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SOME NEW DEVELOPMENTS IN CONTACT PRESSURE OPTIMIZATION (*)

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Relatively few works have dealt with the optimization problems of bodies in contact. The present work is intended as a contribution to the determination of contact pressure distribution in the frame of linear elasticity. Solution of frictionless contact problems are investigated not only on the basis of minimum complementary energy principle, but also on the basis of minimum total potentional energy by the use of an augmented Lagrangian technique. The goal is to optimize the pressure distribution along the contact region. The minimum of maximal pressure is looked for by controlling the pressure distribution. The optimization problem can be handled by a so-called restricted linear programming problem. Effectiveness of the augmented Lagrangian technique has been proved by axisymmetric and plane stress type numerical examples, i.e., the pressure can be calculated directly, solution of the contact problem can be obtained by means of a relatively small penalty parameter, solution of the optimization problem can be found in a relatively easy way.

1. Introduction

Generally, the classical methods of elasticity cannot be used effectively for solution of the contact problems. Using computers, we can apply various numerical methods for the determination of the contact region and pressure. A detailed review of the variational principles to be used is given by Telega [1]. In the early finite element applications to the contact problems, the Lagrange multiplier approach [2 – 4] and the penalty formulation [5] were used. In the latter case the accuracy of the correct choice of the penalty parameter is the essence of the algorithm. Recently, a mixture of the methods – the so-called augmented Lagrangian technique – has been applied successfully to linear and nonlinear contact problems [6].

A design engineer always makes an effort to avoid singularities in the contact regions in order to reduce the stresses and decrease abrasion and energy dissipation of friction.

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In those cases when joints of bodies are realized by unilateral contact, certain mechanical quantities can be optimized by shaping the contact regions. For example, it might be a contact problem of finding the minimum of the contact pressure maximum value or, in addition, a second requirement that the zero contact pressure should be assured at the boundary of the contact region [7]. Some new results are presented in Sec. 3.

2. Formulation of the contact problem

In the present work it is assumed that the displacements and deformations are small, and the adhesion, friction and dynamic effect between the contacting elements can be disregarded; such contact problems are referred to as normal.

For the sake of simplicity, let us assume that the elastic system consists of two bodies. The α body surfaces will be separated into three domains:

S_p^α, S_u^α part of the body surface bearing a given surface load and given displacement,

$S_c^\alpha = \Omega = \Omega'$ proposed zone of contact.

For the examination of the contact/separation conditions, in the zone of contact we shall consider the projection of the displacements in a prescribed direction only (e.g. normal to the surface). At any point of the region Ω, there is a prescribed direction which, in our case, is the normal $\mathbf{n}^1$ to surface of the first body, if

$$d = u_N^2 - u_N^1 + h = 0, \qquad p \geq 0, \qquad x \in \Omega_p \tag{2.1}$$

what means contact.

Separation (gap) occurs, if

$$d = u_N^2 - u_N^1 + h \geq 0, \qquad p = 0, \qquad x \in \Omega_0, \tag{2.2}$$

where x denotes a point of Ω with the coordinates (x_1, x_2, x_3). p is contact pressure, h initial separation (gap) along the normal $\mathbf{n}^1$, $u_N^\alpha = \mathbf{u}^\alpha \cdot \mathbf{n}^1$, $\alpha = 1,2$ the displacement in the prescribed direction. d will be termed the relative displacement, that is the distance after deformation, or in other words the gap. The zones Ω_0 and Ω_p are not known beforehand, $\Omega = \Omega_0 + \Omega_p$. From the Eqs. (2.1), (2.2) it follows, that

$$p \geq 0, \qquad d \geq 0, \qquad pd = 0, \qquad x \in \Omega. \tag{2.3}$$

If one of the bodies – let us suppose it to be the first one – can move as a rigid body, the equilibrium equations must be satisfied

$$
(2.4)\qquad
\begin{aligned}
\mathbf{F} &= \mathbf{F}_0 - \int_{\Omega} p\mathbf{n}^1 \, dS = \mathbf{0},\\
\mathbf{M} &= \mathbf{M}_0 - \int_{\Omega} \mathbf{R} \times \mathbf{n}^1 p \, dS = \mathbf{0},
\end{aligned}
$$

where $\mathbf{F}_0$, $\mathbf{M}_0$ are the resultant force and moment at the origin of the coördinate system, and $\mathbf{R}$ is the position vector, $\times$ is the symbol of a vector product.

