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TRANSPORT EQUATION FOR SHOCK STRENGTH IN HYPERELASTIC RODS (*)

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The singular surface theory and perturbation method of solution are used to examine a $1-D$ shock wave propagation problem in a semi-infinite rod of slowly varying cross-sectional area. The isentropic approximation is used. The weak nonlinear shock propagates into a region, which is homogeneously deformed and at rest. A numerical analysis for decreasing and increasing cross-sectional areas, and for a special type of nonlinear elastic material is conducted.

1. Introduction

The problem of evolution laws for the shock amplitude in a one-dimensional thin rod was examined, among others, by Scott and Fu in two papers:

[1] (*Rod with constant cross-sectional area*).

Using the singular surface theory and a perturbation method with the parameter ε characterising the initial shock strength. Application of the shock fitting method to construct solutions known from the simple wave theory.

[2] (*Rod with a slowly varying cross-sectional area*).

Applying the shock fitting method to the modulated simple wave solutions.

Contrary to the singular surface theory, which can deal only with the local evolutionary behaviour, the shock fitting method gives the possibility to include the whole history of the loading programme. It is very important and useful for finding the solution of the problem. Both the methods yield the same asymptotic laws within their common range of validity [1].

The aim of the paper is to find the solution of the problem of a one-dimensional rod with slowly varying cross-sectional area by using the singular

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surface theory, and to compare the results with those obtained in [2]. We use exactly the notation applied in [1] and [2] and extend the results obtained in [1]. For this reason, the notation *additional term(s)* in expressions and formulas means that the formula without it is *exactly* the same as in [1] (for the constant cross-sectional area), and the terms represent the additional effects of the slowly varying cross-sectional area.

2. Basic equations

A semi-infinite *thin* rod with slowly varying cross-sectional area $A(X)$ occupies the material region $X \geq 0$. We assume the elementary approximate theory of "longitudinal" shock waves in the rod. A one-dimensional motion is considered

$$x = X + u(X,t), \qquad e = u_X, \qquad v = \dot{u}, \tag{2.1}$$

where x is the position, and u, e, v are respectively the displacement, the strain and the velocity.

It is assumed that only one component of the displacement vector parallel to the X-axis is different from zero and it is a function of one variable X (position of the particle in the reference configuration) only. In the simplest approximate theory we assume that the nominal stress over a cross-section is *uniform and purely axial*, i.e. plane cross-sections remain *undistorted* by the motion (2.1). Such situation predicts the propagation of *plane waves* only. All the above assumptions make impossible an exact description of wave propagation in the rod, because they do not allow for the motions in the directions perpendicular to the X-axis, which must occur at the same time as the dilatational (main) motion due to the Poisson's ratio coupling (in the linear theory) and by means of many other elastic constants in the nonlinear theory.

The nominal stress tensor corresponding to the motion (2.1) is evidently given by (cf. (4.2), (4.3))

$$\begin{aligned} T_{11} &= T = 2\rho_R(1+e)(\sigma_1 + 2\sigma_2 + \sigma_3), \\ T_{22} &= T_{33} = 2\rho_R(\sigma_1 + \sigma_2 + (1+e)^2(\sigma_2 + \sigma_3)), \end{aligned}$$

and the shear stresses are equal zero.

We assume in this paper (following Fu and Scott in [1, 2]), that the rod is *thin* so that the motion is essentially one-dimensional and we can neglect the accompanying normal stresses $T_{22} = T_{33}$. If only $T_{11} \neq 0$ and $A(X) =$ const, the traction boundary conditions on the lateral surface of the rod are satisfied *identically*. In an analogous situation and for slowly varying $A(X)$, one component of the stress vector parallel to the X-axis, $\hat{t}_1 = \hat{t} = T_{11}N_1 = T\sin\alpha$, does not vanish; $N_1 = \sin\alpha$ denotes here the component of the unit vector N perpendicular to the lateral surface (Fig. 1). Assuming that $A(X)$ varies slowly, then $\sin\alpha \approx 0$ and the traction boundary conditions on the lateral surface are also *approximately* satisfied.

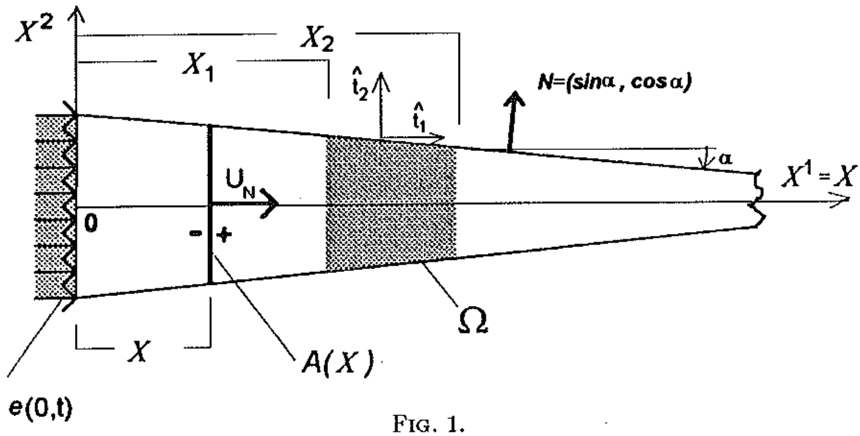


FIG. 1.

The equation expressing the balance of momentum in *the integral form* for the region Ω is

$$(2.2)\qquad \frac{d}{dt}\int_{X_1}^{X_2} \rho_R v A\, dX = (TA)|_{X=X_2} - (TA)|_{X=X_1},$$

where T and ρ_R are, respectively, the nominal stress $T_{11} = T$ and the density in the reference configuration. In the isentropic approximation considered here the nominal stress is then a function of the strain e only, and *the local form* of Eq. (2.2) is

$$(2.3)\qquad \rho_R A\,\dot{v} = \frac{\partial(TA)}{\partial X} \quad\Rightarrow\quad Ee_X = \rho_R v_t \underbrace{-T\frac{d}{dX}\ln(A(X))}_{\text{additional term}}.$$

Differentiating this equation with respect to X we obtain

$$(2.4)\qquad Ee_{XX} + \widetilde{E}e_X^2 = \rho_R v_{tX} \underbrace{-T\frac{d^2}{dX^2}\ln(A(X)) - E\frac{d}{dX}\ln(A(X))}_{\text{additional terms}},$$

where $E = \dfrac{\partial T(e)}{\partial e}$, $\widetilde{E} = \dfrac{\partial^2 T(e)}{\partial e^2}$, and both these equations without *additional term(s)* are exactly the equations obtained for $A(X) = \text{const}$ (Ref. [1]).

