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ON MICROPOLAR FLUID MODEL FOR BLOOD FLOW THROUGH AN ARTERY WITH MILD STENOSIS

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Blood flow through an artery with various shapes of mild constrictions has been analysed by characterising it as a micropolar fluid. Microstructural effects on the pressure gradient, resistance to the flow and the wall shear stress are determined. The results are numerically computed and discussed.

1. Introduction

Analytical and experimental investigations of blood flow through arteries are considered to be very important since various cardiovascular diseases are closely associated with the flow conditions in the blood vessels. The normal flow of blood is disturbed due to some abnormal growths, e.g. stenosis in the lumen of the artery. The specific reason for the initiation of such growth is actually not known but it is obvious that its presence may lead to serious cardiovascular diseases. At low shear rates, while blood is flowing through an arterial tube of small diameter, it exhibits non-Newtonian behaviour while at high shear rates, commonly formed in large arteries, blood behaves like a Newtonian fluid (cf. COKELET [1], HUCKABA *et al.* [2], BUGLIARELLO [3]). Recently, many researchers have studied the flow characteristics of blood in an artery with mild stenosis by considering blood both as a Newtonian and non-Newtonian fluid (cf. SHUKLA *et al.* [4, 5], HALDER [6], HALDER *et al.* [7], CHAKRAVARTY *et al.* [8]). ERINGEN [9] developed the theory of micro-fluids, which is applied to flow in rheologically complex fluids, such as liquid crystals, polymeric suspensions and animal blood. A subclass of these fluids which can support couple stresses and body couples and exhibit microrotational effects and microrotational inertia are termed the micropolar fluids (ERINGEN [10]). ARIMAN *et al.* [11], KANG *et al.* [12], PARVATHAMMA *et al.* [13] have constructed models of microcirculation considering the micropolar character of the fluid, which are useful in explaining certain aspects of blood flow through capillaries. Micro-structural and peripheral layer viscosity effects on the flow of blood through artery with mild stenosis have been studied by TANDON *et al.* [14].

It is known that heart produces a periodic or pulsating flow on the arterial side of the circulatory system. The amplitude of the flow pulse is the largest in the aorta and becomes gradually smaller as the system branches (RODKIEWICZ [15]). OKA [16] conjectured that the blood flow in microvessels can be treated approximately as a steady flow, where the pressure gradient becomes constant in time. In the present paper we study the effect of various shapes of mild constrictions on the characteristics of steady blood flow through the arterial tube of a sufficiently small diameter. In our analysis, we treat the blood as a micropolar fluid and consider the shape of the constrictions to be different from the usually assumed cosine curve (TANDON *et al.* [14). The shape of the constriction is not symmetric about its maximum.

2. ANALYSIS OF THE PROBLEM

We consider axially symmetric, laminar, steady, one-dimensional flow of the blood in a circular rigid tube past a mild stenosis. Blood is considered to be a micropolar, incompressible fluid. The assumption is well justified in the small blood vessels, where the shear rates are low (PARVATHAMMA *et al.* [13]). The geometry of the stenosis is assumed to be manifested in the arterial segment according to CHAKRAVARTY *et al.* [8] by

$$(1)\qquad \begin{aligned} \frac{R(x)}{R_0} &= 1 - A[L_0^{n-1}(x-d) - (x-d)^n], \qquad \text{for} \quad d < x < d + L_0\,, \\ \frac{R(x)}{R_0} &= 1, \qquad \text{otherwise,} \end{aligned}$$

where $R(x)$ is the radius of the artery in the stenotic region, R_0 is the radius of the normal artery, L_0 is the length of the stenosis, n (≥ 2) is a parameter determining the shape of the stenosis, d indicates its location, and A is given by

$$(2)\qquad A = \frac{\varepsilon}{R_0 L_0^n} \cdot \frac{n^{n/(n-1)}}{(n-1)}\,.$$

Here ε is the maximum height of the stenosis located at

$$(3)\qquad X = d + \frac{L_0}{n^{1/(n-1)}}\,,$$

such that $\varepsilon / R_0 \ll 1$ (Fig. 1).