2.1. Construction of the equation/inequality system by using Green's functions

The deflections along the indicated direction are defined by the following equations

$$
(2.5)\qquad
\begin{aligned}
u_N^1(x) &= -\int_{\Omega'} H^1(x,x')p(x') \, dS' + f^1(x) + r(x),\\
u_N^2(x) &= \int_{\Omega'} H^2(x,x')p(x') \, dS' + f^2(x),
\end{aligned}
$$

where $H^1(x,x')$ $(i = 1,2)$ is Green's function, $f^i(x)$ $(i = 1,2)$ is the displacement due to the given loads, $r(x)$ is the displacement due to rigid body type of motion:

$$
(2.6)\qquad r(x) = [\boldsymbol{\lambda}_F + \boldsymbol{\lambda}_M \times \mathbf{R}(x)] \cdot \mathbf{n}^1(x),
$$

where $\boldsymbol{\lambda}_F = [\lambda_{F1}, \lambda_{F2}, \lambda_{F3}]$, $\boldsymbol{\lambda}_M = [\lambda_{M1}, \lambda_{M2}, \lambda_{M3}]$ are the rigid-body approach vectors, components of which are translations $(\boldsymbol{\lambda}_F)$ along the reference axis, and rotations $(\boldsymbol{\lambda}_F)$ also about the reference axis. The gap after deformation (2.5) is

$$
(2.7)\qquad d = \int_{\Omega'} \Big(H^1(x,x') + H^2(x,x')\Big)p(x') \, dS' + f^2(x) + f^1(x) + h(x) - r(x) \geq 0.
$$

The variational principle is applied to determine the contact between the two elastic bodies [1]. In this case the modified complementary energy is [15]

$$(2.8)\qquad L=L(p(x),\boldsymbol{\lambda}_F,\boldsymbol{\lambda}_M)=\frac{1}{2}\int_{\Omega}\int_{\Omega'}p(x)\Big(H^1(x,x')+H^2(x,x')\Big)p(x')dS'dS + \int_{\Omega}p(f^2-f^1+h)\,dS-\boldsymbol{\lambda}_F\cdot\mathbf{F}-\boldsymbol{\lambda}_M\cdot\mathbf{M},$$

and the variational equations and inequalities are $\delta_{\lambda_F,\lambda_M}L=0$ (equilibrium equations for a body, with rigid body-like displacement (2.4)); $\delta_p L\leq 0$, $p\geq 0$, $x\in\Omega$, which gives the contact and separation conditions: (2.1)–(2.3).

In the functional (2.8) $\boldsymbol{\lambda}_F,\boldsymbol{\lambda}_M$ are rigid body approach vectors, $\mathbf{F}=\mathbf{0}$, $\mathbf{M}=\mathbf{0}$ are the equilibrium equations (2.4), points x, x' belong to the proposed zone of contact Ω, Ω'.

For a more accurate determination of the state of stress, the approximation of the contact pressure in the form of C° class function is recommended

$$(2.9)\qquad p=p(x)=\mathbf{P}^T\mathbf{p}=[P_1,\,P_2,\,\ldots,\,P_k]\mathbf{p},\qquad x\in\Omega,$$

where $\mathbf{p}$ is the column matrix of contact pressure at the nodes, and $P_i=P_i(x)$ $(i=1,\ldots,k)$ are the coordinate (global basis) functions. By substitution of (2.9) into (2.4) we have the equilibrium equation

$$(2.10)\qquad \mathbf{q}-\mathbf{G}_R^T\mathbf{p}=\mathbf{0},$$

where $\mathbf{q}$ is the known main vector of the external load $\mathbf{q}^T=\left[\mathbf{F}_0^T,\mathbf{M}_0^T\right]$, and

$$(2.11)\qquad \mathbf{G}_R^T=\int_{\Omega}\begin{bmatrix}\cdots\mathbf{n}^1(x)P_i(x)\cdots\\ \cdots\mathbf{R}(x)\times\mathbf{n}^1(x)P_i(x)\cdots\end{bmatrix}dS$$

is the geometrical matrix.

The discretized modified complementary energy which is the Lagrangian function, is the following:

$$(2.12)\qquad L=L(\mathbf{p},\boldsymbol{\lambda})=\frac{1}{2}\mathbf{p}^T\mathbf{H}\mathbf{p}-\mathbf{p}^T\mathbf{t}-\boldsymbol{\lambda}^T(\mathbf{G}_R^T\mathbf{p}-\mathbf{q}),$$

where

$$(2.13)_1\quad \mathbf{H}=\int_{\Omega}\int_{\Omega'}\mathbf{P}(x)\Big(H^1(x,x')+H^2(x,x')\Big)\mathbf{P}^T(x')\,dS'dS=\mathbf{H}^1+\mathbf{H}^2$$

is the influence matrix (positive definite), and

$$\mathbf{t} = \int_{\Omega} \mathbf{P}(f^1 + f^2)dS - \int_{\Omega} \mathbf{P}h\, dS = \mathbf{f} - \mathbf{h} \tag{2.13$_2$}$$

is the known displacement vector resulting from the external load and initial gap of the bodies, $\boldsymbol{\lambda}^T = [\boldsymbol{\lambda}_F^T, \boldsymbol{\lambda}_M^T]$ is the vector for characterization of the rigid body-like displacement.

