \varphi_y X1 - z_{j-1} - \varphi_{xj-1} Y1_{j-1} - \varphi_{yj-1} X2_{j-1}),\\
B_3 &= c_{zh}(\dot z_{j+1} + \dot\varphi_{xj+1} Y2_{j+1} + \dot\varphi_{yj+1} X1_{j+1} - \dot z - \dot\varphi_x Y2 - \dot\varphi_y X2)\\
&\quad + k_{zh}(z_{j+1} + \varphi_{xj+1} Y2_{j+1} + \varphi_{yj+1} X1_{j+1} - z - \varphi_x Y2 - \varphi_y X2),\\
B_4 &= c_{zh}(\dot z_{j+1} + \dot\varphi_{xj+1} Y1_{j+1} + \dot\varphi_{yj+1} X1_{j+1} - \dot z - \dot\varphi_x Y1 - \dot\varphi_y X2)\\
&\quad + k_{zh}(z_{j+1} + \varphi_{xj+1} Y1_{j+1} + \varphi_{yj+1} X1_{j+1} - z - \varphi_x Y1 - \varphi_y X2),\\
B_5 &= c_{zw}(\dot z_{i+1} + \dot\varphi_{xi+1} Y1_{i+1} + \dot\varphi_{yi+1} X1_{i+1} - \dot z - \dot\varphi_x Y2 - \dot\varphi_y X1)\\
&\quad + k_{zw}(z_{i+1} + \varphi_{xi+1} Y1_{i+1} + \varphi_{y1+1} X1_{i+1} - z - \varphi_x Y2 - \varphi_y X1),\\
B_6 &= c_{zw}(\dot z_{i+1} + \dot\varphi_{xi+1} Y1_{i+1} + \dot\varphi_{yi+1} X2_{i+1} - \dot z - \dot\varphi_x Y2 - \dot\varphi_y X2)\\
&\quad + k_{zw}(z_{i+1} + \varphi_{xi+1} Y1_{i+1} + \varphi_{yi+1} X2_{i+1} - z - \varphi_x Y2 - \varphi_y X2),\\
B_7 &= c_{zw}(\dot z + \dot\varphi_x Y1 + \dot\varphi_y X1 - \dot z_{i-1} - \dot\varphi_{xi-1} Y2_{i-1} - \dot\varphi_{yi-1} X1_{i-1})\\
&\quad + k_{zw}(z + \varphi_x Y1 + \varphi_y X1 - z_{i-1} - \varphi_{xi-1} Y2_{i-1} - \varphi_{yi-1} X1_{i-1}),\\
B_8 &= c_{zw}(\dot z + \dot\varphi_x Y1 + \dot\varphi_y X2 - \dot z_{i-1} - \dot\varphi_{xi-1} Y2_{i-1} - \dot\varphi_{yi-1} X2_{i-1})\\
&\quad + k_{zw}(z + \varphi_x Y1 + \varphi_y X2 - z_{i-1} - \varphi_{xi-1} Y2_{i-1} - \varphi_{yi-1} X2_{i-1});
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
B_{fx1} &= B_{fx2} = c_{fxh}(\dot\varphi_x - \dot\varphi_{xj-1}) + k_{fxh}(\varphi_x - \varphi_{xj-1}),\\
B_{fy1} &= B_{fy2} = c_{fyh}(\dot\varphi_y - \dot\varphi_{yj-1}) + k_{fyh}(\varphi_y - \varphi_{yj-1}),\\
B_{fx3} &= B_{fx4} = c_{fxh}(\dot\varphi_{xj+1} - \dot\varphi_x) + k_{fxh}(\varphi_{xj+1} - \varphi_x),\\
B_{fy3} &= B_{fy4} = c_{fyh}(\dot\varphi_{yj+1} - \dot\varphi_y) + k_{fyh}(\varphi_{yj+1} - \varphi_y),\\
B_{fx5} &= B_{fx6} = c_{fxw}(\dot\varphi_{xi+1} - \dot\varphi_x) + k_{fxw}(\varphi_{xi+1} - \varphi_x),\\
B_{fy5} &= B_{fy6} = c_{fyw}(\dot\varphi_{yi+1} - \dot\varphi_y) + k_{fyw}(\varphi_{yi+1} - \varphi_y),\\
B_{fx7} &= B_{fx8} = c_{fxw}(\dot\varphi_x - \dot\varphi_{xi-1}) + k_{fxw}(\varphi_x - \varphi_{xi-1}),\\
B_{fy7} &= B_{fy8} = c_{fyw}(\dot\varphi_y - \dot\varphi_{yi-1}) + k_{fyw}(\varphi_y - \varphi_{yi-1}).
\end{aligned}
\tag{3.9}
$$