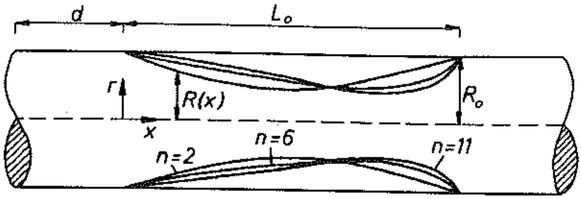

FIG. 1. Geometry of the stenosis.

Under the assumption of slow viscous motion, neglecting the inertia terms, the basic equations of motion in cylindrical polar coordinates are

$$(\mu_v + k_v)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{k_v}{r}\frac{\partial}{\partial r}(rw) = \frac{dp}{dx}, \tag{4}$$

$$\gamma_v\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rw)\right] - k_v\frac{\partial V}{\partial r} - 2k_v w = 0, \tag{5}$$

where V is the axial velocity, dp/dx is the pressure gradient, μ_v is the viscosity of the fluid, w is the suspending particle rotation, k_v is the relative rotational viscosity and γ_v is the viscosity gradient of total rotation.

The corresponding boundary conditions are

$$\begin{aligned} &V \quad \text{is finite} \quad \text{and} \quad w = 0 \quad \text{at} \quad r = 0,\\ &V = 0 \quad \text{and} \quad \frac{1}{r}\frac{\partial}{\partial r}(rw) = 0 \quad \text{at} \quad r = R. \end{aligned} \tag{6}$$

Solutions of Eqs. (4) and (5) with the boundary conditions (6) are given by

$$w = -\frac{1}{2(k_v + 2\mu_v)}\frac{dp}{dx}\left[r - \frac{2}{\lambda}\frac{I_1(\lambda r)}{I_0(\lambda R)}\right], \tag{7}$$

$$V = -\frac{1}{4(\mu_v + k_v)}\frac{dp}{dx}\left[(R^2 - r^2)\frac{2(\mu_v + k_v)}{(k_v + 2\mu_v)} + \frac{4k_v}{(k_v + 2\mu_v)}\frac{I_0(\lambda r) - I_0(\lambda R)}{\lambda^2 I_0(\lambda R)}\right], \tag{8}$$

where

$$\lambda^2 = \frac{k_v}{\gamma_v}\frac{(k_v + 2\mu_v)}{(\mu_v + k_v)} \tag{9}$$

and $I_n(x)$ is the modified Bessel function of order n.

The volumetric flow rate Q of the fluid across any cross-section in the stenotic region of the tube is

$$Q = \int_0^R 2\pi r \cdot V\, dr.$$

Substituting the expression for V given in (8) and integrating we obtain

$$(10)\qquad Q=-\frac{\pi}{4(\mu_v+k_v)}\frac{dp}{dx}\left[\frac{(\mu_v+k_v)}{(k_v+2\mu_v)}R^4+\frac{8k_v}{(k_v+2\mu_v)}\frac{R}{\lambda^2 I_0(\lambda R)}\cdot\left\{\frac{1}{\lambda}I_1(\lambda R)-\frac{R}{2}I_0(\lambda R)\right\}\right],$$

where λ^2 is given by Eq. (9).

If Q_0 is the flow rate of the fluid in the tube in absence of stenosis, then

$$(11)\qquad Q_0=-\frac{\pi}{4(\mu_v+k_v)}\left(\frac{dp}{dx}\right)_0\left[\frac{(\mu_v+k_v)}{(k_v+2\mu_v)}R_0^4+\frac{8k_v}{(k_v+2\mu_v)}\frac{R_0}{\lambda^2 I_0(\lambda R_0)}\cdot\left\{\frac{1}{\lambda}I_1(\lambda R_0)-\frac{R_0}{2}I_0(\lambda R_0)\right\}\right],$$

where $\left(\frac{dp}{dx}\right)_0$ is the presssure gradient of the fluid in the unconstricted tube. If Q and Q_0 occur in the same system, then $Q=Q_0$ and thus we obtain from (10) and (11)