The quadratic programming problem is

$$\min \left\{ \frac{1}{2}\mathbf{p}^T\mathbf{H}\mathbf{p} - \mathbf{p}^T\mathbf{t} \;\Big|\; \mathbf{p} \geq \mathbf{0},\;\; \mathbf{G}_R^T\mathbf{p} - \mathbf{q} = \mathbf{0} \right\} \tag{2.14}$$

from which, in sense of the Kuhn – Tucker condition, the equations-inequalities

$$\begin{aligned} \mathbf{p} \geq \mathbf{0}, \qquad \mathbf{d} &= \frac{\partial L}{\partial \mathbf{p}} = \mathbf{H}\mathbf{p} - \mathbf{G}_R\boldsymbol{\lambda} - \mathbf{t} \geq \mathbf{0}, \\ \mathbf{p}^T\mathbf{d} = 0, \qquad \frac{\partial L}{\partial \boldsymbol{\lambda}} &= -\mathbf{G}_R^T\mathbf{p} + \mathbf{q} = \mathbf{0}, \end{aligned} \tag{2.15}$$

can be constructed. Here $\mathbf{d}$ is the vector of the gap after deformation. We can see that, instead of the conditions (2.3), we get the discretized conditions $\mathbf{p} \geq \mathbf{0}$, $\mathbf{d} \geq \mathbf{0}$, $\mathbf{p}^T\mathbf{d} = 0$, that is, the constraints of the contact/separation appear in the final form as the product integral of $d(x)$ by the coordinate function $P_i(x)$ $(i = 1, \ldots, k)$ for nodal contact pressure, which is positive or zero, depending on the i-th nodal contact pressure being zero or positive.

The quadratic programming problem can be solved by the standard Wolf and Beale method or other modified simplex-type algorithms [8] or special iteration methods [9].

2.2. Procedure on the basis of the Minimum of the Total Potential Energy with augmented Lagrangian technique

To solve the contact problem, the minimum of the total potential energy of the system Π is sought for by using the geometrical inequality conditions, i.e.

$$\min \left\{ \Pi \;\Big|\; d(x) \geq 0 \;\; x \in S_c \right\}, \tag{2.16}$$

where

$$\Pi(\mathbf{u}) = \sum_{\alpha=1}^{2} \left(\frac{1}{2} \int_{V^\alpha} \boldsymbol{\varepsilon}(\mathbf{u}) \cdot\cdot\, \mathbf{D} \cdot\cdot\, \boldsymbol{\varepsilon}(\mathbf{u}) dV - \int_{V^\alpha} \mathbf{f}^B \cdot \mathbf{u} dV - \int_{S_p^\alpha} \tilde{\mathbf{p}} \cdot \mathbf{u} dS \right), \tag{2.17}$$

$\boldsymbol{\varepsilon}(\mathbf{u})$, $\mathbf{D}$ are the strain and elasticity constants tensors, $\mathbf{f}^B$, $\tilde{\mathbf{p}}$, $\mathbf{u}$ are the body force vector, the stress vector given on surface S_p^α and the displacement field, respectively. Double dot $\cdot\cdot$ denotes the double scalar product.

Solution of (2.16) can be obtained effectively by means of the augmented Lagrangian technique [6]. In this case the functional is

$$L_a = \Pi(\mathbf{u}) - \int_{\Omega} p d(\mathbf{u})\, dS + \frac{1}{2} \int_{\Omega_p} c\,(d(\mathbf{u}))^2\, dS, \tag{2.18}$$

where $c \gg 0$ is the penalty parameter. For the process of minimization of L_a, the value of $p = p^{(k)}$ is fixed during the k-th augmented iterational step. Knowing the solution of equation $\delta_{\mathbf{u}} L_a = 0$, the new value of p should be calculated by the formulae

$$p^{(k+1)} = \langle p^{(k)} - c d^{(k)}(\mathbf{u}) \rangle, \tag{2.19}$$

where

$$\langle a \rangle = \frac{1}{2}\Big[a + |a|\Big], \tag{2.20}$$

and k is the number of the augmented iterational step.

After discretisation, using the substructural finite element system, the function L_a has the form:

$$\begin{aligned} L_a = &\sum_{\alpha=1}^{2} \frac{1}{2} \mathbf{u}^{\alpha T} \mathbf{K}^\alpha \mathbf{u}^\alpha - \mathbf{u}^{\alpha T} \mathbf{b}^\alpha \\ &+ \frac{1}{2} [\mathbf{u}^{1T}\ \mathbf{u}^{2T}] \left(\begin{bmatrix} \mathbf{C} & -\mathbf{C} \\ -\mathbf{C} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{u}^1 \\ \mathbf{u}^2 \end{bmatrix} + 2 \begin{bmatrix} -\mathbf{C} \\ \mathbf{C} \end{bmatrix} \mathbf{h} - 2 \int_{\Omega} \begin{bmatrix} -\mathbf{L}^T \\ \mathbf{L}^T \end{bmatrix} p\, dS \right) \end{aligned} \tag{2.21}$$

for which we can get the following system of equations:

$$\begin{aligned} (\mathbf{K}^1 + \mathbf{C})\mathbf{u}^1 - \mathbf{C}\mathbf{u}^2 &= \mathbf{b}^1 + \mathbf{C}\mathbf{h} + \mathbf{f}_p, \\ -\mathbf{C}\mathbf{u}^1 + (\mathbf{K}^2 + \mathbf{C})\mathbf{u}^2 &= \mathbf{b}^2 - \mathbf{C}\mathbf{h} - \mathbf{f}_p, \end{aligned} \tag{2.22}$$

where

$$u_N^\alpha(x) = \mathbf{L}(x)\mathbf{u}^\alpha, \qquad h = \mathbf{L}(x)\mathbf{h},$$

$$\mathbf{C} = \int_{\Omega_p} \mathbf{L}^T c \mathbf{L}\, dS, \qquad \mathbf{f}_p = -\int_{\Omega} \mathbf{L}^T p\, dS, \qquad c \gg 0,$$

$\mathbf{K}^\alpha$ is the stiffness matrix, $\mathbf{b}^\alpha$ the loading vector, $\mathbf{L}(x)$ is the shape function for approximation in normal direction, $\mathbf{u}^\alpha$ is the nodal point displacement vector of the body α. Practically the displacements $\mathbf{u}^{\alpha}$ in the region S_c are divided into components parallel and orthogonal to $\mathbf{n}^1$. In this case $\mathbf{C}$ is the matrix of "springs" having normal directions. Solution of (2.22) can be done by the iterational KALKER procedure [9] with the control of the sign of p. The checking points are the three Gauss integrational points of the contact elements. The reason is, that quadratic elements are used for approximation of the displacement field [16]. In these points we can interpret the vector of contact pressure $\mathbf{p}$. Knowing $p^{(k+1)}$, the $(k+1)$-th displacements are obtained from the solution of (2.22). For the fulfillment of the conditions of non-negative pressure distribution and that of the non-penetration of the contact bodies into each other, the matrix $\mathbf{C}$ should be changed *iterp* and *iterd* times during the Kalker iteration process [15].

3. OPTIMIZATION OF THE CONTACT PRESSURE DISTRIBUTION

Only a few studies can be found in the literature for the contact pressure optimization.

Shape of rigid punch was defined by CONRY and SEIREG [10]. The shape of an axisymmetric punch was approximated by quadratic polynomials. They achieved to get a special shape by which they could approximate the constant value of $p_0 = F/S$, where F is the load and S means the area of the cross-section of the rigid punch. PÁCZELT and HERPAI [11] determined the initial gap between cylindrical shells from the condition of constant pressure. HAUG and KWAK [12] applied the finite element method during an iteration algorithm to find the minimum of contact pressure maximum and the variation of the boundary of the contacting bodies. KIKUCHI and TAYLOR [13] have determined the value of $\min(p_{\max})$ by taking the integral of the gap function as the isoparametric constraint into consideration. The thorough mathematical investigation of the subject can be found in the book of HASLINGER and NEITTAANMAKI [14]. Now we shall investigate the contact optimization problem, where the pressure distribution will be controlled.

3.1. Controlling of the contact pressure

The contact pressure between bodies without optimization may vary significantly within the contact region. In addition to minimizing the maximum contact pressure it is our aim to control the contact pressure [7, 15, 17]. The region Ω_{opt} may differ from the actual region Ω_p or from the original region Ω. Assignment of Ω_{opt} depends on the operation of the whole structure.

If it is required that the pressure at any point x of region Ω_{opt} should be less than $v(x)p_{\max}$, the control condition will be defined by the following inequality:

$$\chi(x) = v(x)p_{\max} - p(x) \geq 0, \qquad x \in \Omega_{\text{opt}}, \tag{3.1}$$

where the controller function $v(x)$ satisfies the condition $0 \leq v(x) \leq 1$, and $p_{\max} \overset{\text{def}}{=} \max p(x)$. It is evident that $p(x) \leq p_{\max}$. In the paper [7, 15] three different forms of the controller functions were considered: constant, of a trapezoid shape and function of class C^1.

In the case of single variable $x \equiv x_1$, if in all the region $x \in \Omega_{\text{opt}}$, $v(x) = 1$, then the controller function yields constant pressure, if

$$v(x) = 1, \qquad 0 \leq x \leq L_1,$$
$$v(x) = \frac{L_2}{L_2 - L_1} - \frac{x}{L_2 - L_1}, \qquad L_1 \leq x \leq L_2.$$

Then we speak about the trapezoidal control; and if we use the controller function $v(x)$ given in Tables 1, 3, then the controller function of the class C^1 is defined. In this case $dv/dx = 0$ at $x = L_1$ and $x = L_2$.

In the two-dimensional case for optimization of the shape of the bearing roller [15], the controller function has the form $v(x_1, x_2) = v(x_1) \cdot v(x_2)$, where $v(x_1) = 1$ and $v(x_2) = v(x)$ of Tables 1, 3.