The geometrical quantities $X1$, $X2$, $Y1$ and $Y2$ involved in (3.8) and (3.9), for internal RFE, are defined as follows:

$$
X1_{...} = -\frac{l_x}{2}, \qquad Y2_{...} = \frac{l_x}{2}, \tag{3.10}
$$

$$
Y1_{...} = -\frac{l_y^2}{2}, \qquad Y2_{...} = \frac{l_y^2}{2}. \tag{3.11}
$$

For elements located at the edges of the plate parallel to the x-axis (that is for $i = 1$ and $i = n_y$), the quantities $X1$ and $X2$ are defined by the relations (3.10), and $Y1$ and $Y2$ – by the relations (3.12),

$$Y1_{\text{edge } x} = -\frac{l_y^2}{4}, \qquad Y2_{\text{edge } x} = \frac{l_y^2}{4}. \tag{3.12}$$

For elements located at the edges of the plate parallel to the y-axis (that is for $j = 1$ and $j = n_x$), the quantities $Y1$ and $Y2$ are defined by the relations (3.11), and $X1$ and $X2$ – by the relations

$$X1_{\text{edge } y} = -\frac{l_x}{4}, \qquad X2_{\text{edge } y} = \frac{l_x}{4}. \tag{3.13}$$

By performing the substitutions (3.8) and (3.9) and using the system of notations assumed, the set of equations describing the vibrations of RFE with the indices i, j can be expressed in the form

$$\begin{aligned}
&m\,\ddot{z} + B_1 + B_2 - B_3 - B_4 - B_5 - B_6 + B_7 + B_8 = 0,\\
&J_x\,\ddot{\varphi}_x + Y1(B_2 - B_4 + B_7 + B_8) + Y2(B_1 - B_3 - B_5 - B_6) + B_{fx1}\\
&\qquad + B_{fx2} - B_{fx3} - B_{fx4} - B_{fx5} - B_{fx6} + B_{fx7} + B_{fx8} = 0,\\
&J_y\,\ddot{\varphi}_y + X1(B_1 + B_2 - B_5 + B_7) + X2(-B_3 - B_4 - B_6 + B_8) + B_{fy1}\\
&\qquad + B_{fy2} - B_{fy3} - B_{fy4} - B_{fy5} - B_{fy6} + B_{fy7} + B_{fy8} = 0.
\end{aligned} \tag{3.14}$$

If we denote

$$m_{\text{RFE}} = \frac{\rho l b h}{n_x n_y}, \tag{3.15}$$

the mass m and the moments of inertia J_x and J_y of the RFE element with indices i, j located in the interior of the plate are defined as

$$m = m_{\text{RFE}}, \qquad J_x = \frac{m_{\text{RFE}}}{12}(l_y^2 + h^2), \qquad J_y = \frac{m_{\text{RFE}}}{12}(l_x^2 + h^2). \tag{3.16}$$

For elements located at the edges of the plate parallel to the x-axis (except the elements located at the corners), we have

$$m = \frac{m_{\text{RFE}}}{2}, \qquad J_x = \frac{m_{\text{RFE}}}{24}\left(\frac{l_y^2}{4} + h^2\right), \qquad J_y = \frac{m_{\text{RFE}}}{24}(l_x^2 + h^2), \tag{3.17}$$

and, in addition

$$\begin{aligned}
B_k &= 0,\\
B_{fxk} &= B_{fyk} = 0,
\end{aligned} \tag{3.18}$$

for $k = 2,4,7,8$, if $i = 1$ and
for $k = 1,3,5,6$, if $i = n_y$.

