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TRIGONOMETRICAL REPRESENTATION OF TRANSFER MATRIX FOR LAYERED ELASTIC MATERIAL

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A harmonic wave of a fixed frequency propagates across the periodic system of elastic layers. The elementary cell consists of three layers. The transfer matrix M may be expressed by two real parameters φ, ψ and a set P of 64 further scalar parameters $M = M(\varphi, \psi, P)$. The parameters are uniquely defined for the particular M and may be calculated from a system of trigonometrical equations. It has been proved numerically that, for materials and dimensions given in advance, this function for each integer n satisfies the identity $[M(\varphi, \psi, P)]^n = M(n\varphi, n\psi, P)$. The derived identity drastically simplifies the calculation of displacements and stresses in the periodically layered medium.

1. Introduction

In the previous paper [2] the author considered the harmonic wave of fixed frequency propagating in a system of elastic layers periodic in space. The elementary cell consisted of two layers, and the 4×4 transfer matrix M for the elementary cell could be expressed by $2\times4+1$ physical quantities: elastic moduli, thicknesses, densities and propagation direction. Since the transfer matrix has 16 complex components, there exist some relations (symmetries) between the components. These relations resulted in drastic simplification of algebra and analytical proof of important formula for M^k.

In the present paper each cell consists not of two, but of three layers. In this situation the transfer matrix M can be expressed by $3 \times 4 + 1 = 13$ physical quantities. Therefore the number of relations between the components of M is smaller than in the case of a cell consisting of two layers only. In other words, M is less symmetric. This fact results in extremely complex computations, that probably exclude the possibility of analytic proof.

In contrast to [2], only numerical analysis is given. The proposed identity has been checked numerically for many dimensions, and many different physical properties of the three layers constituting the elementary cell.

2. Transfer matrix

In order to obtain the formulae for a cell consisting of three layers, we present short preliminary calculations. Consider two elastic materials separated by the plane $x = x_2$. The layer situated between x_1 and x_2, its displacement, strain and stress will be identified by the label 1, whereas the layer between x_2 and x_3 will be identified by the label 2. The waves are not perpendicular to the y-axis, Fig. 1. Full lines denote the longitudinal waves, and dashed lines – the transverse waves. The longitudinal wave propagating in Medium 1, reflecting on a plane produces the reflected longitudinal and transverse waves propagating in Medium 1 and transmitted longitudinal and transverse waves propagating in the Medium 2.

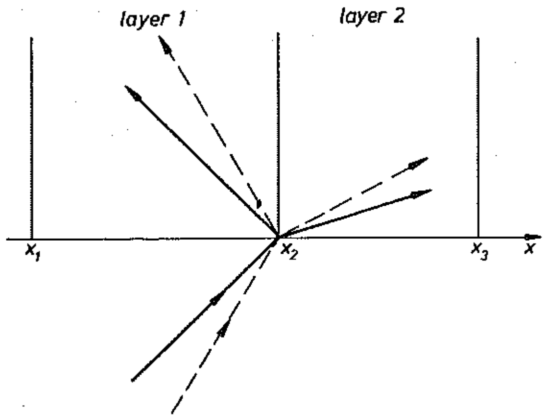

Fig. 1.

In both regions the displacements are represented by the displacement potentials φ_K and ψ_K, $K = 1, 2$. The potential φ_K represents the sum of all longitudinal waves propagating in the layer K, and the potential ψ_K – the sum of all transverse waves propagating in the layer K. We consider the case when the displacement is in the x, y plane. The wave with displacement in the direction perpendicular to xy-plane was considered in [3]. Since the problems are linear, the present paper together with [3] describe the general case.

We expect the following potentials in the regions $K = 1, 2$:

$$\begin{aligned}\varphi_K &= A_K \exp i\,[-\omega t + p_K(x - x_K) + sy] \\ &\qquad + B_K \exp i\,[-\omega t - p_K(x - x_K) + sy], \\ \psi_K &= C_K \exp i\,[-\omega t + q_K(x - x_K) + sy] \\ &\qquad + D_K \exp i\,[-\omega t - q_K(x - x_K) + sy],\end{aligned} \tag{2.1}$$

where p_K, q_K, s are constant parameters. The amplitudes are A_K, B_K, C_K, and D_K. The terms proportional to A_K, B_K represent two harmonic longitudinal waves of amplitudes A_K, B_K. The terms proportional to C_K, D_K represent two transverse waves of amplitudes C_K, D_K. All waves possess the same frequency ω. If a wave does not exist, then the corresponding amplitude equals zero. The parameter s defines the incidence angle.