3.2. Formulation of the optimization problem to define the contact pressure and the initial gap

During optimization the inequality (2.7) can be replaced by the following condition

$$d(x) = \int_{\Omega'_{\text{opt}}} H(x, x')p(x')\, dS' + f^2(x) - f^1(x) + h(x) + \Delta h(x) - r(x) \geq 0, \tag{3.2}$$

where h – initial gap, Δh – change of the initial gap due to variation of the shape of contact body.

In the case of one-dimensional region Ω_{opt} controlling $\chi \geq 0$ is possible without problems if condition $d(x) = 0$ is satisfied. In two-dimensional case it is not always possible. For example, at the bearing rollers the geometrical condition $d(x) = 0$ can be prescribed only in the subregion Ω_v of Ω_{opt}, while in the complementary part $\Omega_{nv} = \Omega_{\text{opt}} - \Omega_v$ only inequality $d(x) \geq 0$ can be prescribed in advance. It is supposed that the change of gap in region Ω_{nv} is expressed by the function of Δh of region Ω_v

$$\Delta h_{nv} = f(\Delta h). \tag{3.3}$$

It follows from the character of the problem that

$$\chi = v(x)p_{\max} - p(x) = 0, \quad x \in \Omega_v \qquad \text{while} \quad \chi \geq 0 \quad \text{at} \quad x \in \Omega_{nv}.$$

The optimization problem can be written briefly in the form

$$(3.4) \qquad \min\Big\{p_{\max} \,\Big|\, \mathbf{F} = \mathbf{0},\ \mathbf{M} = \mathbf{0},\ p(x) \geq 0,\ \chi(x) \geq 0,\ d(x) \geq 0, \\ p(x)d(x) = 0,\ x \in \Omega_{\text{opt}}; \\ \Delta h = \Delta h(x),\ x \in \Omega_v,\ \Delta h_{nv} = f(\Delta h),\ x \in \Omega_{nv}\Big\};$$

The discretised optimization problem (3.4) corresponds to the following linear programming problem.

$$(3.5) \qquad \min\Big\{p_{\max} \,\Big|\, \mathbf{G}_R^T\mathbf{p} - \mathbf{q} = \mathbf{0},\ \mathbf{p} \geq \mathbf{0},\ \boldsymbol{\chi} = \mathbf{v}p_{\max} - \mathbf{p} \geq \mathbf{0}, \\ \mathbf{d} = \mathbf{H}\mathbf{p} - \mathbf{G}_R\boldsymbol{\lambda} - \mathbf{t} + \mathbf{Q}\Delta\mathbf{h} \geq \mathbf{0},\ \Delta\mathbf{h} \geq \mathbf{0},\ \mathbf{p}^T\mathbf{d} = 0, \\ \boldsymbol{\lambda} = \boldsymbol{\lambda}^+ - \boldsymbol{\lambda}^-,\ \boldsymbol{\lambda}^+, \boldsymbol{\lambda}^- \geq \mathbf{0}\Big\}.$$

In order to solve (3.5), three algorithms have been developed and numerical results presented in the case of shape optimization of the bearing roller [15].

If the solution of the given contact problem is derived by means of the equation (2.22), then the optimization problem is

$$(3.6) \qquad \min\Big\{p_{\max} \,\Big|\, \mathbf{p} \geq \mathbf{0},\ \boldsymbol{\chi} \geq \mathbf{0}, \\ \mathbf{d} = \mathbf{u}_N^2 - \mathbf{u}_N^1 + \mathbf{h}_0 + \mathbf{Q}\Delta\mathbf{h} \geq \mathbf{0},\ \mathbf{p}^T\mathbf{d} = 0\Big\},$$

where $\mathbf{d}$ belongs to the integration points, and $\mathbf{h}_0$ – vector of the initial gap.

The problem (3.6) can be solved by a modified version of the iteration procedure shown in [15]. In this case, one of the bodies has prescribed rigid body-like displacement (translation or rotation). The iterational process is working if all the points of region Ω_v are in contact. Some results concerning "theory for the control" can be found in [17].

STEPS:

1. $k = 1$, $\mathbf{h}_0$ – initial gap vector.
2. $\Delta\mathbf{h}^{(k-1)} = \mathbf{0}$.
3. Solution (2.22) by substitution of $\mathbf{h} = \mathbf{h}_0 + \mathbf{Q}\Delta\mathbf{h}^{(k-1)}$.
4. Determination of $\mathbf{p}^{(k-1)}$ on the basis of (2.19), (2.20) $\Rightarrow p_{\max}$.
5. Calculation of new pressure in the region of $x \in \Omega_v$, $\boldsymbol{\chi}_v = \mathbf{v}p_{\max} - \mathbf{p}_v^{(k)} = 0 \Rightarrow \mathbf{p}_v^{(k)} \Rightarrow \mathbf{p}^{(k)}$.
6. Determination of the residual in Ω_v, $\mathbf{m}^{(k)} = \mathbf{u}_N^2\left(\mathbf{p}^{(k)}\right) - \mathbf{u}_N^1\left(\mathbf{p}^{(k)}\right) + \mathbf{h}_0$.