The region ahead of the shock wave is homogeneously deformed and at rest, that means $e_X^+ = 0$, $v^+ = 0$, $T(e) = \text{const} \;\Rightarrow\; E^+ = \widetilde{E}^+ = 0$.

Considering the jumps of (2.3) and (2.4) we arrive at the following equations

$$(2.5)\qquad [Ee_x] = \rho_R[v_t] - \underbrace{\rho_R U_N^2[e]\frac{d(\ln A(X))}{dx}}_{\text{additional term}},$$

$$(2.6)\qquad [Ee_{xx}]+[\widetilde{E}e_x^2] = \rho_R[v_{tx}] \underbrace{-[Ee_x]\frac{d(\ln A(X))}{dx} - \rho_R U_N^2[e]\frac{d^2}{dX^2}(\ln A(X))}_{\text{additional terms}}.$$

The kinematical condition of compatibility [4], [1] for any quantity $f(X,t)$ in one-dimensional shock wave propagation is

$$(2.7)\qquad \frac{d}{dt}[f] = [\dot{f}] + U_N[f_X], \qquad \frac{d}{dt} = U_N\frac{d}{dX},$$

where the displacement derivative d/dt is related to the space derivative d/dX following the wave front by the second relation.

Replacing $f(X,t)$ in (2.7) in turn by u, v, u_X we can obtain, by repeated use of (2.7), all jumps ($[v]$, $[v_t]$, $[v_{tX}]$) which we need in (2.5), (2.6) to have two coupled equations for the shock amplitude $[e]$ and amplitudes of the accompanying higher-order discontinuities $[e_x]$ and $[e_{xx}]$,

$$(2.8)\qquad 2\rho_0 U_N^2\frac{d[e]}{dX} + \rho_0 U_N[e]\frac{dU_N}{dX} + (E^- - \rho_0 U_N^2)[e_X] + \underbrace{\rho_0 U_N^2[e]\frac{d}{dX}\ln A(X)}_{\text{additional term}} = 0,$$

$$(2.9)\qquad 2\rho_0 U_N^2\frac{d[e]}{dX} + \widetilde{E}^-[e_X]^2 + \rho_0 U_N[e]\frac{dU_N}{dX} - \rho_0 U_N^2\frac{d^2[e]}{dX^2} - \rho_0 U_N\frac{dU_N}{dX}\frac{d[e]}{dX} + (E^- - \rho_0 U_N^2)[e_{XX}] + \underbrace{E^-[e_X]\frac{d}{dX}\ln A(X) + \rho_0 U_N^2[e]\frac{d^2}{dX^2}\ln A(X)}_{\text{additional terms}} = 0.$$

3. Approximate solution

Expanding $[e]$, $[e_X]$ and $[e_{XX}]$ into the power series of a small dimensionless parameter ε, we obtain (Ref. [1])

$$(3.1)\qquad \begin{aligned} [e] &= \varepsilon Y_1 + \varepsilon^2 Y_2 + \ldots, \\ [e_X] &= Z_0 + \varepsilon Z_1 + \ldots, \\ [e_{XX}] &= W_0 + \varepsilon W_1 + \ldots . \end{aligned}$$

On substituting (3.1) into (2.8) and (2.9) and then equating the coefficients of the same powers of ε, we obtain a hierarchy of differential equations $(\varepsilon^0, \varepsilon)$

$$\begin{aligned} &\frac{dZ_0}{dX} + \frac{1}{2}c_1 Z_0^2 + \underbrace{\frac{1}{2}Z_0 \frac{d}{dX}\ln A(X)}_{\text{additional term}} = 0, \\ &\frac{dY_1}{dX} + \frac{1}{4}c_1 Y_1 Z_0 + \underbrace{\frac{1}{2}Y_1 \frac{d}{dX}\ln A(X)}_{\text{additional term}} = 0, \end{aligned} \tag{3.2}$$

where $c_1 = \widetilde{E}^+/E^+$, and

$$[e]\Big|_{X=0} = h, \quad [e_X]\Big|_{X=0} = k \quad \Rightarrow \quad Z_0(0) = k, \quad \varepsilon Y_1(0) = h. \tag{3.3}$$

The differential equations governing the amplitude of the higher order discontinuity $[e_X]$ accompanying the shock $[e]$ is the *well-known Bernoulli equation* for the function $Z_0(X)$,

$$\frac{dZ_0}{dX} + f(X)Z_0^2 + g(X)Z_0 = 0, \tag{3.4}$$

with solution

$$Z_0(X) = k\left[1 + \frac{c_1 k}{2}\sqrt{\frac{A(X)}{A(0)}}\int_0^X \sqrt{\frac{A(0)}{A(X)}}dX\right]^{-1}. \tag{3.5}$$

In the second equation of the system (3.2) the variables are separable

$$\frac{dY_1}{dX} + \left[\frac{1}{4}c_1 Z_0(X) + \frac{1}{2}\frac{d}{dX}\ln A(X)\right]Y_1 = 0, \tag{3.6}$$

and the function $\varepsilon Y_1(X)$ is

$$Y_1\varepsilon = h\sqrt{\frac{A(0)}{A(X)}}\left\{\exp\left[-\frac{c_1 k}{4}\int_0^X \frac{dX}{1 + \frac{c_1 k}{2}\sqrt{\frac{A(X)}{A(0)}}\int_0^X \sqrt{\frac{A(0)}{A(X)}}dX}\right]\right\}. \tag{3.7}$$

In the case $A(X) = A(0)$ both the above solutions are *exactly the same* as in [1]. We have obtained the solution (3.7) instead of the solution given by Fu and Scott ([2]), by using the *modulated* simple wave solutions and the *shock fitting* methods.

4. Application to Murnaghan material

Analysis of the problem is restricted to a special kind of *second order* elastic material, called the Murnaghan material

$$(4.1)\qquad \rho_R\sigma = \frac{1+2m}{24}(I_1-3)^3 + \frac{\lambda+2\mu+4m}{8}(I_1-3)^2 + \frac{8\mu+n}{8}(I_1-3) - \frac{m}{4}(I_1-3)(I_2-3) - \frac{4\mu+n}{8}(I_2-3) + \frac{n}{8}(I_3-1),$$

where $I_1 = B_{ii}$, $I_2 = (B_{ii}B_{jj} - B_{ij}B_{ij})/2$, $I_3 = \det(B_{ij})$ are the invariants of the left Cauchy-Green strain tensor $\mathbf{B}$, λ and μ are Lamé coefficients, and l, m, n are the elastic constants of second order.

Numerical analysis shows that only shocks of a relatively small (of order up to 10^{-3}) intensity can propagate in this material [5].