The planes of constant phase are

$$\pm p_K(x - x_K) + sy = \text{const}, \qquad \pm q_K(x - x_K) + sy = \text{const}.$$

The direction perpendicular to the plane of constant phase is the propagation direction. Denote by α_{LK} and α_{TK} the angles between the propagation direction and the normal to the layers in the k-th layer. There is

$$\text{(2.2)} \qquad \operatorname{tg} \alpha_{LK} = \pm s/p_K, \qquad \operatorname{tg} \alpha_{TK} = \pm s/q_K.$$

The parameters p_K, q_K may be derived from the equations of motion.

The displacement components u_x, u_y, u_z in each layer may be calculated from the formulae

$$\text{(2.3)} \qquad \begin{aligned} u_{Kx} &= \partial\Phi_K/\partial x - \partial\Psi_K/\partial y, \\ u_{Ky} &= \partial\Phi_K/\partial y + \partial\Psi_K/\partial x, \qquad u_{Kz} = 0. \end{aligned}$$

The above potentials must satisfy the equations of motion

$$\text{(2.4)} \qquad \partial^2\Phi_k/\partial x^2 + \partial^2\Phi_K/\partial y^2 = \frac{1}{c_{LK}^2}\partial^2\Phi_K/\partial t^2,$$

$$\text{(2.5)} \qquad \partial^2\Psi_k/\partial x^2 + \partial^2\Psi_K/\partial y^2 = \frac{1}{c_{LT}^2}\partial^2\Psi_K/\partial t^2,$$

where c_{LK} and c_{TK} are the longitudinal and transverse wave speeds, respectively, and t is time. It follows that the five parameters s, p_K, q_K in Eq. (2.1) are not arbitrary, but must satisfy the relations

$$\text{(2.6)} \qquad p_K^2 + s^2 = \omega^2/c_{LK}^2, \qquad c_{LK}^2 = (\lambda_K + 2\mu_K)/\varrho_K,$$

$$\text{(2.7)} \qquad q_K^2 + s^2 = \omega^2/c_{TK}^2, \qquad c_{TK}^2 = \mu_K/\varrho_K,$$

where ϱ_K is the density of the material. If in particular the parameters s and ω are given in advance, then the parameters p_k, q_k are defined by (2.6), (2.7). From (2.2) it follows that the angles α_{LK}, α_{TK} are defined by s and the properties of the K-th layer. Only one wave direction in one layer is arbitrary, the other wave directions must match the first one (Snellius rule).

The expressions for the potentials allow to calculate the displacements and stresses in both layers as functions of x. Of particular interest are the values on the interface $x = x_2$. Simple calculations result in the formulae for the displacements and stresses at this point in the first and second layer

$$(2.8)\qquad \begin{bmatrix} -iu_x \\ -iu_y \\ \tau_{xx} \\ \tau_{xy} \end{bmatrix}_1 = \begin{bmatrix} p_1 & -p_1 & -s & -s \\ s & s & q_1 & -q_1 \\ -z_1 & -z_1 & 2\mu q_1 s & 2\mu q_1 s \\ -2\mu_1 p_1 s & 2\mu_1 p_1 s & w_1 & w_1 \end{bmatrix}$$

$$\times \begin{bmatrix} \exp ip_1 l_1 & 0 & 0 & 0 \\ 0 & \exp(-ip_1 l_1) & 0 & 0 \\ 0 & 0 & \exp iq_1 l_1 & 0 \\ 0 & 0 & 0 & \exp(-iq_1 l_1) \end{bmatrix} = \begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{bmatrix},$$

$$(2.9)\qquad \begin{bmatrix} -iu_x \\ -iu_y \\ \tau_{xx} \\ \tau_{xy} \end{bmatrix}_2 = \begin{bmatrix} p_2 & -p_2 & -s & -s \\ s & s & q_2 & -q_2 \\ -z_2 & -z_2 & 2\mu q_2 s & 2\mu q_2 s \\ -2\mu_2 p_2 s & 2\mu_2 p_2 s & w_2 & w_2 \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{bmatrix},$$

where

$$(2.10)\qquad w_K = \mu_K(s^2 - q_K^2), \qquad l_1 = x_2 - x_1,$$

$$(2.11)\qquad z_K = \lambda_K(p_K^2 + s^2) + 2\mu_K p_K^2 .$$

At both sides of the boundary between the regions, the stress vector and the displacement vector have the same values. In order to express the amplitudes A_2, B_2, C_2, D_2 by the amplitudes A_1, B_1, C_1, D_1, the inverse matrix of that in (2.9) must be calculated. As the final result, the relation

$$(2.12)\qquad \begin{bmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{bmatrix} = M_1 \begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{bmatrix}$$

is obtained. The transfer matrix M_1 allows to calculate the amplitudes of the waves propagating in the Region 2 provided the amplitudes of the waves propagating in the Region 1 are known. The complex-valued components of