Using the condition that the gap does not change at point N of the region Ω_v, the value of the residual is calculated at this point. Let this value be a, i.e. $\mathbf{m}_N^{(k)T} = [0, 0 \ldots, a, \ldots, 0]$. Thus the effect of the gap change is $\mathbf{Q}\Delta\mathbf{h}^{(k)} = -\left(\mathbf{m}^{(k)} - \mathbf{m}_N^{(k)}\right)$.

7. Checking of the convergence condition

$$\Theta = \frac{m_j^{(k)} - m_j^{(k-1)}}{m_j^k},$$

$$\text{if } \Theta_j \leq \vartheta \text{ then stop } (\vartheta - \text{given value});$$

$$\text{if } \Theta_j \geq \vartheta, \; k = k + 1, \text{ back to step 3.}$$

4. Numerical examples

An axisymmetric contact problem is shown in Fig. 1. Materials of both bodies are the same, with Young moduli E $= 2 \cdot 10^5$ MPa and Poisson

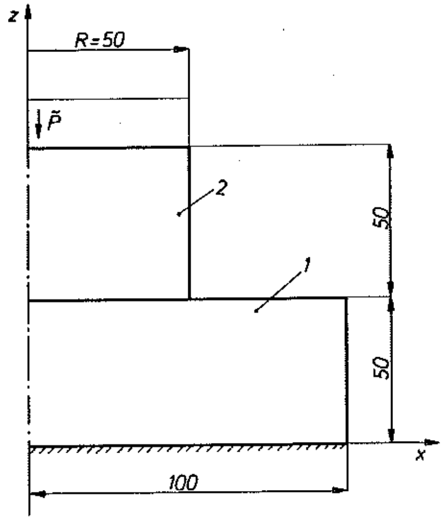

FIG. 1. Contact problem of axisymmetric bodies.

ratio $\nu = 0.3$. The upper surface of body 2 is loaded by a constant intensity pressure, $\tilde{p} = 100$ MPa. By using the finite element mesh shown in Fig. 2, with the penalty contact elements (penalty parameter $c = E \cdot 10^4$) (Fig. 3) but without optimization, the peak stress value $p_{\max} = 272.53$ MPa at the edge of the punch was obtained. After choosing different controlling functions $v(x)$, we have got different maximum pressure and initial gap functions. The gap changes could be approximated by the form of $\Delta h = \mathbf{L}(x)\Delta\mathbf{h}$. From

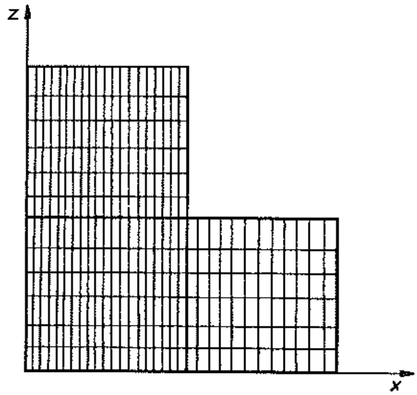

FIG. 2. Finite element mesh of axisymmetric bodies. We used quadratic isoparametric ring element for approximation of displacement fields.

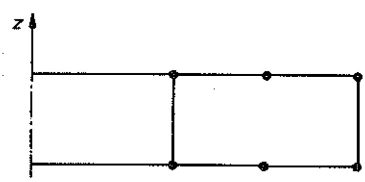

FIG. 3. Penalty contact element with penalty parameter c in the vertical direction.

the rigid-like movement of the body 2 it follows that the resultant of the optimized contact pressure $F_0 = p_{\max} \int_0^R 2\pi x v(x)\, dx$ is equal to the resultant of the known external loading $F_0 = R^2 \pi \widetilde{p}$. Once the value of $p_{\max}$ is known, the loading $p = p_{\max} v(x)$ is applied as an extra load, independently to both solids which are now separated ($\mathbf{C} = \mathbf{0}$), i.e. the gap can be determined from the equation

$$d(x) = u_N^2(x, \widetilde{p}, p) - u_N^1(x, p) + \Delta h(x) \tag{4.1}$$

after discretisation.

The results can be seen in the Table 1 and Fig. 4.

The structure was investigated for the loading case of prescribed axial displacement $w = -0.05$ mm of the upper surface of the body 2.

In this case, the restricted linear programming problem (3.6) was solved using the iterational technique. The problem has been solved by different control functions. Table 2 summarizes the values of L_1, $p_{\max}$, $\Delta h_{\max}$, the sums of squares of the gap variations

$$TOLH = \sum_{j=1}^{KONT} \left(\Delta h_j^{(k)} - \Delta h_j^{(k-1)}\right)^2, \tag{4.2}$$

Table 1.