For elements located at the edges of the plate parallel to the y-axis (except the elements located at the corners), we have

$$(3.19)\qquad m = \frac{m_{\text{RFE}}}{2}, \qquad J_x = \frac{m_{\text{RFE}}}{24}(l_y^2 + h^2), \qquad J_y = \frac{m_{\text{RFE}}}{24}\left(\frac{l_x^2}{4} + h^2\right),$$

the relations (3.18) being additionally satisfied
for $k = 1,2,5,7$, if $j = 1$,
for $k = 3,4,6,8$, if $j = n_x$.

For elements located at the corners we have

$$(3.20)\qquad m = \frac{m_{\text{RFE}}}{4}, \qquad J_x = \frac{m_{\text{RFE}}}{48}\left(\frac{l_y^2}{4} + h^2\right), \qquad J_y = \frac{m_{\text{RFE}}}{48}\left(\frac{l_x^2}{4} + h^2\right),$$

and the expressions B_k, B_{fxk}, B_{fyk} take the following values:

In the case of $i = 1, j = 1$:
values which are in agreement with relation $(3.18)_1$ for $k = 1,4,5,7,8$;
values which are in agreement with relation $(3.18)_2$ for $k = 1,2,4,5,7,8$.
The quantity B_2 becomes

$$(3.21)\qquad B_2 = c_{31}\left(\dot z + \dot\varphi_x\, Y1 + \dot\varphi_y\, X1 - \dot z_{21}\right) + k_{31}(z + \varphi_x Y1 + \varphi_y X1 - z_{21}).$$

In the case of $i = 1$, $j = n_x$:
relation $(3.18)_1$ is satisfied for $k = 2,3,6,7,8$;
relation $(3.18)_2$ is satisfied for $k = 2,3,4,6,7,8$;
B_4 takes the value

$$(3.22)\qquad B_4 = c_{33}\left(\dot z + \dot\varphi_x\, Y1 + \dot\varphi_y\, X2 - \dot z_{23}\right) + k_{33}(z + \varphi_x Y1 + \varphi_y X2 - z_{23}).$$

In the case of $i = n_y$, $j = 1$:
relation $(3.18)_1$ is satisfied for $k = 2,3,5,6,7$;
relation $(3.18)_2$ is satisfied for $k = 1,2,3,5,6,7$;
B_1 assumes the value

$$(3.23)\qquad B_1 = c_{35}\left(\dot z + \dot\varphi_x\, Y2 + \dot\varphi_y\, X1 - \dot z_{25}\right) + k_{35}(z + \varphi_x Y2 + \varphi_y X1 - z_{25}).$$

In the case of $i = n_y$, $j = n_x$:
relation $(3.18)_1$ is satisfied for $k = 1,4,5,6,8$;
relation $(3.18)_2$ is satisfied for $k = 1,3,4,5,6,8$;
B_3 assumes the value

$$(3.24)\qquad B_3 = c_{37}\left(\dot z + \dot\varphi_x\, Y2 + \dot\varphi_y\, X2 - \dot z_{27}\right) + k_{37}(z + \varphi_x Y2 + \varphi_y X2 - z_{27}).$$

This way of constructing a mathematical model of vibration of the plate is very convenient for formulating a computation algorithm. In its expanded form, the vibrations of the plate should be described by a set of $3 \times (n_x + 1) \times (n_y + 1)$ differential equations. The above relations have been used to formulate a procedure of vibration analysis for the partial model V, that is for the plate.