M are given by the relations

$$
\begin{aligned}
M_{11} &= [a_2(2\mu_1 p_1 s^2 - p_1 w_2) + b_2(2\mu_2 q_2 s^2 + q_2 z_1)]\exp ip_1 l_1 ,\\
M_{12} &= [a_2(-2\mu_1 p_1 s^2 + p_1 w_2) + b_2(2\mu_2 q_2 s^2 + q_2 z_1)]\exp(-ip_1 l_1),\\
M_{13} &= [a_2 s(w_2 - w_1) + 2b_2 q_1 q_2 s(\mu_2 - \mu_1)]\exp iq_1 l_1 ,\\
M_{14} &= [a_2 s(w_2 - w_1) - 2b_2 q_1 q_2 s(\mu_2 - \mu_1)]\exp(-iq_1 l_1),\\
M_{31} &= [-2a_2 p_1 p_2 s(\mu_2 - \mu_1) + b_2 s(z_2 - z_1)]\exp ip_1 l_1 ,\\
M_{32} &= [2a_2 p_1 p_2 s(\mu_2 - \mu_1) + b_2 s(z_2 - z_1)]\exp(-ip_1 l_1),\\
M_{33} &= [a_2(2\mu_2 p_2 s^2 - p_2 w_1) + b_2(2\mu_1 q_1 s^2 + q_1 z_2)]\exp iq_1 l_1 ,\\
M_{34} &= [a_2(2\mu_2 p_2 s^2 - p_2 w_1) - b_2(2\mu_1 q_1 s^2 + q_1 z_2)]\exp(-iq_1 l_1),
\end{aligned} \tag{2.13}
$$

$$
\begin{aligned}
M_{21} &= \overline{M_{12}}, \quad M_{22} = \overline{M_{11}}, \quad M_{23} = -\overline{M_{14}}, \quad M_{24} = -\overline{M_{13}},\\
M_{41} &= -\overline{M_{32}}, \quad M_{42} = -\overline{M_{31}}, \quad M_{43} = \overline{M_{34}}, \quad M_{44} = -\overline{M_{33}},
\end{aligned} \tag{2.14}
$$

$$
a_2 = \frac{1}{2}\frac{1}{2\mu_2 p_2 s^2 - p_2 w_2}, \qquad b_2 = \frac{1}{2}\frac{1}{2\mu_2 q_2 s^2 + q_2 z_2}. \tag{2.15}
$$

The matrix of symmetry (2.14) will be called w-symmetric. The product of two w-symmetric matrices is w-symmetric.

The following identity may be obtained from (2.10)

$$
\begin{aligned}
&p_1(A_1\overline{A_1} - B_1\overline{B_1}) + q_1(C_1\overline{C_1} - D_1\overline{D_1})\\
&\qquad = p_2(A_2\overline{A_2} - B_2\overline{B_2}) + q_2(C_2\overline{C_2} - D_2\overline{D_2}).
\end{aligned} \tag{2.16}
$$

It expresses the fact that the energy fluxes in both regions are equal.

3. Elementary cell

Assume that a fixed set of the above layers is repeated in space. The smallest such set constitutes the elementary cell, and may consist of any number of layers. We consider here the case when the cell consists of three layers: the layer of material m_a of thickness l_a, material m_b of thickness l_b, and material m_c of thickness l_c, Fig. 2. The generalisation to larger number of layers is trivial. The purpose of the present Section is to calculate the transfer matrix for the elementary cell.

Concentrate our attention at the right-hand side of (2.13). Replace the suffixes 1 by the suffixes a and the suffixes 2 by b. Denote the resulting transfer matrix by M_{ab}. Replace in turn the suffixes 1 by the suffixes b and

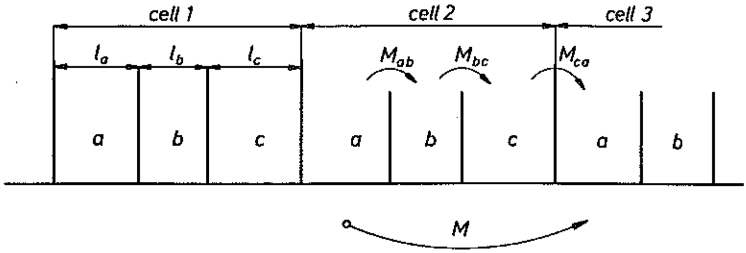

FIG. 2.

the suffixes 2 by c and denote the resulting transfer matrix by M_{bc}. Replace finally the suffixes 1 by the suffixes c and the suffixes 2 by a and denote the resulting transfer matrix by M_{ca}.