$v(x)$ $L_2 = R = 50\,\mathrm{mm}$					
L_1 [mm]	0	20	30	40	50
$p_{\max}$ [PMa]	333.2	196.9	154.3	123.2	100
$\Delta h_{\max}$ [μm]	93.67	64.72	48.71	32.27	10.87
$p(x) = v(x)p_{\max}$	$v(x) = 1,\quad 0 \le x \le L_1$ $v(x) = 1-3[(x-L_1)/(L_2-L_1)]^2+2[(x-L_1)/(L_2-L_1)]^3,\quad L_1 \le x \le L_2$				

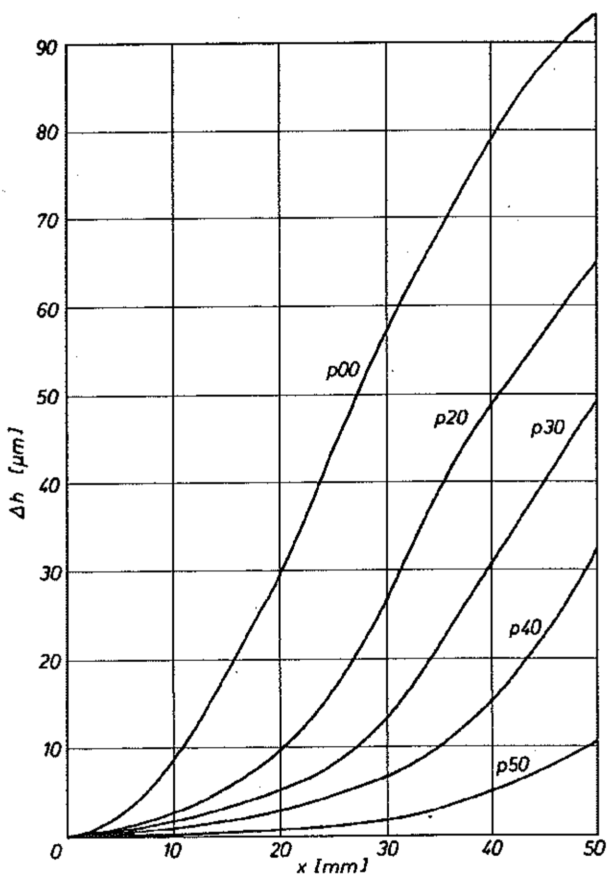

FIG. 4. The change of the gap functions in the case of a given load $\widetilde{p} = 100\,\mathrm{MPa}$. The curve pxy belongs to $L_1 = xy$.

and the iterational steps Iterp, Iterd which were needed for the KALKER algorithm [15] belonging to the solution of (3.6) at the k-th iterational step. The iteration was performed at the convergence criterion $TOLH \leq 0.5 \cdot 10^{-6}$. The gap functions can be seen in Fig. 5.

Table 2.

Given displacement $w = -0.05$ mm						
L_1 [mm]	Iter (k)	p_{max} [MPa]	Δh_{max} [μm]	$TOLH$	Iterp	Iterd
50	1	338.9	36.82	$0.9209 \cdot 10^{-2}$	1	1
	2	167.2	18.15	$0.2367 \cdot 10^{-2}$	6	3
	3	113.5	12.33	$0.2301 \cdot 10^{-3}$	2	1
	4	111.7	12.14	$0.2584 \cdot 10^{-6}$	1	1
40	1	338.9	85.57	$0.5830 \cdot 10^{-1}$	1	1
	2	186.2	47.00	$0.1184 \cdot 10^{-1}$	10	3
	3	119.0	30.03	$0.2292 \cdot 10^{-2}$	9	3
	4	114.6	28.94	$0.9594 \cdot 10^{-5}$	7	3
	5	114.4	28.89	$0.1540 \cdot 10^{-7}$	2	1
30	1	338.9	101.9	0.1179	1	1
	2	146.1	43.96	$0.3813 \cdot 10^{-1}$	10	3
	3	132.4	39.83	$0.1936 \cdot 10^{-3}$	7	2
	4	121.0	36.40	$0.1335 \cdot 10^{-3}$	4	1
	5	119.8	36.04	$0.1454 \cdot 10^{-5}$	7	3
	6	119.7	36.01	$0.1079 \cdot 10^{-7}$	2	1
20	1	338.9	105.8	0.1634	1	1
	2	164.4	51.34	$0.4332 \cdot 10^{-1}$	8	2
	3	134.6	42.03	$0.1267 \cdot 10^{-2}$	8	2
	4	129.2	40.35	$0.4119 \cdot 10^{-5}$	4	1
	5	128.3	40.07	$0.1110 \cdot 10^{-5}$	4	1
	6	128.2	40.03	$0.2525 \cdot 10^{-7}$	3	1
0	1	338.9	90.84	0.1635	1	1
	2	221.5	59.38	$0.1960 \cdot 10^{-1}$	8	1
	3	183.7	49.24	$0.2039 \cdot 10^{-2}$	9	4
	4	170.3	45.64	$0.2565 \cdot 10^{-3}$	20	3
	5	165.5	44.35	$0.3277 \cdot 10^{-4}$	6	2
	6	163.8	43.90	$0.3989 \cdot 10^{-5}$	5	2
	7	163.2	43.75	$0.4636 \cdot 10^{-6}$	9	3

The same problem was investigated for plane stress as well, with the thickness of $b = 10$ mm. The results are shown in the Table 3 and Fig. 6. It can be seen from the above examples that some very small changes of the gaps can cause large differences in the pressure distribution.