The deformation gradient for the one-dimensional motion (2.1) is given by

$$(4.2)\qquad F_{i\alpha} = \begin{bmatrix} 1+e & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The components of the tensors $\sigma_{ik}^{\alpha\beta}$, $\sigma_{ikl}^{\alpha\beta\gamma}$, which we need here are (cf. [5])

$$(4.3)\qquad \begin{aligned} \sigma_{11}^{11} &= 2\sigma_1 + 4\sigma_2 + 2\sigma_3 + 4\sigma_{11}(1+e)^2 + 16\sigma_{12}(1+e)^2, \\ \sigma_{111}^{111} &= 8\sigma_{111}(1+e)^3 + 12\sigma_{11}(1+e) + 48\sigma_{12}(1+e), \end{aligned}$$

where $\sigma_i = \dfrac{\partial\sigma}{\partial I_i}$, $\sigma_{ik} = \dfrac{\partial^2\sigma}{\partial I_i \partial I_k}$, $\sigma_{ikl} = \dfrac{\partial^3\sigma}{\partial I_i \partial I_k \partial I_l}$, and after substitution of the elastic constants we have

$$(4.4)\qquad \begin{aligned} \sigma_{11}^{11} &= (\lambda+2\mu)[(1+3e)+2\eta e], \\ \sigma_{111}^{111} &= (\lambda+2\mu)[2\eta(1+6e)+3(1+e)], \end{aligned}$$

where

$$(4.5)\qquad \eta = \frac{l+2m}{\lambda+2\mu}, \qquad c_1 = \frac{\widetilde{E}}{E} = \frac{\sigma_{111}^{111}}{\sigma_{11}^{11}} = \frac{2\eta(1+6e)+3(1+e)}{(1+3e)+2\eta e}.$$

5. Comparison of the results with solutions obtained in [2]

Equation (3.7) can be integrated numerically for the assumed form of the cross-sectional area $A(X)$. It is easy to obtain the solution in a closed form for the case

$$A(\widehat{X}) = \frac{A(0)}{(1+\beta\widehat{X})^2}, \qquad \widehat{X} = \frac{X}{X_0}, \tag{5.1}$$

where $\widehat{X}$ is the dimensionless length and $X_0 = 1$ m, and β is the real number which describes the rate of decrease of the cross-sectional area. Substituting (5.1) into (3.7) and integrating twice we obtain the solution

$$[e] \approx Y_1\varepsilon = h\frac{(1+\beta\widehat{X})}{\sqrt{1+\left(\beta+\frac{c_1kX_0}{2}\right)\widehat{X}+\frac{\beta}{2}\frac{c_1kX_0}{2}\widehat{X}^2}} \times \left(\frac{1+\frac{\beta q\widehat{X}}{(p+q-1)}}{1+\frac{\beta q\widehat{X}}{(p+q+1)}}\right)^{p/2}, \tag{5.2}$$

where

$$p = \frac{\beta}{\sqrt{\beta^2+\left(\frac{c_1kX_0}{2}\right)^2}}, \qquad q = \frac{\left(\frac{c_1kX_0}{2}\right)}{\sqrt{\beta^2+\left(\frac{c_1kX_0}{2}\right)^2}}. \tag{5.3}$$

The second term in (5.2) with power $p/2$ follows directly from the integral

$$\int_0^{\widehat{X}} \frac{d\widehat{X}}{4+(4\beta+2c_1kX_0)\widehat{X}+\beta c_1kX_0\widehat{X}^2}, \qquad \Delta = b^2-4ac = 16\beta^2+(2c_1kX)^2 > 0. \tag{5.4}$$

There exists only one solution, because $\Delta > 0$ (cf. [7]).

We follow now a procedure proposed by SCOTT and FU [2]. For a triangular initial wave profile, the solution for (5.1) is

$$[e] = h\frac{1+\beta\widehat{X}}{\sqrt{1+\frac{c_1kX_0}{2}\widehat{X}\underbrace{+\frac{c_1kX_0}{2}\frac{\beta}{2}\widehat{X}^2}_{\text{shock fitting method}}}}. \tag{5.5}$$

It is easy to notice that both the last results are *equivalent only for small distances of travel* (singular surface method can deal only with local evolutionary behaviour) and arbitrary value of β, or *for long distances* and $\beta \ll 1$

(cf. Fig. 6a). For constant cross-sectional area, *on the contrary*, the perturbation method and the shock fitting method (for triangular profile) yield the same evolution laws for the shock amplitude.

$$[e] = \frac{h}{\sqrt{1 + \frac{c_1 k X_0}{2}\widehat{X}}} . \tag{5.6}$$

Let us now consider a different situation, when the cross-sectional area, contrary to (5.1), decreases according to (5.7)

$$A(\widehat{X}) = A(0)(1 + \beta\widehat{X})^4 . \tag{5.7}$$

Applying directly the formula (3.7) to (5.7) we obtain

$$Y_1\varepsilon = \frac{h}{(1+\beta\widehat{X})^2} \exp\left\{ -\frac{c_1 k}{4} \left(\int\limits_0^{\widehat{X}} \frac{d\widehat{X}}{1 + \frac{c_1 k X_0}{2}(1+\beta\widehat{X})\widehat{X}} \right) \right\} . \tag{5.8}$$

Taking $\beta = 0$ in (5.8) we can obtain the result (5.6), but after integration the limits are *different*. Additionally we pay the attention to the fact that $c_1 k X_0/2$ is positive (cf. Sec. 5). The discriminant Δ can change its sign and three situations are possible (cf. [7]):

$$\text{(i)} \quad \Delta = b^2 - 4ac = \left(\frac{c_1 k X_0}{2}\right)^2 - 4\beta \frac{c_1 k X_0}{2} = 0 \;\Rightarrow\; \beta = \frac{1}{4}\frac{c_1 k X_0}{2} , \tag{5.9}$$

with the solution

$$Y_1\varepsilon = \frac{h}{(1+\beta\widehat{X})^2} \exp\left(\frac{1}{2\beta\widehat{X} + 1} - 1 \right) , \tag{5.10}$$

$$\text{(ii)} \quad \Delta > 0 \;\Rightarrow\; \frac{c_1 k X_0}{2} > 4\beta , \tag{5.11}$$

and the solution

$$Y_1\varepsilon = \frac{h}{(1+\beta\widehat{X})^2} \left(\sqrt{\frac{2\beta\widehat{X} + 1 + \psi}{2\beta\widehat{X} + 1 - \psi} \cdot \frac{1-\psi}{1+\psi}} \right)^{\psi} , \tag{5.12}$$

with

$$\psi = \sqrt{1 - \frac{4\beta}{c_1 k \widehat{X}_0/2}} .$$

The last case is connected with

$$(5.13) \qquad \text{(iii)} \qquad \Delta < 0 \quad \Rightarrow \quad \frac{c_1 k X_0}{2} < 4\beta,$$

and we obtain the solution in the following form:

$$(5.14) \qquad Y_1\varepsilon = \frac{h}{(1+\beta\widehat{X})^2} \exp\left(-\frac{1}{\sqrt{-\psi}}\left(\arctan\frac{2\widehat{X}+1}{\sqrt{-\psi}} - \arctan\frac{1}{\sqrt{-\psi}}\right)\right).$$

According to SCOTT and FU ([2]), for a triangular initial wave profile, there exists only one solution for (5.7), i.e.

$$(5.15) \qquad [e] = \frac{h}{(1+\beta\widehat{X})^2\sqrt{1+\dfrac{c_1 k X_0}{2}\cdot\dfrac{\widehat{X}}{(1+\beta\widehat{X})}}}$$

and it is valid for all the three above cases.

