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INTERNAL AND COMBINATION RESONANCES IN A KINEMATICALLY EXCITED SYSTEM OF NON-PRISMATIC RODS (*)

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Internal and periodic combination resonances in a system of three coupled non-prismatic rods with articulated joints are analyzed. The resonances are of a parametric nature. The transverse vibrating system is placed on the vertically moving support. In the equation of motion two kinds of nonlinearities of geometrical nature appear. The considered problems may have a practical significance for the paraseismic phenomena when the weak excitation may cause great effects because of the autoparametric resonances.

1. Introduction

Vibrating nonlinear coupled systems are rich in many kinds of resonances (e.g. internal or combination resonances) [1]. Examples of the kinematically forced mechanical systems in which parametric or autoparametric resonances occur are given in the papers [2–7]. Appearance of autoparametric resonance is due to the coupling of elements of the system.

Usually the resonance phenomena in mechanical systems are undesirable. Hence, our aim is to avoid the resonance states or to minimize their disadvantageous effects. One of the methods is the optimal structural design which can maximize the frequency range without resonances. However, if such procedure does not eliminate the resonance phenomena, their effects should be minimized by minimization of some objective functions (measures of the phenomenon), more often the amplitudes of vibration in a steady state of resonance.

In this paper a parametric optimization of a plane system of non-prismatic, viscoelastic rods subjected to conditions of internal or combination resonance is considered. Transverse harmonic load acts on the horizontal rod and the system is placed on a vertically moving support. Such a system is

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an essential element of buildings and structures, e.g. the engine rooms. Because of couplings of the rods through periodically changing internal forces which are transverse forces at the ends of the neighbouring rods, the resonances have an autoparametric nature. The analysis of these problems in kinematically excited systems is important. Kinematic excitation of buildings and structures occurs in connection with seismic or paraseismic excitations. Sources of paraseismic vibration can be: motion of vehicles, running of machines, shootings in quarries. Vibrations of supports of buildings or structures produce a kinematic excitation. The response depends on the dynamic properties and character of the structure.

2. Equation of motion

Equations of motion for the system of three non-prismatic, visco-elastic (Kelvin – Voigt model) rods connected with articulated joints with concentrated masses M (Fig. 1) are derived in [7, 9]. Transverse harmonic load

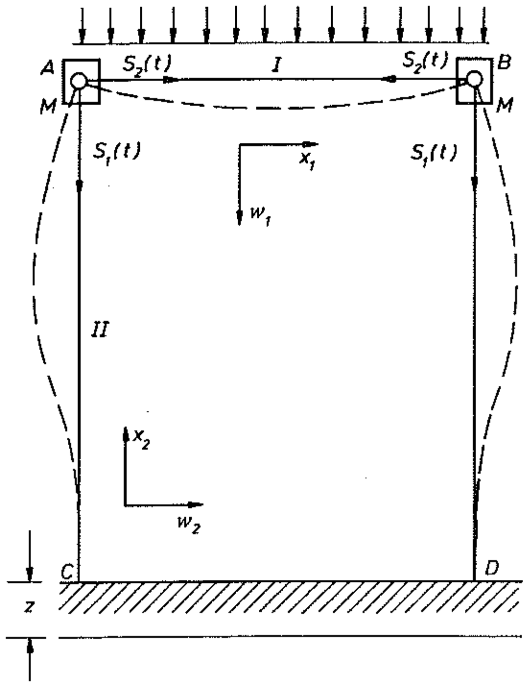


FIG. 1. The model of vibrating system.

$q(t) = \gamma_1 \sin \omega t$ acts on the horizontal element and the system is placed on a vertically moving support, its motion being represented by $z(t) = \gamma \sin \omega t$.