4. The results of the analysis

The principal aim of this presentation of the computation results is to show the convergence of the iteration procedure used. The problem of convergence of iteration procedures for linear systems was studied by analytical means in [7] and [8]. In the present paper it will be illustrated by quoting the numerical results obtained.

Figure 4 shows, as a function of time, the displacement z_{21} of the mass m_{21}, and Fig. 5 – the displacement of the corner of the plate connected with that mass by means of an elastic pad with damping, the indices of which are 3, 1 (see Fig. 2).

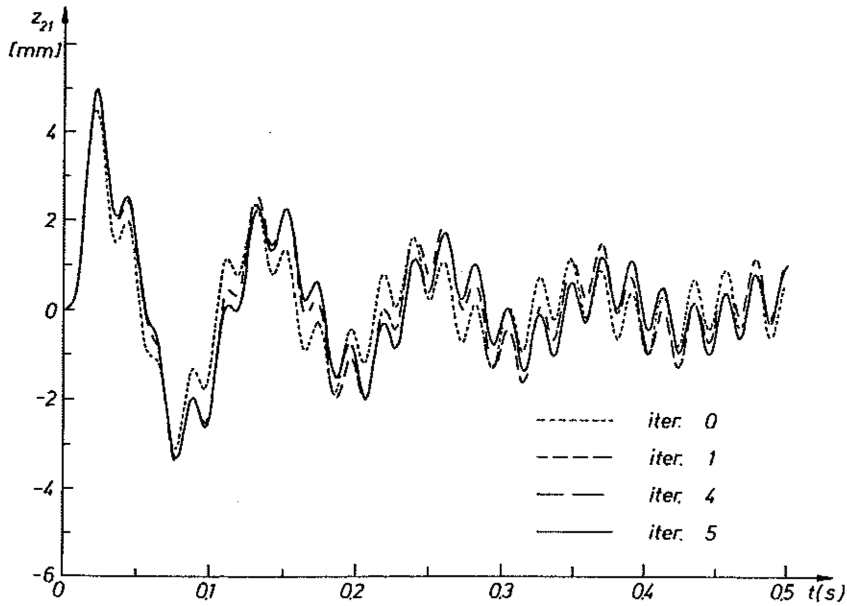

FIG. 4. Displacements of a mass m_{21} obtained in consecutive iterations.

In both diagrams it is seen that the iterations 4 and 5 coincide almost perfectly, therefore the computation process may be stopped (iterations 2

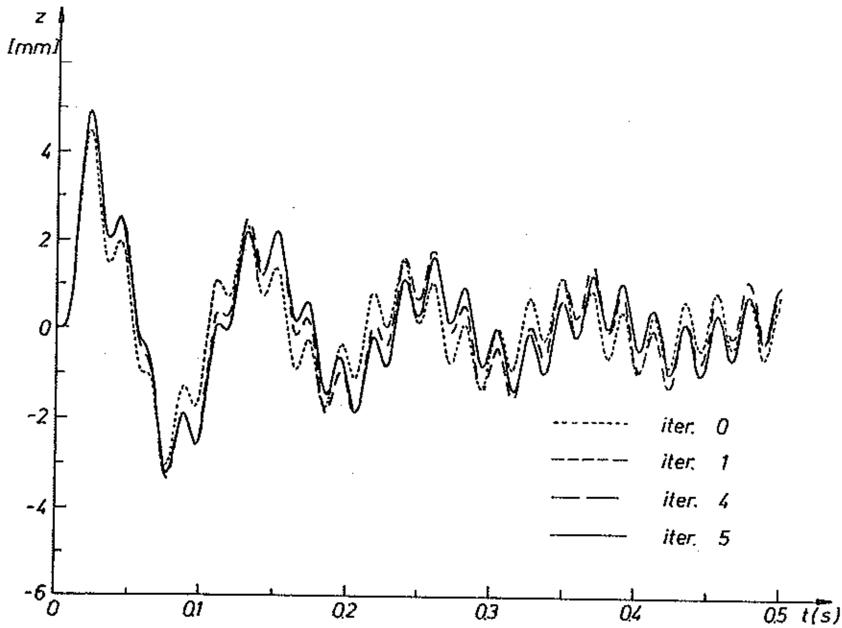

FIG. 5. Displacements of the corner of the plate supported on the mass m_{21}.