$$(12)\qquad \left(\frac{dp}{dx}\right)\Big/\left(\frac{dp}{dx}\right)_0=\frac{aR_0^4+8b\frac{R_0}{\lambda^2}\frac{1}{I_0(\lambda R_0)}\left\{\frac{1}{\lambda}I_1(\lambda R_0)-\frac{R_0}{2}I_0(\lambda R_0)\right\}}{aR^4+8b\frac{R}{\lambda^2}\frac{1}{I_0(\lambda R)}\left\{\frac{1}{\lambda}I_1(\lambda R)-\frac{R}{2}I_0(\lambda R)\right\}},$$

where

$$(13)\qquad a=\frac{\mu_v+k_v}{k_v+2\mu_v},\qquad b=\frac{k_v}{k_v+2\mu_v}.$$

Now, for large values of x, the modified Bessel function may be expressed by the approximate formula

$$(14)\qquad I_n(x)\simeq\frac{e^x}{\sqrt{2\pi x}}.$$

Thus from (12), on using (14), we obtain the relative local pressure gradient as

$$(15)\qquad \frac{\left(\frac{dp}{dx}\right)}{\left(\frac{dp}{dx}\right)_0}=\frac{aR_0^4+\frac{8bR_0}{\lambda^2}\left(\frac{1}{\lambda}-\frac{R_0}{2}\right)}{aR^4+\frac{8bR}{\lambda^2}\left(\frac{1}{\lambda}-\frac{R}{2}\right)},$$

where R/R_0 is given by Eq. (1).

From Eq. (10)

$$\frac{dp}{dx} = -\frac{4(\mu_v+k_v)}{\pi}Q\left[aR^4+\frac{8b}{\lambda^2}R\frac{1}{I_0(\lambda R)}\left\{\frac{1}{\lambda}I_1(\lambda R)-\frac{R}{2}I_0(\lambda R)\right\}\right]^{-1}. \tag{16}$$

Integrating Eq. (16) and applying (14) we obtain the resistance to flow λ expressed in the form

$$\lambda = \frac{P_i - P_0}{Q} = \frac{4(\mu_v + k_v)}{\pi}\int_{x=0}^{L}\frac{dx}{aR^4 + \frac{8b}{\lambda^2}R\left(\frac{1}{\lambda} - \frac{R}{2}\right)} \tag{17}$$

$$= \frac{4(\mu_v + k_v)}{\pi}\left[\frac{L - L_0}{aR^4 + \frac{8b}{\lambda^2}R_0\left(\frac{1}{\lambda} - \frac{R_0}{2}\right)} + \int_{d}^{L_0+d}\frac{dx}{aR^4 + \frac{8b}{\lambda^2}R\left(\frac{1}{\lambda} - \frac{R}{2}\right)}\right],$$

where $p = P_i$ at $x = 0$ and $p = p_0$ at $x = L$, L being the length of the artery.

In the case of no stenosis ($R = R_0$), the resistance to flow λ_N (normal artery) is given by

$$\lambda_N = \frac{4(\mu_v + k_v)}{\pi}\frac{L}{aR_0^4 + \frac{8b}{\lambda^2}R_0\left(\frac{1}{\lambda} - \frac{R_0}{2}\right)}. \tag{18}$$

From (17) and (18) we obtain the non-dimensional form of resistance to the flow as

$$\frac{\lambda}{\lambda_N} = 1 - \frac{L_0}{L} + \frac{aR_0^4 + \frac{8b}{\lambda^2}R_0\left(\frac{1}{\lambda} - \frac{R_0}{2}\right)}{L} \cdot \int_{d}^{L_0+d}\frac{dx}{aR^4 + \frac{8b}{\lambda^2}R\left(\frac{1}{\lambda} - \frac{R}{2}\right)}, \tag{19}$$

where R/R_0 is given by (1), and constants a, b are given by (13).