Symmetric steady vibration of this system in the state of internal or combination resonance is considered. The internal coupling forces are longitudinal forces of the neighbouring rods. The equations of motion are obtained by means of the Lagrange equations of the second kind:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{T}_i}\right) - \frac{\partial L}{\partial T_i} = Q_i + Q_{S_i} + Q_{\Delta S_i} + Q_{D_i}, \qquad i = 1, 2, \tag{2.1}$$

where L is a Lagrangian of the system, Q_i are generalized forces connected with external loadings and kinematic excitation, Q_{S_i} are generalized forces connected with internal coupling forces $S_i(t)$ (Fig. 1), $Q_{\Delta S_i}$ are generalized forces connected with nonlinear damping and nonlinear inertia which are taken into consideration in our description, Q_{D_i} are generalized forces connected with internal dissipation forces. If we assume that the transverse displacements of rods are

$$w_i(x_i, t) = Y_i(x_i) T_i(t), \qquad (i = 1, 2), \tag{2.2}$$

we obtain on the basis of [7, 8, 9] the system of differential equations in the form

$$\begin{aligned} \ddot{T}_1 + (B/A) T_1 &= (C/A) T_2 T_1 - (D/A)\, \dot{T}_1 - (E/A)\, \dot{T}_1\, T_1^2 \\ &\quad - ((F + F_M)/A)(T_1\, \dot{T}_1^2 + T_1^2\, \ddot{T}_1) - (G/A)\, \ddot{z} + (\Gamma_1/A) \sin \omega t, \\ \ddot{T}_2 + (\overline{B}/\overline{A}) T_2 &= -(\overline{C}/\overline{A}) T_1 T_2 - (\overline{D}/\overline{A})\, \dot{T}_2 - (\overline{E}/\overline{A})\, \dot{T}_2\, T_2^2 \\ &\quad - ((\overline{F} + \overline{F}_M)/\overline{A})(T_2\, \dot{T}_2^2 + T_2\, \ddot{T}_2) - (\overline{G}/\overline{A})\, \ddot{z}\, T_2. \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} A &= \int_0^{l_1} m_1(x_1) Y_1^2(x_1)\, dx_1, & \overline{A} &= \int_0^{l_2} m_2(x_2) Y_2^2(x_2)\, dx_2, \\ B &= E_1 \int_0^{l_1} I_1(x_1) [Y_1''(x_1)]^2\, dx_1, & \overline{B} &= E_2 \int_0^{l_2} I_2(x_2) [Y_2''(x_2)]^2\, dx_2, \\ D &= \eta_1 \int_0^{l_1} I_1(x_1) Y_1''^2(x_1)\, dx_1, & \overline{D} &= \eta_2 \int_0^{l_2} I_2(x_2) Y_2''^2(x_2)\, dx_2, \end{aligned} \tag{2.4}$$

$$C = \frac{\partial}{\partial x_2} \left[E_2 I_2(x_2) Y_2''(x_2)\right]_{x_2 = l_2} \int_0^{l_1} Y_1'^2(x_1)\, dx_1,$$

$$\overline{C} = \frac{\partial}{\partial x_1} \left[E_1 I_1(x_1) Y_1''(x_1)\right]_{x_1 = 0} \int_0^{l_2} Y_2'^2(x_2)\, dx_2,$$

(2.4) [cont.]

$$E = k\left\{\int_0^{l_1}[Y_1(x_1)]^2\,dx_1\right\}^2, \qquad \overline{E} = k\left\{\int_0^{l_2}[Y_2(x_2)]^2\,dx_2\right\}^2,$$

$$F = \int_0^{l_1} m_1(x_1)\left[\int_{x_1}^{l_1/2} Y_1'^2(\xi)\,d\xi\right]^2 dx_1,$$

$$\overline{F} = \int_0^{l_2} m_2(x_2)\left[\int_0^{x_2} Y_2'^2(\xi)\,d\xi\right]^2 dx_2,$$

$$F_M = M\left[\int_0^{l_1} Y_1'^2(x_1)\,dx_1\right]^2, \qquad \overline{F}_M = M\left[\int_0^{l_2} Y_2'^2(x_2)\,dx_2\right]^2,$$

$$G = \int_0^{l_1} m_1(x)Y_1(x)\,dx,$$

$$\overline{G}_1 = \int_0^{l_2} m_2(x)\,dx\int_0^{x}\left\{\frac{\partial Y_2}{\partial \xi}\right\}^2 d\xi,$$

$$\Gamma_1 = \int_0^{l_1} \gamma_1 Y_1(x_1)\,dx_1, \qquad \overline{G}_2 = 21.448\frac{M}{l_2^2}, \qquad \overline{G} = \overline{G}_1 + \overline{G}_2.$$

Here the following notations have been adopted ($i = 1, 2$): l_i are the lengths of the rods, m_i are the linear mass densities, E_i are the Young's moduli, I_i are the cross-sectional moments of inertia, η_i are the coefficients of internal damping, k is the coefficient of nonlinear damping. The coefficients G and $\overline{G}$ are connected with forces generated by harmonic vertical displacement of the support.