and 3 have not been shown for the sake of clarity). This conclusion is confirmed by the analysis of other functions of time (not discussed in the present paper), obtained for variables appearing in the partial model I and models II to IV.

Vibration analysis of the partial model V (the plate) shows similar the differences between the iterations 0, 1 and 4, 5 for elements located at the corners and in the interior of the plate region. Convergence was "most difficult" to be obtained for the edges of the plate, although Fig. 6 shows that the analysis may be limited, also in this case, to about 5 iterations. This figure represents the vibration process of the middle point of the shorter edge of the plate (between the supports indexed 3,1 and 3,5).

Figures 7 and 8 illustrate the convergence of the procedure in another way, showing the form of the plate obtained for consecutive iterations at two arbitrarily selected instants of time $t = 0.35$ [s] and $t = 0.45$ [s]. To enhance the differences between the iterations, the dimensions of the plate (axes x and y) are expressed in meters, and the vertical displacements – in milimeters.

Despite the expansion of the vertical axis, the forms of the plate obtained in the iterations 4 and 5 are almost identical (and they coincide over a considerable part of the plate).

In the analysis, the results of which are presented above, the rigidity

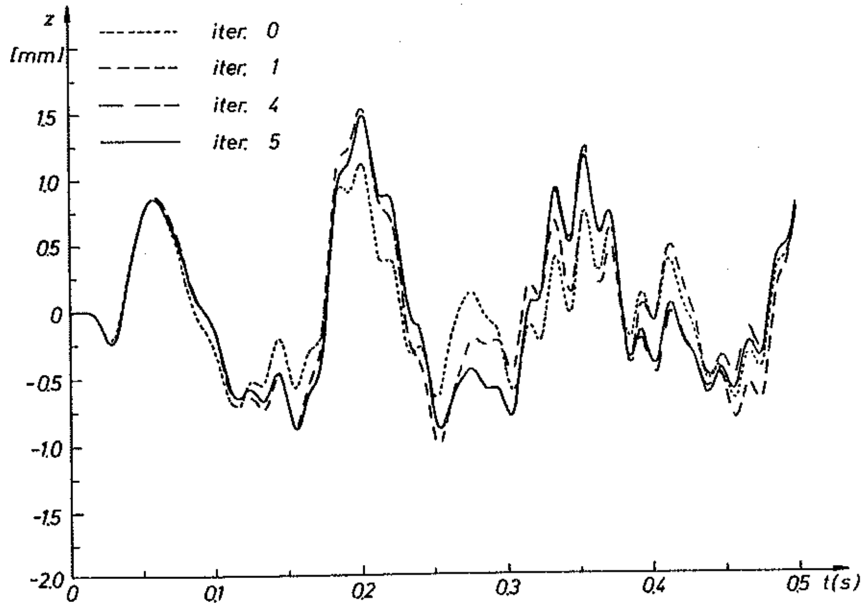

FIG. 6. Displacements of the shorter edge of the plate.