The fluid flowing past a stationary solid boundary always exerts a shear stresses on the boundary. Thus there is a non-zero shear rate dv/dr at the wall, and hence a non-zero shear stress at the surface of the stenosis is given by the formula

$$\tau_R = \left[-\mu_v\frac{dv}{dr}\right]_{r=R}, \tag{20}$$

which, on using (8), becomes

$$\tau_R = \frac{\mu_v}{(\mu_v + k_v)}\frac{dp}{dx}\left[-aR + \frac{b}{\lambda}\frac{I_1(\lambda R)}{I_0(\lambda R)}\right]. \tag{21}$$

If the value of τ_R at the throat of the stenosis, i.e. at $x = d + \frac{L_0}{n^{1/(n-1)}}$, is τ_S and τ_N is the shear stress at the wall in absence of stenosis, then using (14), the non-dimensional form of τ_S will be

$$(22) \qquad \tau = \frac{\tau_S}{\tau_N} = \left(\left[-aR_0\left(1-\frac{\varepsilon}{R_0}\right)+\frac{b}{\lambda}\right]\left[aR_0^4+\frac{8b}{\lambda^2}R_0\left(\frac{1}{\lambda}-\frac{R_0}{2}\right)\right]\right)$$
$$\bigg/\left(\left[aR_0^4\left(1-\frac{\varepsilon}{R_0}\right)^4+\frac{8b}{\lambda^2}R_0\left(1-\frac{\varepsilon}{R_0}\right)\left\{\frac{1}{\lambda}-\frac{R_0}{2}\left(1-\frac{\varepsilon}{R_0}\right)\right\}\right]\left(-aR_0+\frac{b}{\lambda}\right)\right).$$

3. Numerical results and discussions

Numerical computations are done wih the use of the data given below and exhibited in Figs. 2–7. For different shapes of the stenosis we consider $n = 2, 6, 11$. We use also the values of the constants given by (Tandon *et*

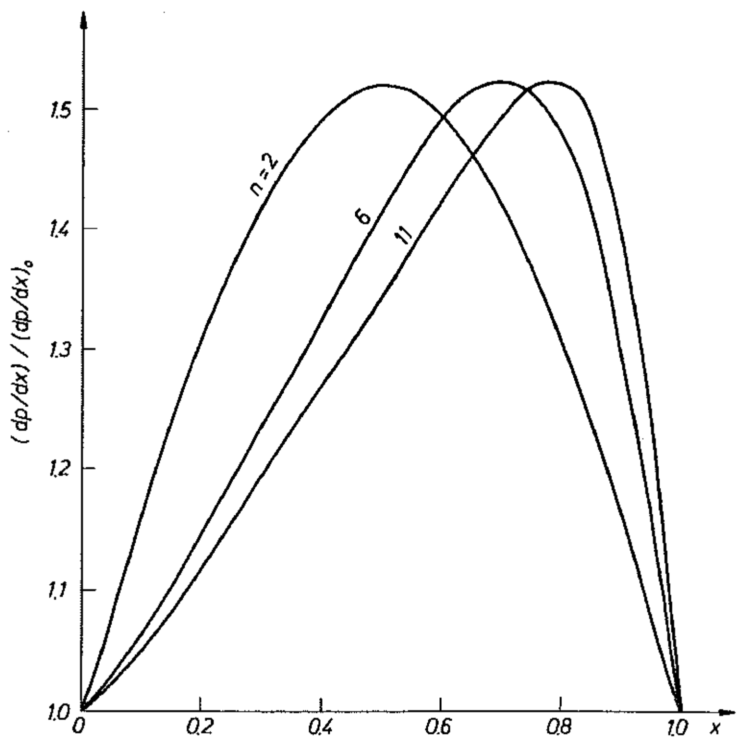

Fig. 2. Distribution of relative pressure gradient for different n.

al. [17], CHATURANI *et al.* [18])

$$\mu_v = 0.8,\ 1,\ 1.4\,\text{cp}, \qquad L = 5\,\text{cm}, \qquad L_0 = 1\,\text{cm},$$
$$k_v = 0.82,\ 0.98,\ 1.14\,\text{cp}, \qquad R_0 = 0.16\,\text{cm}, \qquad d = 0,$$
$$\gamma_v = 12 \times 10^{-8}\,\text{g cm/s}, \qquad \frac{\varepsilon}{R_0} = 0.1,\ 0.2, \ldots, 0.6.$$

The variation of relative local pressure gradient with the length of the stenosis are shown in Fig. 2. It is seen that in each case of $n = 2, 6, 11$, the maximum pressure gradient is attained at the throat of the stenosis. It is also observed that the maximum pressure gradient for the case $n = 2$ (symmetric shape) is attained at $x = 0.5$, while for the cases $n = 6$ and $n = 11$, such maxima are respectively located at $x = 0.7$ and $x = 0.79$.