The coefficients (2.4) are functions of the shape parameters. The rods are of square cross-sections. In this paper we assume that the transverse dimension a_1 of element I varies as a quadratic function of x_1 (Fig. 2):

$$a_1(x_1) = \alpha_1\phi_1(x_1, \kappa_1), \qquad \phi_1(x_1, \kappa_1) = 4\kappa_1\left(\frac{x_1^2}{l_1^2} - \frac{x_1}{l_1}\right) + 1,$$

(2.5)
$$\kappa_1 = \frac{\alpha_1 - \beta_1}{\alpha_1}, \qquad a_1(0) = a_1(l_1) = \alpha_1,$$
$$a_1(l_1/2) = \beta_1, \qquad \kappa_1 \in (-\infty, 1].$$

We also assume that the transverse dimension a_2 of element II varies as a

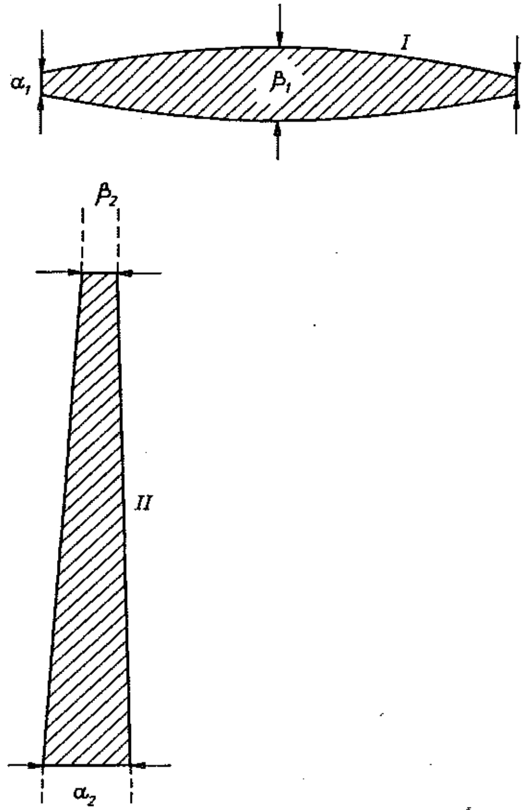


FIG. 2. The shape of elements I and II.

linear function of x_2 (Fig. 2):

$$(2.6)\quad \begin{gathered} a_2(x_2) = \alpha_2\phi_2(x_2,\kappa_2), \qquad \phi_2(x_2,\kappa_2) = 1 - \kappa_2\frac{x_2}{l_2}, \\ \kappa_2 = \frac{\alpha_2-\beta_2}{\alpha_2}, \quad a_2(0) = \alpha_2, \quad a_2(l_2) = \beta_2, \quad \kappa_2 \in (-\infty, 1]. \end{gathered}$$

Because of the assumptions that the coupling has a small influence on the modes of transverse vibrations, these modes are obtained as solutions of partial differential equations describing the transverse vibrations of separate rods. Solving the proper boundary value problems, one gets

$$(2.7)\quad \begin{aligned} Y_1(x_1) &= \sin(\pi x_1/l_1), \\ Y_2(x_2) &= \cos\lambda_1\Big\{sin(\lambda_1 x_2/l_2) - \mathrm{sh}(\lambda_1 x_2/l_2) \\ &\qquad - \mathrm{tg}\,\lambda_1[cos(\lambda_1 x_2/l_2) - \mathrm{ch}(\lambda_1 x_2/l_2)]\Big\}. \end{aligned}$$