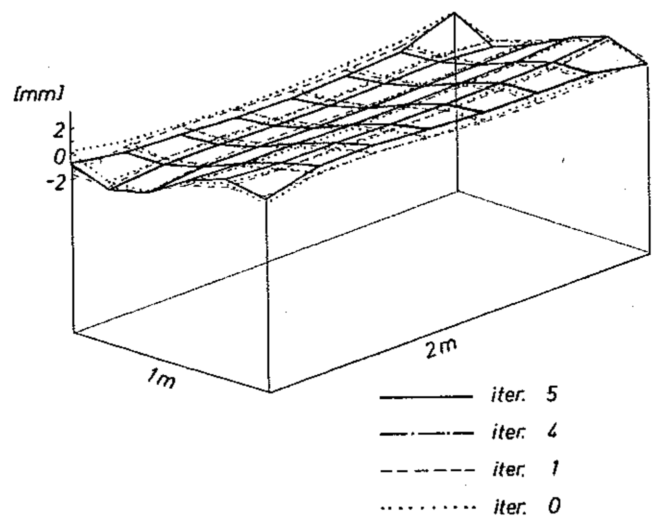

FIG. 7. The form of the plate as obtained in consecutive iterations, at the instant of time $t = 0.35$ s.

of the supports k_{3i} of the plate has been deliberately increased in order to bring out the differences between particular iterations. Theoretical analysis as well as practical realization of the computation show that a decrease in rigidity of the supports k_{3i} weakens the couplings acting in the system and accelerates the convergence of the iteration procedure.

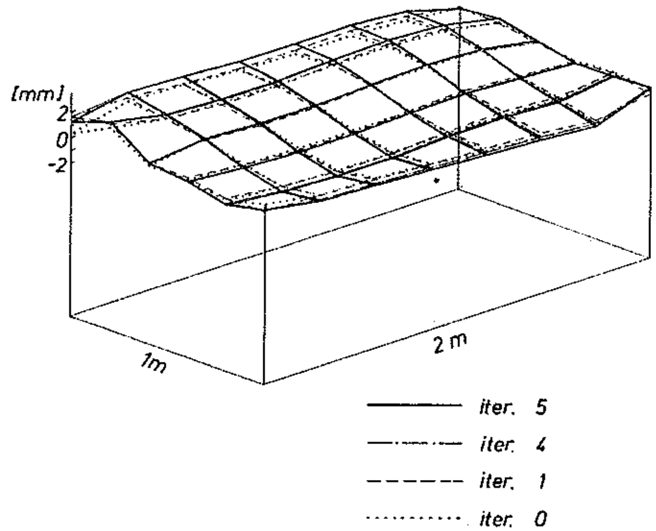

FIG. 8. The form of the plate as obtained in consecutive iterations, at the instant of time $t = 0.45$ s.

Figures 9 and 10 shows the results of the computations performed for the rigidities k_{3i} reduced to one-fifth of their original values (all the other data

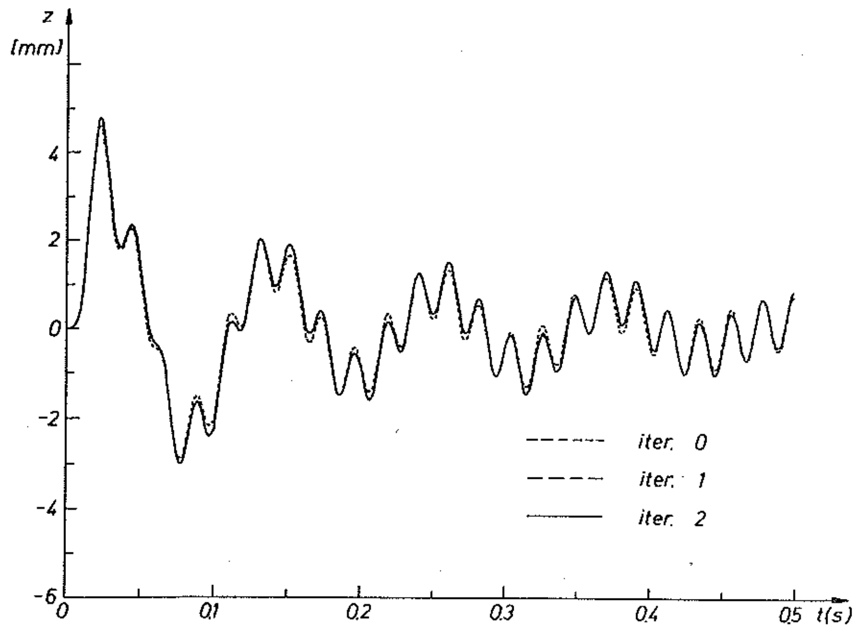

FIG. 9. Displacement of the corner of the plate supported on the mass m_{21}, after a 1:5 reduction in the support rigidity k_{31} (result of weakening of the couplings between the plate and the partial models I to IV).