6. Numerical results and calculations for steel ([5])

The evolution solutions for the shock amplitude discussed in Sec. 5, are examined numerically for a certain kind of steel and for one value of the incident shock strength $h = 0.0017$. The elasticity constants of the first and second order were taken from [5]. The additional constans which we need here are (cf. (4.5))

$$(6.1) \qquad \eta = -6.3 < 0 \qquad \text{and} \qquad c_1 \cong -9.6 < 0.$$

The entropy condition must be satisfied; for this reason the only possible direction of the amplitude vector is $\mathbf{H} = [\mathbf{e}] = h(-1,0,0)$ (cf. [5]). The initial amplitude of the strain derivative $[e_X]$ can be calculated for the amplitude vector $[\mathbf{e}] = (-h,0,0)$ (cf. [2]) as

$$(6.2) \qquad k = \frac{hX_0}{\overline{U}_N T} = \frac{-21\cdot 10^7}{T}\ [\text{sec}],$$

where $h = -0.0125$, $\overline{U}_N = \sqrt{\dfrac{E^+}{\rho_R}} = 5852\dfrac{\text{m}}{\text{sec}}$ is the acoustic wave speed in the unstrained region behind the shock (Fig. 1), and T is the time of the pulse duration. For the Murnaghan material the value of the coefficient (6.3) below is always positive,

$$(6.3) \qquad \frac{c_1 k X_0}{2} = \frac{1.05\cdot 10^{-5}}{T}\ [\text{sec}].$$

The *modulated* simple wave solutions have been obtained under the assumption that the pulse on the boundary varies much faster than the cross-sectional area. This means that the cross-sectional area varies very slowly, and over one wavelength the change is almost negligible. Denoting by L the length scale for the area variation, we assume

$$\frac{T}{L}\sqrt{\frac{E^+}{\rho_R}} \approx O(h) \quad \Rightarrow \quad L \approx \frac{T}{h}\sqrt{\frac{E^+}{\rho_R}}, \tag{6.4}$$

and according to (5.1)

$$\frac{A(L)}{A(0)} = \frac{1}{\left(1+\beta\dfrac{T}{hX_0}\sqrt{\dfrac{E^+}{\rho_R}}\right)^2}. \tag{6.5}$$

Assume for example for steel $T = 10^{-5}$ sec, $\dfrac{c_1 k X_0}{2} = 1.05$, $L \approx 0.3442$ m, $\beta = 2$, $\lambda = \overline{U}_N T = 0.0585$ m, and that the cross-section varies rapidly in the range $\langle 0, L\rangle$ and $\dfrac{A(L)}{A(0)} \approx 2.05 \cdot 10^{-4}$. The cross-sectional area variation in the range of one wavelength remains, however, small:

$$\frac{A(\widehat{X}+\lambda)}{A(\widehat{X})} = \frac{1}{\left(1+\dfrac{\beta\lambda}{1+\beta\widehat{X}}\right)^2}, \quad \text{and} \quad \frac{A(0+\lambda)}{A(0)} \approx 0.81. \tag{6.6}$$

If β decreases, the cross-section area approaches a constants value. We will compare the results for values of the parameter $\beta \in \langle 0^+, 2\rangle$. We also notice that the limit strengths, when $\widehat{X} \to \infty$ for cross-section (5.1), are

$$\begin{aligned} \lim_{X\to\infty}[e]_{(5.5)} &= \frac{\sqrt{2\beta}}{\sqrt{\left(\dfrac{c_1 k X_0}{2}\right)}}, \\ \lim_{X\to\infty}[e]_{(5.2)} &= \frac{\sqrt{2\beta}}{\sqrt{\left(\dfrac{c_1 k X_0}{2}\right)}}\left(\frac{p+q+1}{p+q-1}\right)^{p/2}. \end{aligned} \tag{6.7}$$

Solutions obtained for values of the parameter $\beta \in \langle 0^+, 2\rangle$ and for the pulse duration $T = 10^{-5}$ sec are shown in Figs. 2, 3, 4 and 5. For the ratio