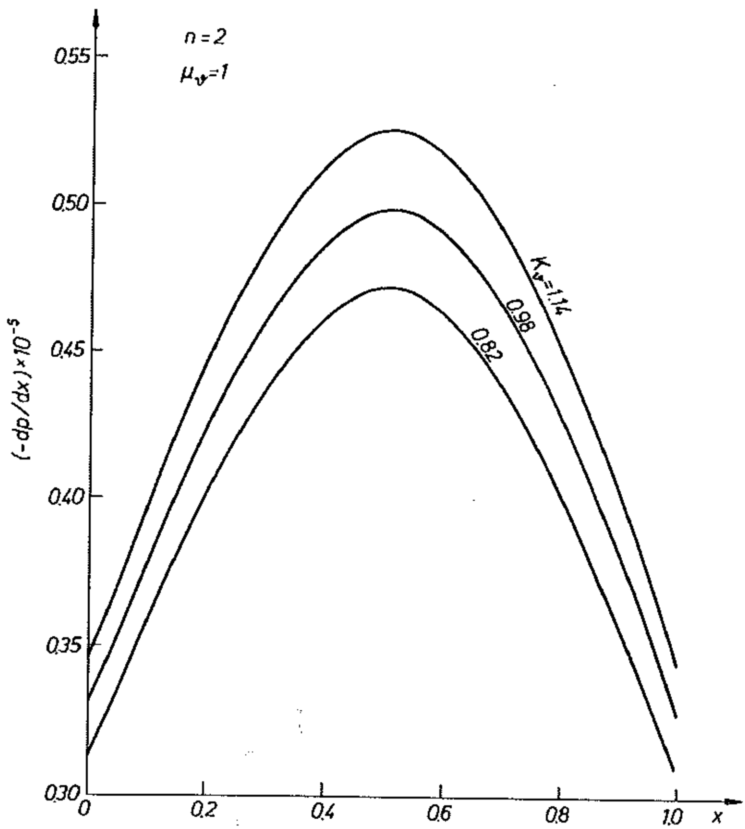

FIG. 3. Distribution of pressure gradient for different K_ϱ.

Figure 3 illustrates the variation of pressure gradient with the length of the stenosis for different values of k_v. It is seen that the pressure gradient increases as the increasing rotational viscosity k_v. ARIMAN *et al.* [11] found that the rotational viscosity increases with the increase of hematocrit value. Thus we may conclude that the pressure gradient increases with the increase of hematocrit value indicating the fall of the pressure. Thus the lowest pressure occurs at the throat of the stenosis due to rise of the hematocrit.

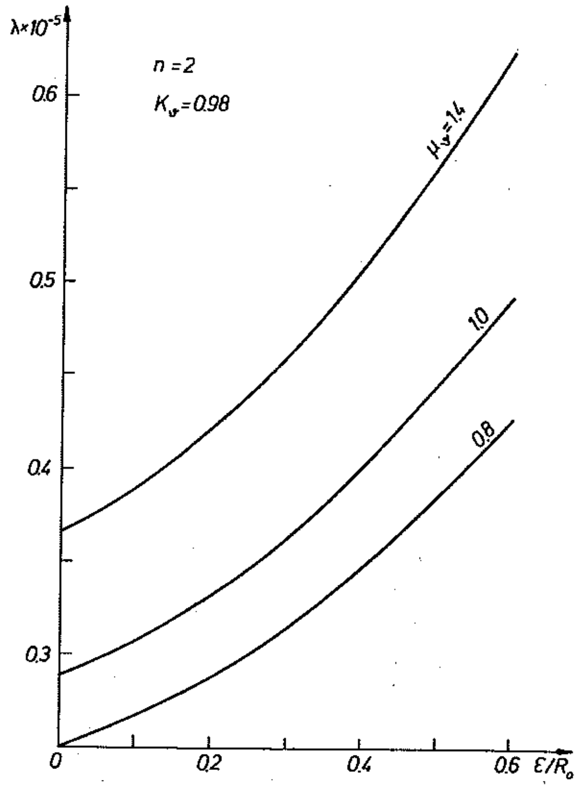

FIG. 4. Variation of flow resistance with ε/R_0 for different μ_ϑ.