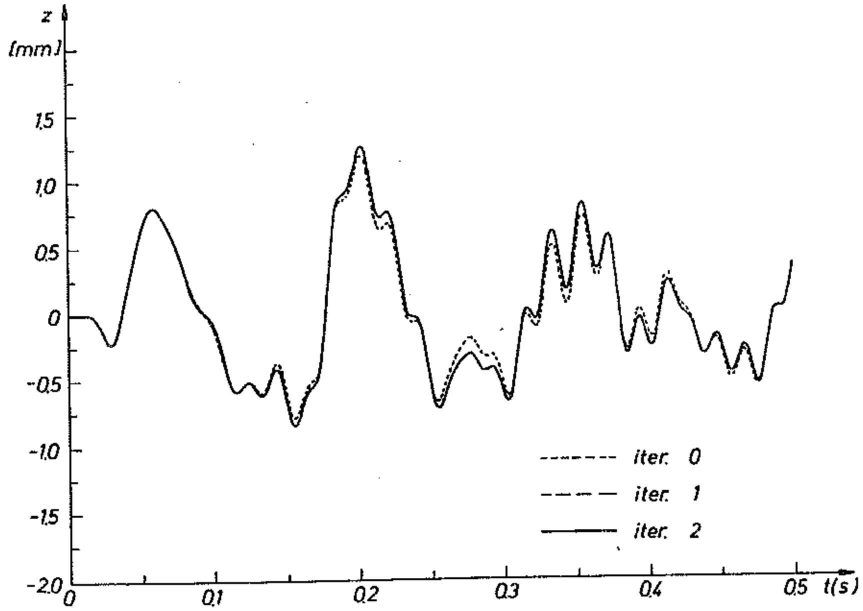

FIG. 10. Displacement of the shorter edge of the plate after a 1:5 reduction in the support rigidity k_{31}; (result of weakening of the couplings between the plate and the partial models I to IV).

remaining unchanged). They show the displacements of the same points as in Figs. 5 and 6. In this case the computation could be limited to the iterations 0 and 1 only.

A similar effect of weakening of the couplings between the partial models I to IV and the plate (partial model V) can also be obtained by increasing the damping in the system, [7, 8].

5. Conclusions

The results just presented confirm the efficiency of the method suggested. For such models the iterative procedure of computation is rapidly convergent. Such a way of analysing the vibration of a complete model is more lucid and facilitates inferences to be drawn. It makes also possible to replace any partial model with another, of a more developed structure, or to take into consideration the nonlinearity of the elastic elements, etc. This reduces to an exchange of one of the procedures of the computation program for another.

A simplification which is often made in the analysis of vibration of system similar to the system represented in Fig. 2 is the assumption of total uncoupling of the system, that is the analysis of the partial models I to IV and introduction of the response of those systems as an excitation acting on the plate. In some cases such a procedure may be justified (see Figs. 9 and 10), but sometimes it may lead to considerable errors (see Fig. 6, for instance). An analysis made in the manner described in the present paper will enable us to avoid such errors.

There is another advantage of the above method for the analysis of combined discrete-continuous systems: the knowledge of the weak couplings acting in the system may be used, by weakening them, to reduce the vibration and the noise. In the physical sense, weakening of the couplings means that vibrations of some subassemblies of the machine are not transferred (or transferred in a small proportion) to other subasemblies and elements.

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