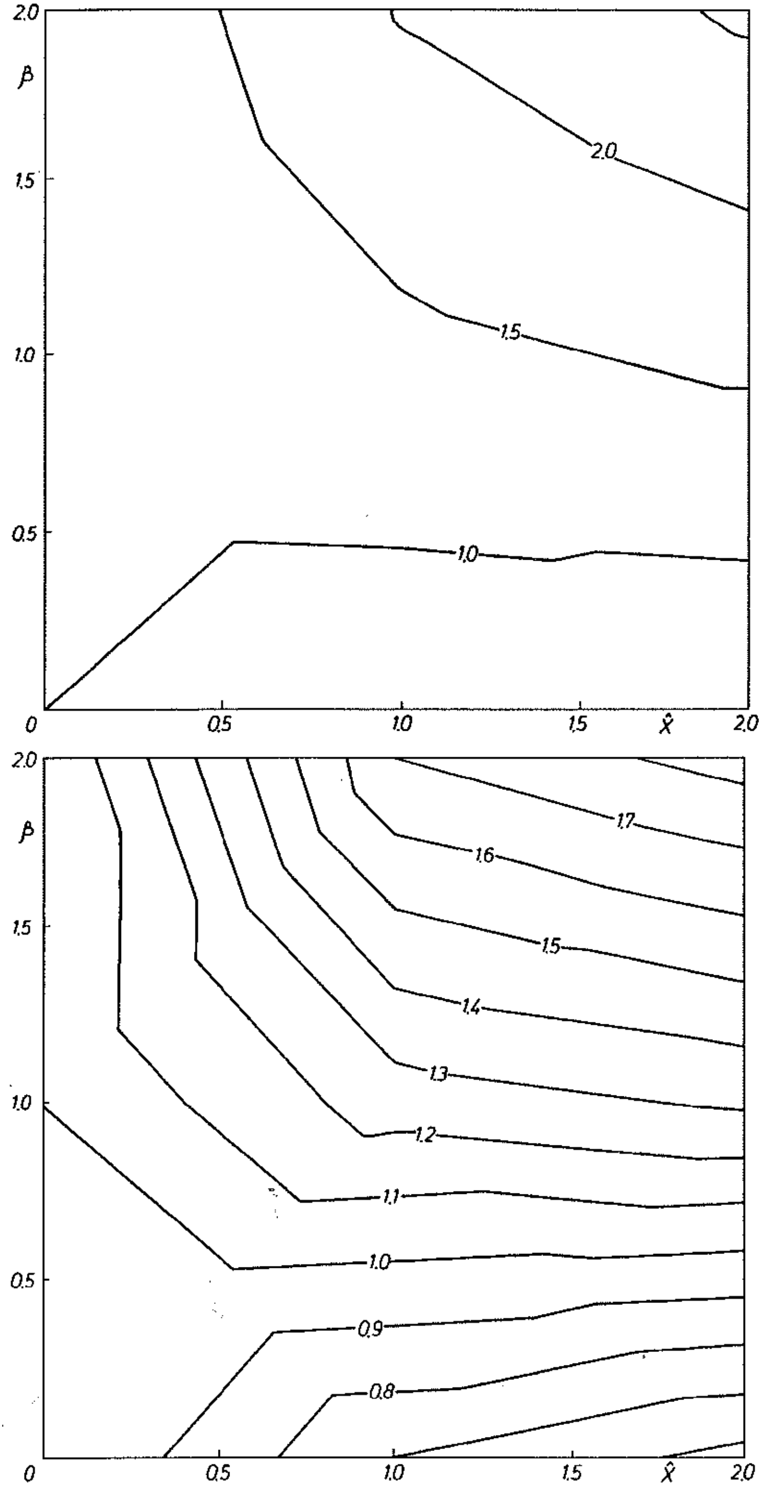


FIG. 2. Contour lines for (5.5) – (up) and for (5.2) – (down), $T = 10^{-5}$, $X = 2$.

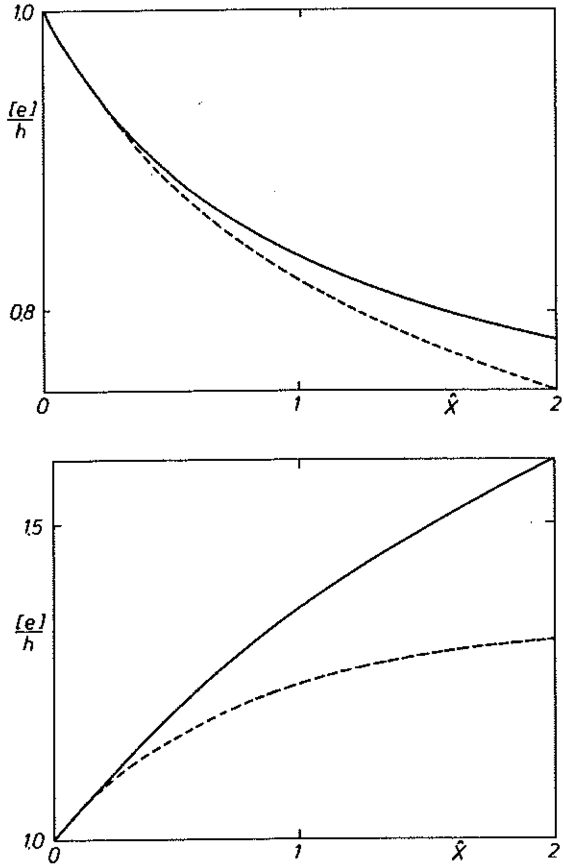
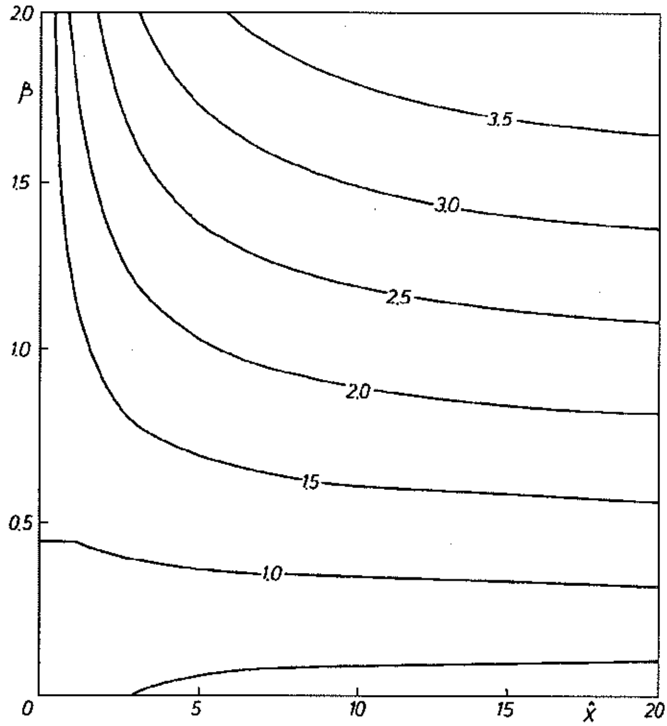


FIG. 3. Sections of the contour lines from Fig. 2, for $\beta = 0.2$ (up) and $\beta = 1$ (down), (5.2) $\cdots$, (5.5) —.

$\dfrac{c_1 k X_0}{2\beta} \geq 10$, (cf. Fig. 6a), both the limits (6.7) are practically equal and solutions (5.5) and (5.2) differ very little in the whole range of propagation. The pulse decreases in this case. For certain value of $\beta_0 \approx 0.5$ the amplitude of (5.5) is constant. For increasing pulse, and for values greater than β_0 there are differences in both the methods, but they depend also on $\dfrac{c_1 k X_0}{2\beta}$. We can see that for $\dfrac{c_1 k X_0}{2} = 1.05$ and for $\beta = 2$, the values obtained from (5.5) are two times greater than those obtained from (5.2).

The contour lines in Fig. 2 and Fig. 4 are practically parallel and horizontal. This means that the value of the amplitude quickly approaches its limiting value.