Variations of flow resistance with ε/R_0 for different values of μ_v have been shown in Fig. 4. The integration at the right-hand side of Eq. (17) is done numerically. It is observed that in each case of $\mu_v = 0.8$, 1, 1.4, the resistance increases with the increase of ε/R_0, and this increase is faster with greater stenosis height. This observation agrees well with the results obtained by SHUKLA *et al.* [4]. It is also seen that the resistance grows rapidly with the increase of μ_v. This result may be considered significant from the clinical point of view. Deviations in the blood flow from the normal conditions may be not only due to the heart failure or blood vessel disorder, but may also result from high values of blood viscosity as observed in many diseases. DINTENFASS [19] studied intensively the high viscosity syndroms in various diseases (e.g. ischaemic, sickle cell, hemolitic anemia, polycythemia etc.).

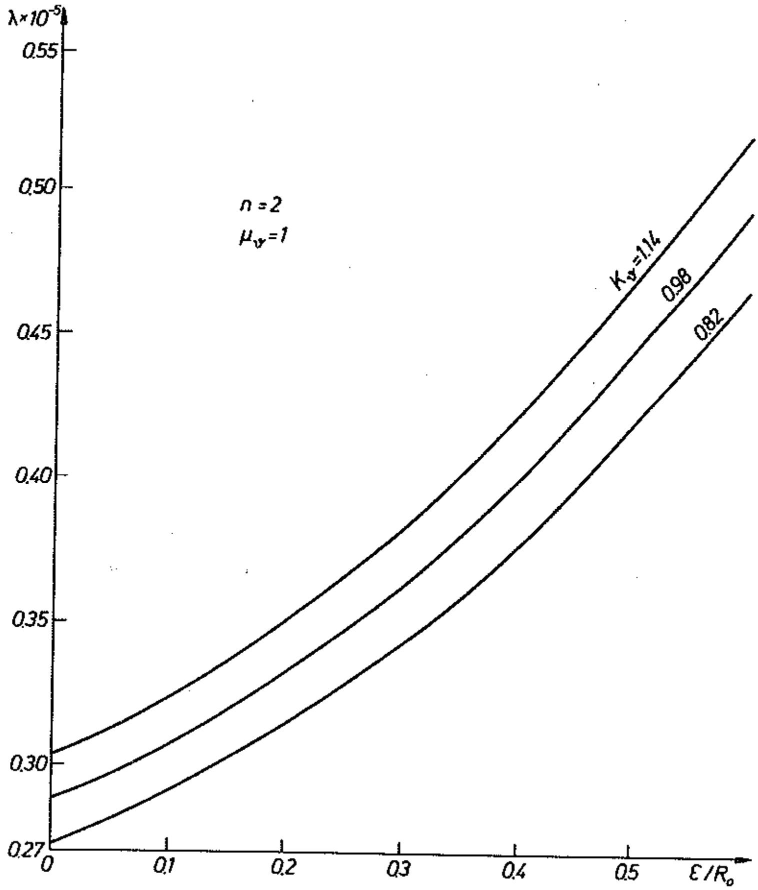

FIG. 5. Variation of flow resistance with ε/R_0 for different K_ϑ.