Therefore in virtue of (2.4), (2.5), (2.6) and (2.7), we obtain

$$
\begin{aligned}
A(\alpha_1,\kappa_1) &= \rho_1 l_1 \alpha_1^2 f_A(\kappa_1), \qquad \overline{A} = \rho_2 l_2 \alpha_2^2 f_{\overline{A}}(\kappa_2),\\
B(\alpha_1,\kappa_1) &= \frac{E_1\alpha_1^4}{l_1^3} f_B(\kappa_1), \qquad \overline{B} = \frac{E_2\alpha_2^4}{l_1^3} f_{\overline{B}}(\kappa_2),\\
D(\alpha_1,\kappa_1) &= \frac{\eta_1\alpha_1^4}{l_1^3} f_B(\kappa_1), \qquad \overline{D} = \frac{\eta_2\alpha_2^4}{l_2^3} f_{\overline{B}}(\kappa_2),\\
C(\alpha_2) &= \frac{E_2\alpha_2^4}{l_1 l_2^3} f_C\,, \qquad \overline{C} = \frac{E_1\alpha_1^4}{l_2 l_1^3} f_{\overline{C}}\,,\\
E &= \frac{k}{l_1^2} f_E\,, \qquad \overline{E} = \frac{k}{l_2^2} f_{\overline{E}}\,, \qquad \Gamma_1 = 2\gamma_1 l_1/\pi\,,\\
F &= \frac{\Pi^4 \rho_1 \alpha_1^2}{4 l_1}(0.04534 - 0.02324\kappa_1 + 0.004652\kappa_1^2),\\
\overline{F} &= \frac{\rho_2 \alpha_2^2 Z_4^2}{l_2}(0.11089 - 0.17116\kappa_2 + 0.070627\kappa_2^2),\\
Z_4 &= \lambda_1^2 \cos^2\lambda_1,\\
F_M &= 24.3523 M/l_1^2, \qquad \overline{F}_M = 33.087 M/l_2^2,\\
G &= l_1 \rho_1 \alpha_1^2 f_G\,, \qquad \overline{G}_1 = \frac{\rho_2\alpha_2}{l_2} f_{\overline{G}}\,,
\end{aligned}
\tag{2.8}
$$

where ρ_i $(i = 1,2)$ are the mass densities and

$$
\begin{aligned}
f_A(\kappa_1) &= 0.391\kappa_1^2 - 0.896\kappa_1 + 0.500,\\
f_B(\kappa_1) &= 2.701\kappa_1^4 - 11.64\kappa_1^3 + 19.01\kappa_1^2 - 14.12\kappa_1 + 4.058,\\
f_{\overline{A}}(\kappa_2) &= 0.1747\kappa_2^2 - 0.5680\kappa_2 + 0.4999,\\
f_{\overline{B}}(\kappa_2) &= 1.253\kappa_2^4 - 7.153\kappa_2^3 + 15.94\kappa_2^2 - 17.06\kappa_2 + 9.890,\\
f_D &= f_B, \qquad f_E = 24.35, \qquad f_G = 0.445\kappa_1^2 - 1.032\kappa_1 + 0.636,\\
f_{\overline{D}} &= f_{\overline{B}}, \qquad f_{\overline{E}} = 33.09, \qquad f_{\overline{G}} = 1.115\kappa_2^2 - 2.930\kappa_2 + 2.144,\\
f_C &= 24.2, \qquad f_{\overline{C}} = -14.4.
\end{aligned}
\tag{2.9}
$$

The next two parts of the paper concern the parametric optimization of the system represented by the model of two degrees of freedom in states of internal resonance, i.e.

$$
\omega \cong \omega_{0j}\,, \qquad \omega_{01} = \frac{m}{k}\omega_{02}, \qquad j = 1,2, \quad m,k = 1,2,3\ldots
$$

or in periodic combination resonance

$$
\omega \cong \frac{M\omega_{01} + N\omega_{02}}{n}, \qquad \omega_{01} = \frac{m}{k}\omega_{02}, \qquad N,M = \pm 1, \pm 2, \ldots\,,
$$

where ω_{0i} $(i = 1,2)$ are the natural frequencies of the rods.

The method of solution depends on the kind of resonance and is adopted from the general theory of quasi-harmonic systems with many degrees of freedom, cf. [1, 8, 10].

3. Internal resonance

We will analyse the special case of internal resonance; we assume that in steady state of internal resonance the following relations hold:

$$\omega \cong \omega_{01}, \qquad \omega_{01} = 2\omega_{02}.$$

The mathematical analysis of Eqs. (2.3) was accomplished by using the modified harmonic balance method, cf. [9].