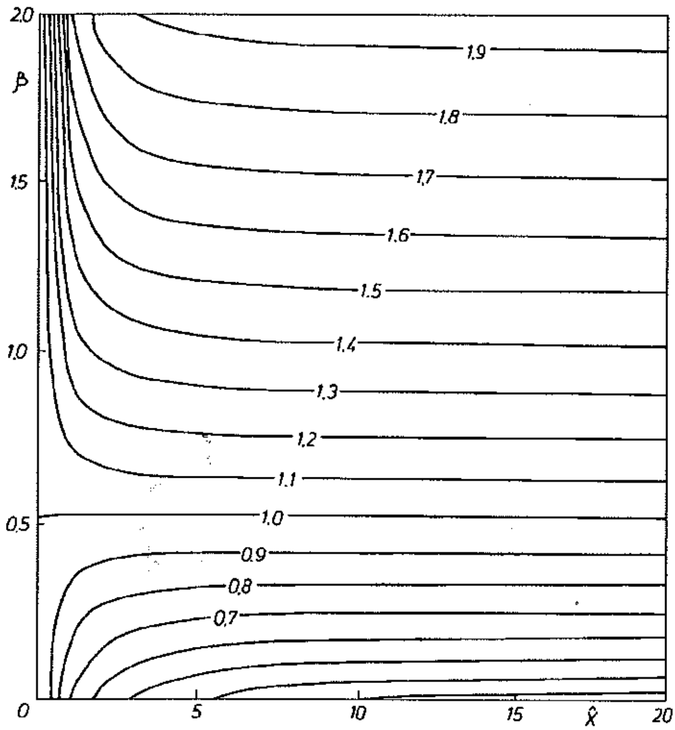


FIG. 4. Contour lines for (5.5) – (up) and for (5.2) – (down), $T = 10^{-5}$, $X = 20$.

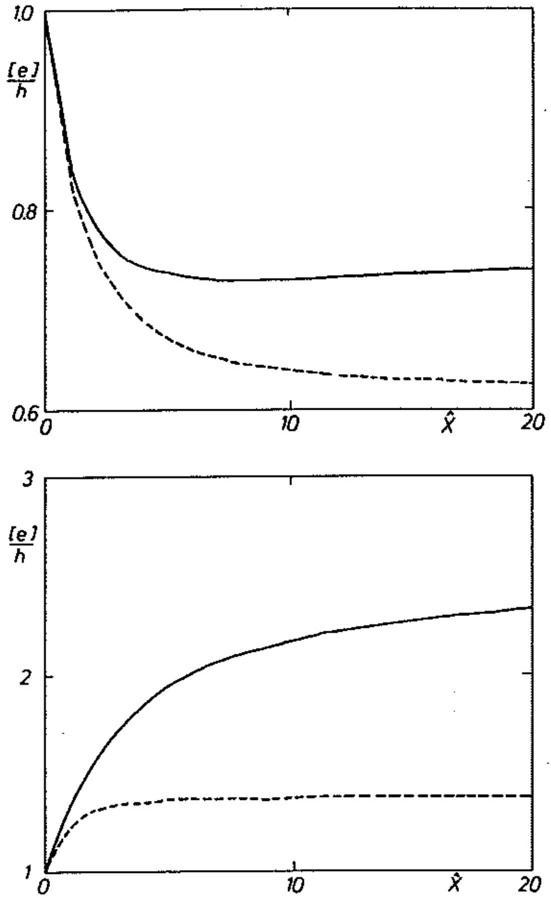


FIG. 5. Sections of the contour planes from Fig. 4, for $\beta = 0.2$ (up) and $\beta = 1$ (down), (5.2) $\cdots$, (5.5) —.

Figure 5 shows the diagram of the ratio κ of the two limits (6.7)

$$\kappa = \frac{\lim\limits_{X\to\infty}[e]_{(5.2)}}{\lim\limits_{X\to\infty}[e]_{(5.5)}} = \left(\frac{p+q+1}{p+q-1}\right)^{p/2}. \tag{6.8}$$

Figure 6 illustrates the comparison of the two methods. The ratio $[e]_{(5.2)}/[e]_{(5.5)}$ as a function of the parameter β and the distance $\widehat{X}$ is plotted in Fig. 6a. The ratio κ of the two limits (6.8) as a function of the parameter β is displayed in Fig. 6b.

We return now to the case of increasing cross-sectional area (5.7). There are three cases possible. We observe a good agreement of both methods for the case (i) (Fig. 7).

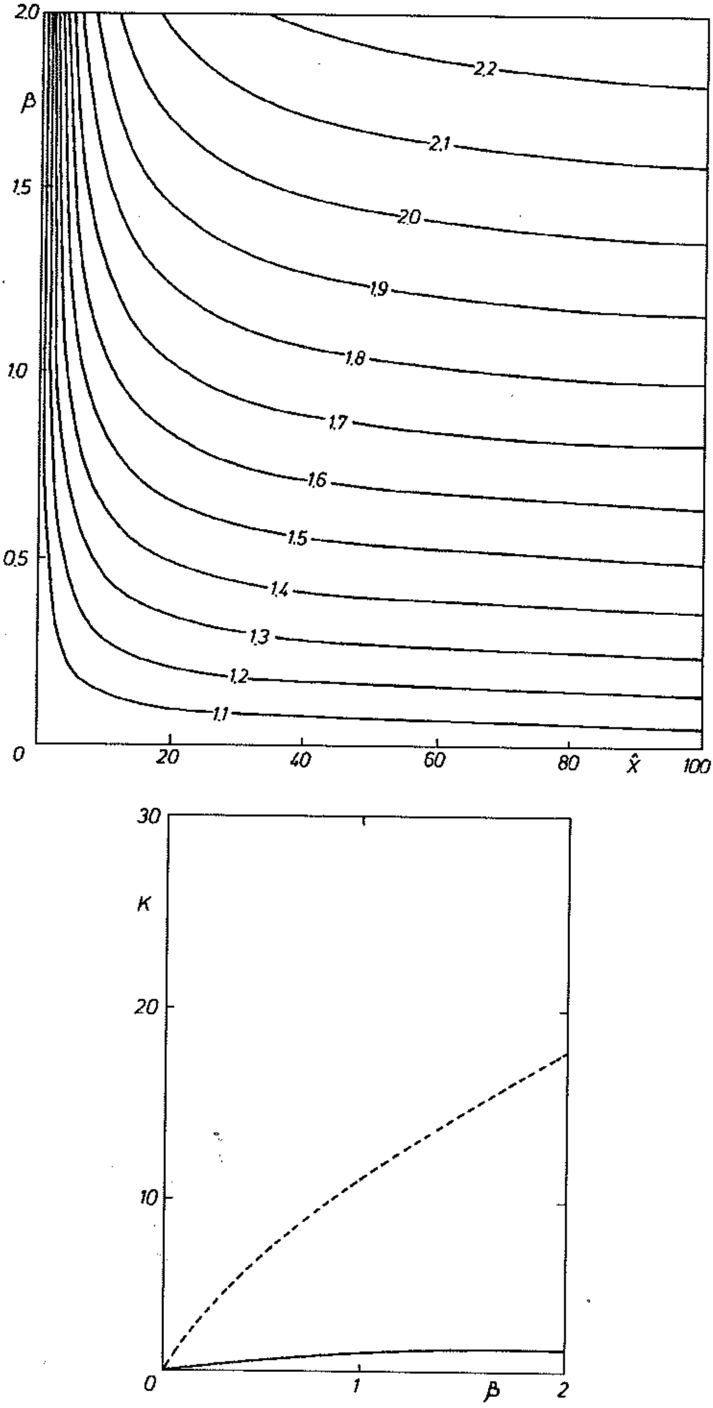


FIG. 6. Comparison of the results; for ratio of values $[e]_{(5.2)}/[e]_{(5.5)}$, ($T = 10^{-5}$) (up), κ for limits (6.8), ($T = 10^{-5}$ — and $T = 10^{-2}$ $\cdots$) (down).