The transfer matrix for the transitions $a \Rightarrow b$ is the matrix M_{ab}; the transition matrix for the transition $b \Rightarrow c$ is the matrix M_{bc}; and the transfer matrix for the transition $c \Rightarrow a$ is the matrix M_{ca}. In all cells the transfer matrices M_{ab}, M_{bc}, M_{ca} are the same. The transfer matrix $\mathbf{M}$ for one cell is the product

$$\mathbf{M} = M_{ca} M_{bc} M_{ab} \,. \tag{3.1}$$

The matrix $\mathbf{M}$ is the product of three w-symmetric matrices, and therefore is w-symmetric. Note that $\mathbf{M}$ has a very complex algebraic structure. Each component of M_{ab} is a sum of four complex numbers. Therefore each component of the product is a sum of 64 complex numbers. Finally $\mathbf{M}$ is formed by 1024 complex numbers. It is impossible to give here the very long, full algebraic expressions for the components of $\mathbf{M}$. In the example the multiplications will be performed numerically.

The formulae (2.12) may now be chained to obtain

$$\begin{bmatrix} A_{3k} \\ B_{3k} \\ C_{3k} \\ D_{3k} \end{bmatrix} = M^k \begin{bmatrix} A_0 \\ B_0 \\ C_0 \\ D_0 \end{bmatrix},$$

$$\begin{bmatrix} A_{3k+1} \\ B_{3k+1} \\ C_{3k+1} \\ D_{3k+1} \end{bmatrix} = M_{ab} M^k \begin{bmatrix} A_0 \\ B_0 \\ C_0 \\ D_0 \end{bmatrix}, \tag{3.2}$$

$$\begin{bmatrix} A_{3k+2} \\ B_{3k+2} \\ C_{3k+2} \\ D_{3k+2} \end{bmatrix} = M_{bc} M_{ab} M^k \begin{bmatrix} A_0 \\ B_0 \\ C_0 \\ D_0 \end{bmatrix}.$$

The displacement field in the subsequent cells may be expressed by the amplitudes in the first cell, the powers of the transfer matrix M^1, M^2, M^3, ..., and the matrices M_{ab}, M_{bc}. It is seen that essential for the calculation of displacement in a system of large number of cells is the calculation of M^k.

In [2] was considered the case, when the cell consisted of two layers only. In this situation the transfer matrix for one cell equals the product of two w-symmetric matrices, and therefore it may be expressed by the nine real physical quantities (frequency ω does not influence $\mathbf{M}$)

$$(3.3) \qquad s, \quad (\lambda_1, \mu_1, \varrho_1, l_1), \quad (\lambda_2, \mu_2, \varrho_2, l_2).$$

In general, the complex-valued w-symmetric matrix possesses 15 independent components (not 16, since the determinant equals 1). It follows that the transfer matrix for elementary cell consisting of two layers possesses additional symmetries, which can be expressed as nonlinear relations between the components. Obviously each M may be represented in the following form (and in thousands of other forms)

$$(3.4) \qquad M_{ij} = a_{ij} sin\varphi + b_{ij} \sin\psi + c_{ij} \cos\varphi + d_{ij} \cos\psi + i[e_{ij} \sin\varphi + f_{ij} \sin\psi + g_{ij} \cos\varphi + h_{ij} \cos\psi],$$

where φ, ψ, a_{ij}, b_{ij}, ..., h_{ij} are constants. For definiteness, not restricting the generality, we may assume $\cos\varphi \le \cos\psi$, $0 \le \varphi < 2\pi$, $0 \le \psi < 2\pi$. In [2] it has been proved analytically that for a cell consisting of two layers, the powers of M for $k = 1, 2, 3, 4, ...$ satisfy the very useful relation

$$(3.5) \qquad (M^k)_{ij} = a_{ij} \sin k\varphi + b_{ij} \sin k\psi + c_{ij} \cos k\varphi + d_{ij} \cos k\psi + i[e_{ij} \sin k\varphi + f_{ij} \sin k\psi + g_{ij} \cos k\varphi + h_{ij} \cos k\psi].$$

The constants φ, ψ, a_{ij}, b_{ij}, ..., h_{ij} are not arbitrary, but must be appriopriately calculated basing on the components of M.

If the cell consists of three layers, the transfer matrix $\mathbf{M}$ for one cell is determined by 13 real quantities

$$(3.6) \qquad s, \quad (\lambda_a, \mu_a, \varrho_a, l_a), \quad (\lambda_b, \mu_b, \varrho_b, l_b), \quad (\lambda_c, \mu_c, \varrho_c, l_c).$$

For a cell consisting of three layers, matrix $\mathbf{M}$ possesses only some (not all) additional symmetries that were discussed in [2]. We shall prove numerically that the relation (3.5) holds even in this case. The direct analytic proof would demand an enormous amount of algebraic computations.