Figure 5 indicates that the resistance to the flow increases with the increase of rotational viscosity k_v. From ARIMAN *et al.* [11] we have seen that the rotational viscosity increases with the increase of the hematocrit value. Thus we arrive at the conclusion that the resistance to the flow increases with the increase of the hematocrit value. The result is important from the clinical point of view. For a patient suffering from coronary artery disease and, in particular, with vascular narrowing, the determination of optimal hematocrit is considered to be of primary interest. As far as the cerebrovascular disease is concerned, it is reported by THOMAS *et al.* [20] that the cerebral blood flow for patients with hematocrits of 47 – 53% is significantly lower than that in patients with hematocrits of 36 – 46%. By reducing the hematocrit level, the cerebral blood velocity, however, may be increased for the higher group. Such improvement of blood flow is mainly due to reduction of viscosity (OKA [16]).

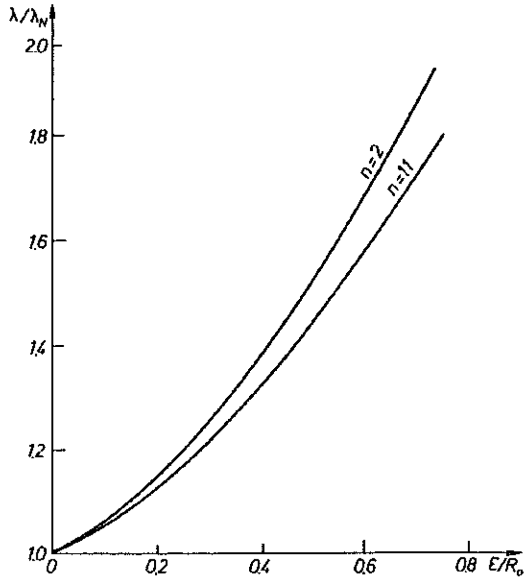

FIG. 6. Variation of λ/λ_N with ε/R_0 for different n.

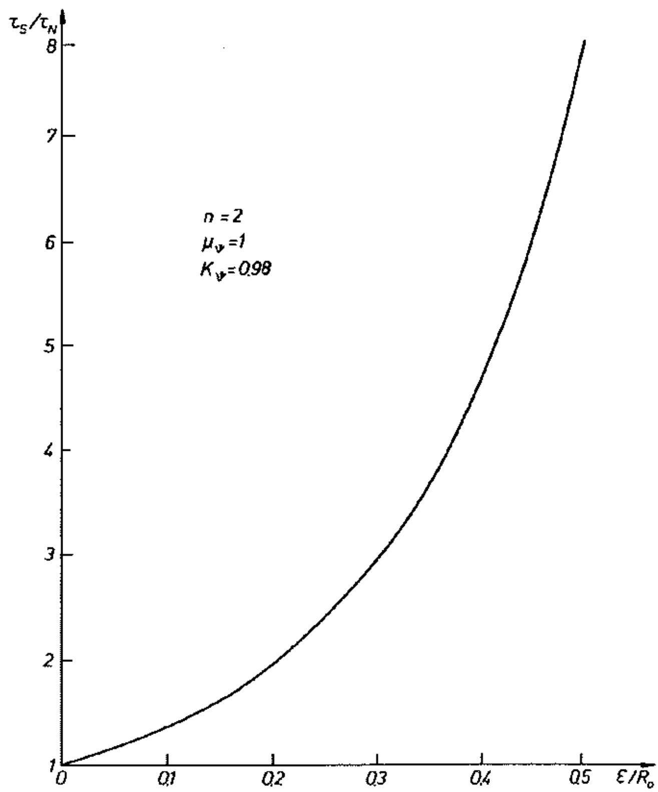

FIG. 7. Variation of τ with ε/R_0.

It is seen form Fig. 6 that the resistance to the flow is maximum for $n = 2$ i.e., for the case of axially symmetric stenosis. The resistance decreases as n increases. It is seen from Fig. 1 that the location of maximum height of the stenosis is displaced towards its right end with the increase in the value of n. Thus the stenosis having that shape, with shifting of the maximum height towards the right end, produces a smaller resistance. This observation is also important with respect to the clinical point of view. This result is in qualitative agreement with CHAKRAVARTY *et al.* [8].

The variation of wall shearing stress with stenosis height are presented in Fig. 7. It is noticed that the wall shear increases with the increase of the stenosis height. This result is similar in nature to that of SHUKLA *et al.* [4].

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