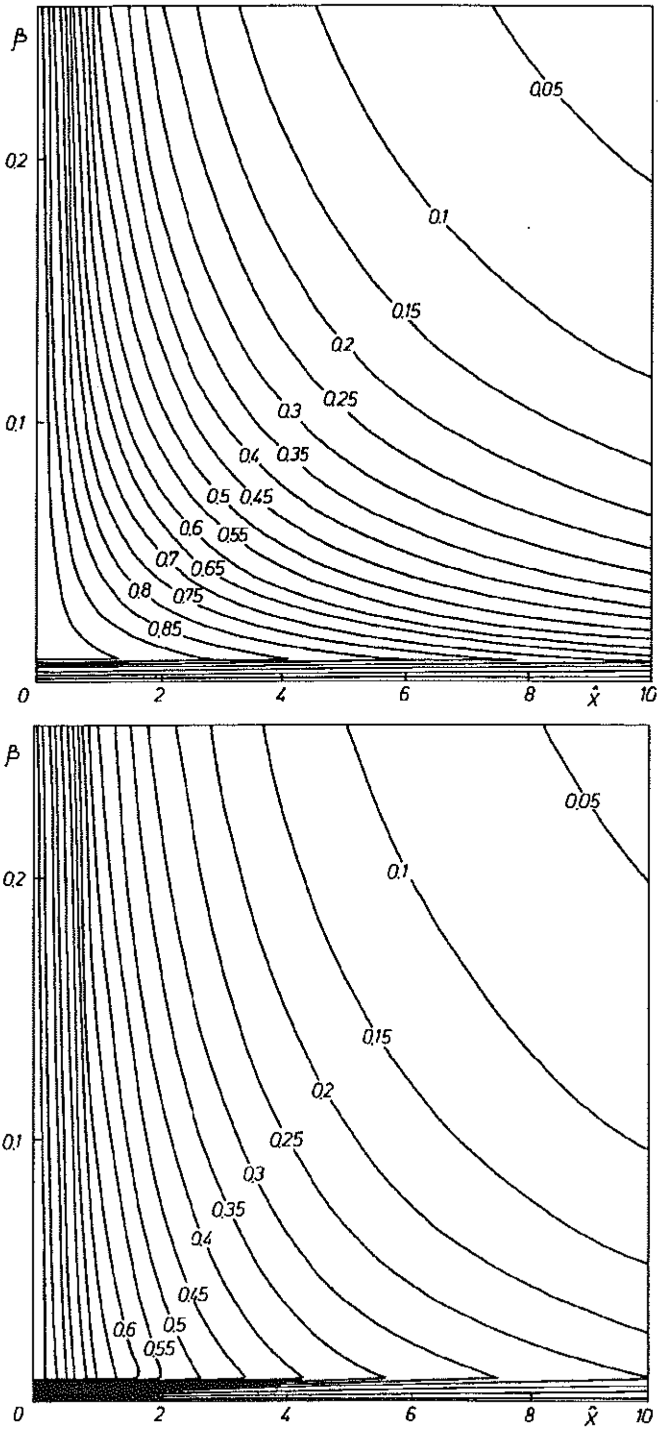
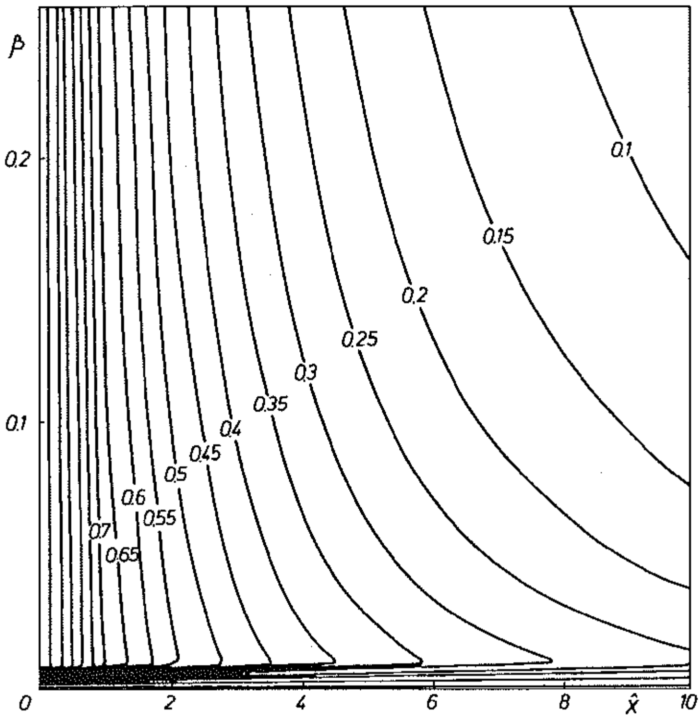


FIG. 7. $\Delta = 0$, $T \in \langle 1, 10^{-5} \rangle$. Contour lines for (5.10) – (up) and for (5.15) – (down).