If (3.5) holds, then the constants φ, ψ, a_{ij}, b_{ij}, ..., h_{ij} are uniquely defined by the following system of real equations

$$
(3.7)\quad
\begin{aligned}
\operatorname{Re}(M)_{ij} &= a_{ij}\sin\varphi + b_{ij}\sin\psi + c_{ij}\cos\varphi + d_{ij}\cos\psi\,,\\
\operatorname{Re}(M^2)_{ij} &= a_{ij}\sin 2\varphi + b_{ij}\sin 2\psi + c_{ij}\cos 2\varphi + d_{ij}\cos 2\psi\,,\\
\operatorname{Re}(M^3)_{ij} &= a_{ij}\sin 3\varphi + b_{ij}\sin 3\psi + c_{ij}\cos 3\varphi + d_{ij}\cos 3\psi\,,\\
\operatorname{Re}(M^4)_{ij} &= a_{ij}\sin 4\varphi + b_{ij}\sin 4\psi + c_{ij}\cos 4\varphi + d_{ij}\cos 4\psi\,,\\
\operatorname{Re}(M^5)_{ij} &= a_{ij}\sin 5\varphi + b_{ij}\sin 5\psi + c_{ij}\cos 5\varphi + d_{ij}\cos 5\psi\,,\\
\operatorname{Re}(M^6)_{ij} &= a_{ij}\sin 6\varphi + b_{ij}\sin 6\psi + c_{ij}\cos 6\varphi + d_{ij}\cos 6\psi\,,
\end{aligned}
$$

$$
(3.8)\quad
\begin{aligned}
\operatorname{Im}(M)_{ij} &= e_{ij}\sin\varphi + f_{ij}\sin\psi + g_{ij}\cos\varphi + h_{ij}\cos\psi\,,\\
\operatorname{Im}(M^2)_{ij} &= e_{ij}\sin 2\varphi + f_{ij}\sin 2\psi + g_{ij}\cos 2\varphi + h_{ij}\cos 2\psi\,,\\
\operatorname{Im}(M^3)_{ij} &= e_{ij}\sin 3\varphi + f_{ij}\sin 3\psi + g_{ij}\cos 3\varphi + h_{ij}\cos 3\psi\,,\\
\operatorname{Im}(M^4)_{ij} &= e_{ij}\sin 4\varphi + f_{ij}\sin 4\psi + g_{ij}\cos 4\varphi + h_{ij}\cos 4\psi\,,\\
\operatorname{Im}(M^5)_{ij} &= e_{ij}\sin 5\varphi + f_{ij}\sin 5\psi + g_{ij}\cos 5\varphi + h_{ij}\cos 5\psi\,,\\
\operatorname{Im}(M^6)_{ij} &= e_{ij}\sin 6\varphi + f_{ij}\sin 6\psi + g_{ij}\cos 6\varphi + h_{ij}\cos 6\psi\,,
\end{aligned}
$$

The above system contains 64+2 unknowns. Obviously, we could consider rather only the equations for powers of M up to 4 (64 equations) and two arbitrary equations concerning the expressions for M^5. We prefer to solve the overdetermined system (3.7)–(3.8) since in this situation the system of equations separates into 16 subsystems, each of them consisting of 6 equations with six unknowns. Such subsystem constitutes e.g. (3.7) for $i = 1$, $j = 3$. From each subsystem may be calculated the constants that are present in this subsystem only, and additionally φ and ψ that are present in all subsystems. We shall demonstrate that from each subsystem, practically the same φ, ψ are obtained.

It is evident, that it is entirely impossible to solve analytically the system (3.7)–(3.8). In [2] some preliminary numerical analysis resulted in purely analytic proof of (3.3). Here the situation is more difficult and we must confine our analysis to numerical treatment only.

4. Numerical calculation of the coefficients

The purpose of this paper is to demonstrate the possibility of special representation (3.4) of the transfer matrix **M** satisfying the identity (3.5). Taking this into account we do not consume space for introduction of the dimensioneless variables and take the data corresponding to essentially different layers. The calculations will be peformed for the following data:

$$\omega = 1, \tag{4.1}$$

$$s = .25, \tag{4.2}$$

$$\begin{aligned} \lambda_a &= 1, & \mu_a &= 1, & \varrho_a &= 1, \\ \lambda_b &= 2, & \mu_b &= 2, & \varrho_b &= 1, \\ \lambda_c &= 4, & \mu_c &= 4, & \varrho_c &= 1. \end{aligned} \tag{4.3}$$