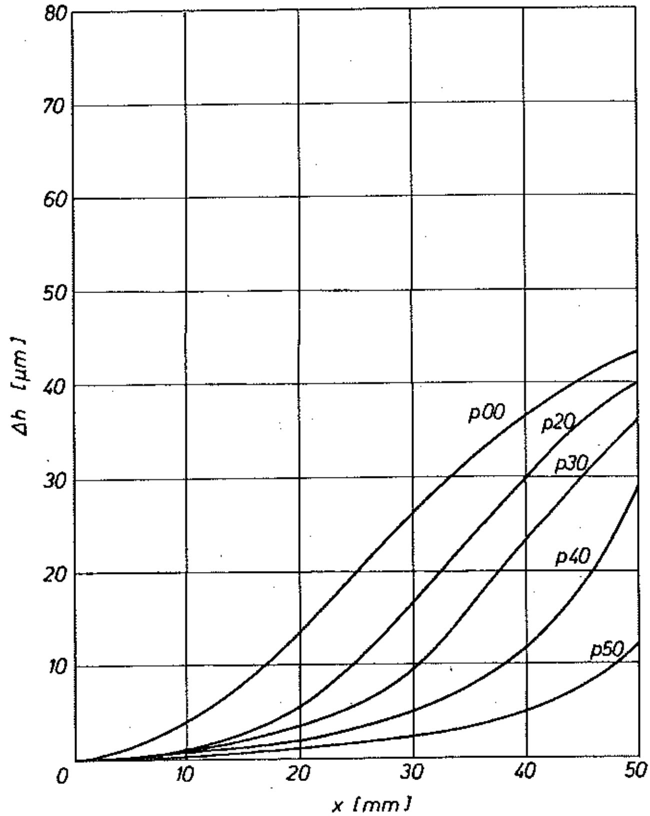

FIG. 5. The change of the gap functions in the case of given vertical displacement $w = -0.05$ mm on the upper surface of body 2. The curve pxy belongs to $L_1 = xy$.

Table 3.

$v(x)$ $L_2 = R = 50$ mm					
L_1 [mm]	0	20	30	40	50
$p_{\max}$ [PMa]	283.3	142.9	125.0	111.1	100
$\Delta h_{\max}$ [μm]	88.35	65.86	52.02	36.02	12.69
$p(x) = v(x)p_{\max}$	$v(x) = 1, \quad 0 \le x \le L_1$ $v(x) = 1-3[(x-L_1)/(L_2-L_1)]^2+2[(x-L_1)/(L_2-L_1)]^3, \quad L_1 \le x \le L_2$				

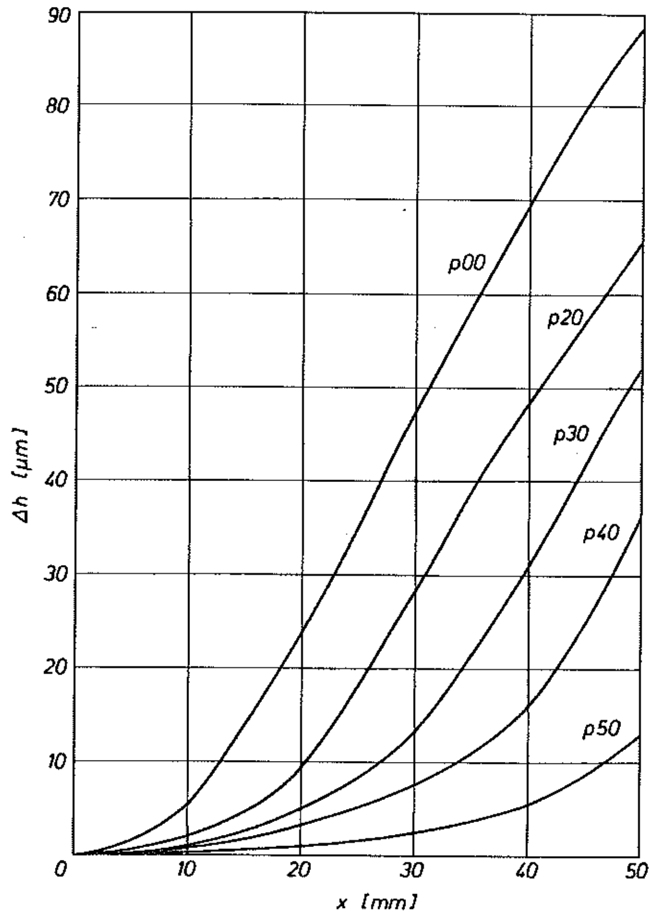

FIG. 6. The change of the gap functions in the case of an elastic system for plane stresses. The curve pxy belongs to $L_1 = xy$.

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