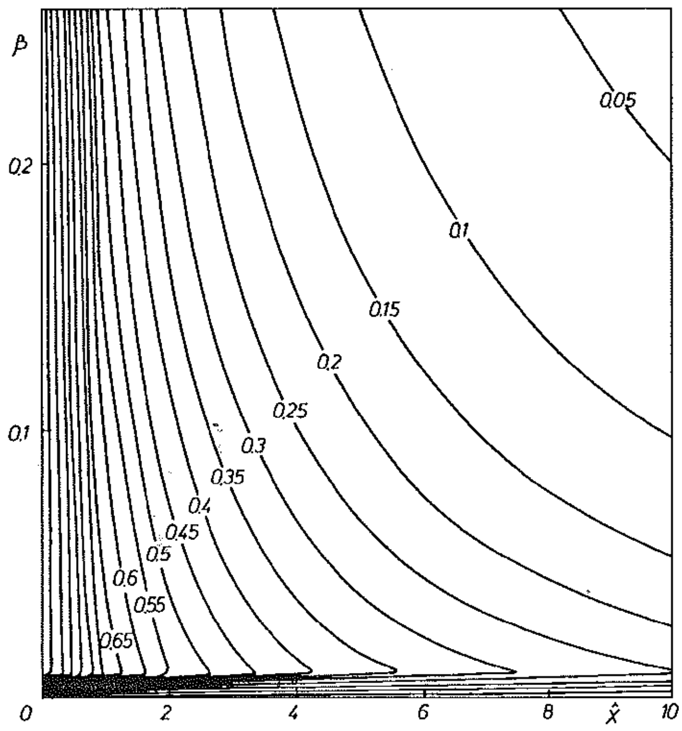


FIG. 8. $\Delta > 0$, $T = 10^{-5}$. Contour lines for (5.12) – (up) and for (5.15) – (down).

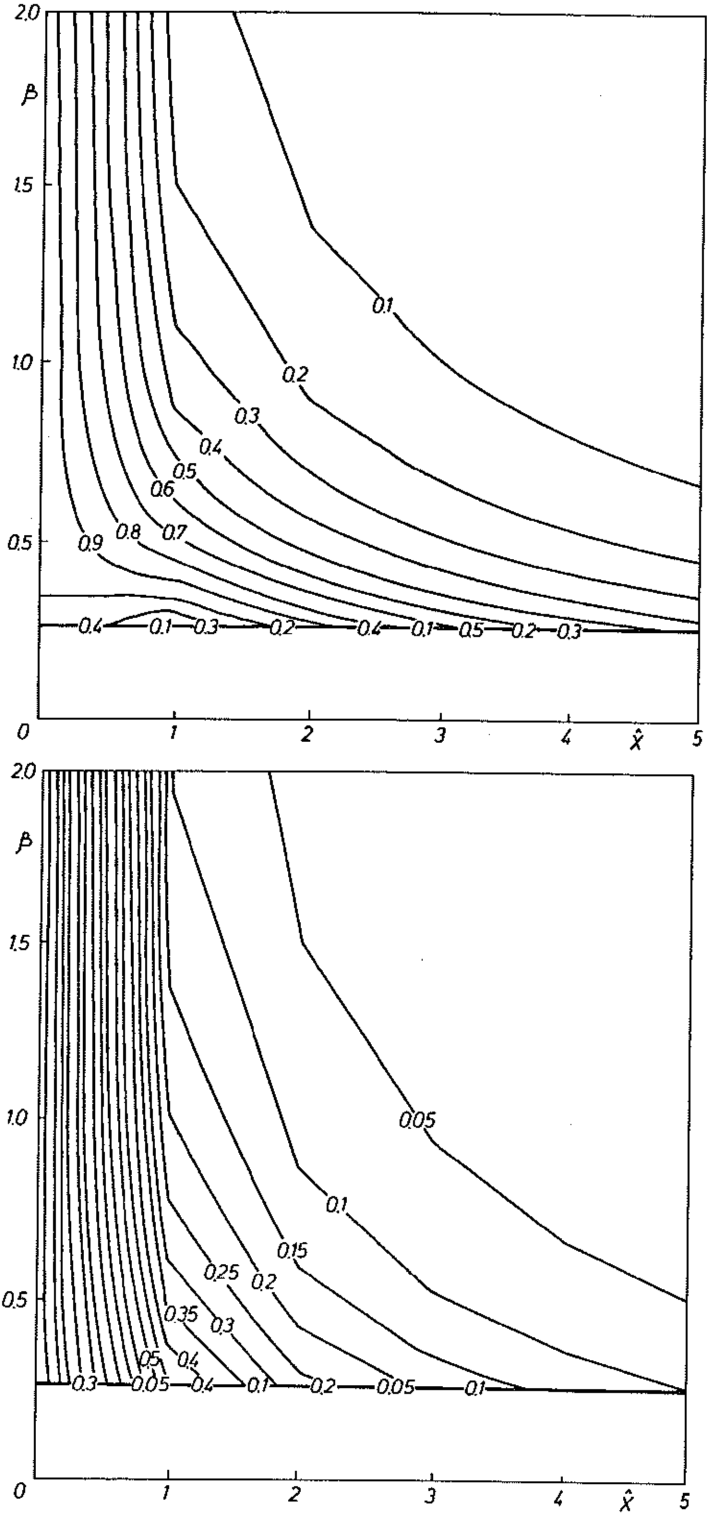


FIG. 9. $\Delta < 0$, $T = 10^{-5}$. Contour lines for (5.14) – (up) and for (5.15) – (down).

The contour lines are no longer horizontal or parallel and we observe that the pulse rapidly vanishes in all three cases. The general character of all three solutions obtained here for the increasing cross-sectional area (5.10), (5.12) and (5.14) is qualitatively different from that of the previous solutions for a decreasing cross-sectional area, since

$$\lim_{\beta\to 0}\left(\frac{h}{(1+\beta\widehat{X})^2}\exp\left\{-\frac{c_1k}{4}\int_0^{\widehat{X}}\frac{d\widehat{X}}{1+\frac{c_1kX_0}{2}(1+\beta\widehat{X})\widehat{X}}\right\}\right) \tag{6.9}$$
$$\neq \frac{h}{(1+\beta\widehat{X})^2}\exp\left\{-\frac{c_1k}{4}\int_0^{\widehat{X}}\lim_{\beta\to 0}\left(\frac{1}{1+\frac{c_1kX_0}{2}(1+\beta\widehat{X})\widehat{X}}\right)d\widehat{X}\right\}.$$

The right-hand side of the above equality and the solution of Eq. (5.15) have the same limit:

$$\lim_{\beta\to 0}\frac{h}{(1+\beta\widehat{X})^2\sqrt{1+\frac{c_1kX_0}{2}\cdot\frac{\widehat{X}}{(1+\beta\widehat{X})}}}=\frac{h}{\sqrt{1+\frac{c_1kX_0}{2}\widehat{X}}}. \tag{6.10}$$

The left-hand side of (6.9) has a *different* limit equal h in all three cases. The equality below which is satisfied for solutions obtained by SCOTT and FU ([2]) in the cases presented here, and for the perturbation solution in the case of (5.1), is not satisfied for increasing cross-sectional area (5.7).

$$\lim_{\beta\to 0}\int_0^{\widehat{X}} f(\widehat{X},\beta)\,d\widehat{X}=\int_0^{\widehat{X}}\lim_{\beta\to 0} f(\widehat{X},\beta)\,d\widehat{X}. \tag{6.11}$$

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