In order to calculate the transfer matrix $\mathbf{M}_{ab}$ for the layers a, b, the following constants must be substituted into (2.13):

$$\begin{aligned} \lambda_1 &= \lambda_a, & \mu_1 &= \mu_a, & \varrho_1 &= \varrho_a, & h_1 &= h_a, & l_1 &= l_a, \\ \lambda_2 &= \lambda_b, & \mu_2 &= \mu_b, & \varrho_2 &= \varrho_b, & h_2 &= h_b, & l_2 &= l_b. \end{aligned} \tag{4.4}$$

There results the transfer matrix $\mathbf{M}_{ab}$

$$\begin{aligned} (M_{ab})_{11} &= 1.10009 + .63048i\,, & (M_{ab})_{12} &= -.12402 + .07108i\,, \\ (M_{ab})_{13} &= .16462 + .23938i\,, & (M_{ab})_{14} &= -.10975 + .15955i\,, \\ (M_{ab})_{31} &= -.13338 - .07644i\,, & (M_{ab})_{32} &= .09238 + .99109i\,, \\ (M_{ab})_{33} &= .68176 + .99109i\,, & (M_{ab})_{34} &= -.04417 + .06421i\,. \end{aligned} \tag{4.5}$$

The remaining components are determined by the w-symmetry. We presented above 5 digits only, but the numerical calculations were performed with accuracy of 16 digits.

In order to calculate the transfer matrices $\mathbf{M}_{bc}$, $\mathbf{M}_{ca}$ for the layers b, c and c, a, we must substitute in turn

$$\begin{aligned} \lambda_1 &= \lambda_b, & \mu_1 &= \mu_b, & \varrho_1 &= \varrho_b, & h_1 &= h_b, \\ \lambda_2 &= \lambda_c, & \mu_2 &= \mu_c, & \varrho_2 &= \varrho_c, & h_2 &= h_c, \end{aligned} \tag{4.6}$$

$$\begin{aligned} \lambda_1 &= \lambda_c, & \mu_1 &= \mu_c, & \varrho_1 &= \varrho_c, & h_1 &= h_c, \\ \lambda_2 &= \lambda_c, & \mu_2 &= \mu_c, & \varrho_2 &= \varrho_c, & h_2 &= h_c. \end{aligned} \tag{4.7}$$

There result the transfer matrices $\mathbf{M}_{bc}$, $\mathbf{M}_{ca}$. We do not quote here their numerical values.

Perform now the multiplication (3.1). There results the transfer matrix for the elementary cell

$$\begin{aligned} M_{11} &= .45995 + .92089i, & M_{12} &= -.14018 - .19368i, \\ M_{13} &= .25107 + .06063i, & M_{14} &= -.26602 - .02116i, \\ M_{31} &= .12622 - .05079i, & M_{32} &= -.03471 + .13640i, \\ M_{33} &= -.44419 + .90350i, & M_{34} &= -.07879 - .07065i. \end{aligned} \tag{4.8}$$

Calculate the powers of M up to M^6 and substitute into the systems of equations (3.7) – (3.8). From these nonlinear systems of trigonometric equations, the coefficients a_{ij}, b_{ij}, ..., h_{ij} may be calculated. Using the method of successive approximations, the following values were obtained:

$$
\begin{aligned}
&a_{11} = 0, && b_{11} = 0,\\
&a_{12} = -.023573, && b_{12} = -.136384,\\
&a_{13} = .083868, && b_{13} = -.075406,\\
&a_{14} = -.113383, && b_{14} = -.074406,\\
&a_{31} = -.045078, && b_{31} = .040529,\\
&a_{32} = .060942, && b_{32} = -.039992,\\
&a_{33} = 0, && b_{33} = 0,\\
&a_{34} = -.056234, && b_{34} = -.033254,\\
&c_{11} = .025163, && d_{11} = .974837,\\
&c_{12} = 0, && d_{12} = 0,\\
&c_{13} = -.255182, && d_{13} = .255182,\\
&c_{14} = .105790, && d_{14} = .105790,\\
&c_{31} = -.137156, && d_{31} = .137156,\\
&c_{32} = .056860, && d_{32} = -.056860,\\
&c_{33} = .974837, && d_{33} = .025163,\\
&c_{34} = 0, && d_{34} = 0;
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
&e_{11} = .050383, && f_{11} = 1.001422,\\
&e_{12} = -.036737, && f_{12} = -.184222,\\
&e_{13} = -.242045, && f_{13} = .250572,\\
&e_{14} = -.084413, && f_{14} = -.071849,\\
&e_{31} = -.130095, && f_{31} = .120965,\\
&e_{32} = .045371, && f_{32} = .038618,\\
&e_{33} = .978637, && f_{33} = .044255,\\
&e_{34} = -.065284, && f_{34} = -.014815,\\
&g_{11} = 0, && h_{11} = 0,\\
&g_{12} = 0, && h_{12} = 0,\\
&g_{13} = -.081471, && h_{13} = .081471,\\
&g_{14} = -.122171, && h_{14} = .122171,\\
&g_{31} = .043789, && h_{31} = -.043789,\\
&g_{32} = -.065665, && h_{32} = .065665,\\
&g_{33} = 0, && h_{33} = 0,\\
&g_{34} = 0, && h_{34} = 0.
\end{aligned}
\tag{4.10}
$$

Each of the above coefficients was obtained only once, from one definite subsystem (3.7), or (3.8). In contrast to this, the values of φ and ψ were obtained from each of the 16 subsystems. Obviously, they were different due to the numerics. However, the differences are small. The solutions are equal (to within seven decimal digits)

$$\varphi = 2.057994, \qquad \psi = 1.065687. \tag{4.11}$$

In order to improve the accuracy in further 16-digit calculations, the mean value of all 16 calculated values of φ, ψ was used. To within 7 digits, the mean value is given by (4.11).

Now we are in a position to check the validity of (3.5). Substitute the above coefficients and calculate twice the k-th power of $\mathbf{M}$:

i. From the formula (3.5).

ii. From (4.8) using the recursive formula $\mathbf{M}^k = \mathbf{M}\,\mathbf{M}^{k-1}$.

In the table below are quoted the values calculated from (3.5), and the values calculated from the recursive formula.

	from (3.5)	from $M^k = MM^{k-1}$
$(M^{100})_{11}$	$.946254 - .293689i$	$.946251 - .293689i$
$(M^{100})_{12}$	$.056704 + .081487i$	$.056703 + .081488i$
$(M^{100})_{13}$	$.175625 + .264275i$	$.175625 + .264277i$
$(M^{100})_{33}$	$.048822 - .989083i$	$.048816 - .989092i$

The remaining components exhibit differences between both values of the same order. Note that the differences should not be attributed to the correctness of the formula (3.5). This conclusion is supported by the values calculated with different accuracies for $k = 200$. For calculations with the accuracy of 7 or 16 digits, the following values are obtained:

	from (3.5)	from $M^k = MM^{k-1}$	digits
$(M^{200})_{11}$	$.834602 - .474581i$	$.834597 - .475578i$	16
$(M^{200})_{33}$	$-.951422 - .069851i$	$.951440 - .069851i$	16
$(M^{200})_{11}$	$.834602 - .474584i$	$.834607 - .474570i$	7
$(M^{200})_{33}$	$-.951419 - .069917i$	$.951398 - .069869i$	7

Comparison of the above results proves that the values calculated from (3.5) are scattered less than the values calculated from the recursive formula. This suggests that the differences should be attributed to errors arising during the recursive computation of M^{200}. The formula (3.5) was checked many times for other dimensions and properties (4.1)–(4.3). In all cases the validity of (3.5) was confirmed.

5. Simplification of computation

Inspection of (4.9) and (4.10) proves that, with high accuracy (first 6 decimal digits equal zero), there is

$$a_{ij} = b_{ij} = 0 \quad \text{for} \quad i = j, \tag{5.1}$$

$$d_{11} = 1 - c_{11}, \qquad c_{33} = d_{11}, \qquad d_{33} = c_{11}, \tag{5.2}$$

$$d_{ij} = -c_{ij}, \qquad h_{ij} = -g_{ij} \quad \text{for} \quad i \neq j. \tag{5.3}$$

The above relations allow to simplify the calculations. In accord with (5.1)–(5.2) there is

$$\begin{aligned} c_{11}\cos\varphi + (1 - c_{11})\cos\psi &= \mathrm{Re}\,(M)_{11}, \\ c_{11}\cos 2\varphi + (1 - c_{11})\cos 2\psi &= \mathrm{Re}\,(M^2)_{11}, \\ (1 - c_{11})\cos\varphi + c_{11}\cos\psi &= \mathrm{Re}\,(M)_{33}, \\ (1 - c_{11})\cos 2\varphi + c_{11}\cos 2\psi &= \mathrm{Re}\,(M^2)_{33}. \end{aligned} \tag{5.4}$$

Let us produce two further equations by adding together the first and the third equations, and the second and the fourth equations,

$$\begin{aligned} \cos\varphi + \cos\psi &= \mathrm{Re}\,(M)_{11} + \mathrm{Re}\,(M)_{33}, \\ \cos 2\varphi + \cos 2\psi &= \mathrm{Re}\,(M^2)_{11} + \mathrm{Re}\,(M^2)_{33}. \end{aligned} \tag{5.5}$$

In this system there are only two unknowns: φ and ψ. Squaring the first equation and transforming the second one, we obtain

$$\begin{aligned} \cos^2\varphi + \cos^2\psi + 2\cos\varphi\cos\psi &= [\mathrm{Re}\,(M)_{11} + \mathrm{Re}\,(M)_{33}]^2, \\ \cos^2\varphi + \cos^2\psi - 1 &= \frac{1}{2}[\mathrm{Re}\,(M^2)_{11} + \mathrm{Re}\,(M^2)_{33}]. \end{aligned} \tag{5.6}$$

Subtracting both the equations, we obtain

$$\begin{aligned} 2\cos\varphi\cos\psi + 1 = {} & [\mathrm{Re}\,(M)_{11} + \mathrm{Re}\,(M)_{33}]^2 \\ & - \frac{1}{2}\left[\mathrm{Re}\,(M^2)_{11} + \mathrm{Re}\,(M^2)_{33}\right]. \end{aligned} \tag{5.7}$$

The parameter ψ may now be eliminated, and the resulting quadratic equation for $\cos\varphi$ may be solved. The two resulting solutions (φ_1, ψ_1), (φ_2, ψ_2) are essentially the same: $\varphi_2 = \psi_1$, $\psi_2 = \varphi_1$. Further calculations are based on the solution satisfying the demand $\cos\varphi \le \cos\psi$. The resulting solution is

$$(5.8) \qquad 2\cos\varphi = \mathrm{Re}\,(M)_{11} + \mathrm{Re}\,(M)_{33} - \Big[2 + \mathrm{Re}\,(M^2)_{11} + \mathrm{Re}\,(M^2)_{33} - [\mathrm{Re}\,(M)_{11} + \mathrm{Re}\,(M)_{33}]^2\Big]^{1/2},$$

$$(5.9) \qquad 2\cos\psi = \mathrm{Re}\,(M)_{11} + \mathrm{Re}\,(M)_{33} + \Big[2 + \mathrm{Re}\,(M^2)_{11} + \mathrm{Re}\,(M^2)_{33} - [\mathrm{Re}\,(M)_{11} + \mathrm{Re}\,(M)_{33}]^2\Big]^{1/2}.$$

Equation (5.4) allows to calculate the coefficient c_{11}. In accord with (5.2), there is

$$(5.10) \qquad \begin{aligned} c_{11} &= \frac{1}{2}\,\frac{(\mathrm{Re}\,(M)_{11} + \mathrm{Re}\,(M)_{33})}{1 + \cos\varphi - \cos\psi}, \qquad d_{11} = 1 - c_{11}, \\ c_{33} &= d_{11}, \qquad d_{33} = c_{11}. \end{aligned}$$

Since the parameters φ, ψ are already known, the calculations are considerably simplified. In accord with (5.1)–(5.3), we have

$$(5.11) \qquad \begin{aligned} a_{ij}\sin\varphi + b_{ij}\sin\psi + c_{ij}(\cos\varphi - \cos\psi) &= \mathrm{Re}\,(M)_{ij}, \\ a_{ij}\sin 2\varphi + b_{ij}\sin 2\psi + c_{ij}(\cos 2\varphi - \cos 2\psi) &= \mathrm{Re}\,(M^2)_{ij}, \\ a_{ij}\sin 3\varphi + b_{ij}\sin 3\psi + c_{ij}(\cos 3\varphi - \cos 3\psi) &= \mathrm{Re}\,(M^3)_{ij}, \end{aligned}$$

$$(5.12) \qquad \begin{aligned} d_{ij}\sin\varphi + e_{ij}\sin\psi + f_{ij}(\cos\varphi - \cos\psi) &= \mathrm{Im}(M)_{ij}, \\ d_{ij}\sin 2\varphi + e_{ij}\sin 2\psi + f_{ij}(\cos 2\varphi - \cos 2\psi) &= \mathrm{Im}(M^2)_{ij}, \\ d_{ij}\sin 3\varphi + e_{ij}\sin 3\psi + f_{ij}(\cos 3\varphi - \cos 3\psi) &= \mathrm{Im}(M^3)_{ij}. \end{aligned}$$

Instead of six nonlinear trigonometric equations (3.7), we face now three linear algebraic equations (5.11), and instead of six nonlinear trigonometric equations (3.8) – three linear algebraic equations (5.12).

In (5.11) the subscripts (ij) equal to (11) or (33) should be excluded, since the coefficients a_{11}, b_{11}, c_{11}, ..., c_{33} have been already calculated. The numerical values of a_{ij}, b_{ij}, ..., f_{ij} up to the first 6 digits do not differ from those calculated from (3.7), (3.8) and quoted in (4.9), (4.10).

Note that in the previous chapter, in order to solve (3.7)–(3.8) we were forced to use approximate computational methods. In this chapter φ, ψ have been calculated from a simple trigonometric equation, and the coefficients a_{ij}, b_{ij}, ... f_{ij} – from the systems consisting of three linear algebraic equations.

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