The amplitude $R_1 = 2\sqrt{X_1}$ of the vibration of element I is given by the equation

$$aX_1^3 + bX_1^2 + cX_1 + d = 0, \tag{3.1}$$

where

$$\begin{aligned} a &= \frac{(F+F_M)^2}{A^2}(\omega_{01} - 3\omega)^2 + \left(\frac{E}{A}\right)^2, \\ b &= 2\frac{DE}{A^2} - 4\frac{F+F_M}{A}(\omega - \omega_{01})(\omega_{01} - 3\omega), \\ c &= 4(\omega_{01} - \omega)^2 + \left(\frac{D}{A}\right)^2, \\ d &= -\frac{1}{4\omega_{01}^2}\frac{1}{A^2}\left(G\omega^2\gamma + \Gamma_1\right)^2. \end{aligned} \tag{3.2}$$

Element II of the vibrating system is parametrically excited by the internal coupling force S_2. In the approximate balance method the coupling force S_1 is neglected. The amplitude $R_2 = 2\sqrt{X_2}$ is given by the following equation

$$aX_2^2 + bX_2 + c = 0, \tag{3.3}$$

where

$$\begin{aligned} a &= \frac{(\overline{F} + \overline{F}_M)^2}{\overline{A}^2}(\omega_{02} - 3\omega/2)^2 + \left(\frac{\overline{E}}{\overline{A}}\right)^2, \\ b &= \frac{\overline{F} + \overline{F}_M}{\overline{A}}(\omega_{02} - \omega/2)(\omega_{02} - 3\omega/2) + 2\frac{\overline{DE}}{\overline{A}^2}, \end{aligned} \tag{3.4}$$

(3.4) [cont.]
$$c = 4\left(\omega_{02} - \frac{\omega}{2}\right)^2 + \left(\frac{\overline{D}}{\overline{A}}\right)^2 - \frac{1}{\omega_{02}^2}\left\{\left(\frac{\overline{G}}{\overline{A}}\frac{\gamma}{2}\omega^2\right)^2 + \left(\frac{\overline{C}}{\overline{A}}\right)^2 A_1 A_1^* - \frac{\overline{G}\,\overline{C}\gamma}{\overline{A}^2}\frac{\omega^2}{\omega_{01}}\frac{(\omega_{01}-\omega)\left(\frac{G}{A}\omega^2\gamma + \frac{\Gamma_1}{A}\right)}{4(\omega_{01}-\omega)^2 + \left(\frac{D}{A}\right)^2\left(1 + \frac{E}{D}A_1A_1^*\right)^2}\right\}.$$

Amplitudes R_1 and R_2 are functions of the shape parameters. In the case of internal resonance, a pure main resonance occurs in element I, and an autoparametric resonance in element II.

4. Combination resonance

We analyse the particular steady state of a combination resonance. We assume that in this special case the following relations hold

$$\omega \cong 2\omega_{01} + \omega_{02}\,, \qquad \omega_{02} = 2\omega_{01}\,.$$

For the analysis, the Tondl method is adopted, cf. [10, 11]. First, the Eqs. (2.3) ($M = 0$, nonlinear damping only) are transformed to the quasi-harmonic form

$$\ddot{T}_s + \omega_{0s}^2 T_s = \mu F_s(\dot{T}_k, T_k) + q_s(\omega t), \qquad s = 1, 2, \tag{4.1}$$

where a small parameter $\mu = D_p^2/\sqrt{C_p\Gamma_1}\,A_p$ (subscript p denotes that the rod is prismatic), and

$$\begin{aligned}
F_1 &= c_1(\kappa_1, \alpha_1)T_2T_1 - d_1(\kappa_1, \alpha_1)\,\dot{T}_1 + e_1(\kappa_1, \alpha_1)\,\dot{T}_1\,T_1^2,\\
F_2 &= -c_2(\kappa_2, \alpha_2)T_1T_2 - d_2(\kappa_2, \alpha_2)\,\dot{T}_2 - e_2(\kappa_2, \alpha_2)\,\dot{T}_2\,T_2^2\\
&\qquad - g_2(\kappa_2, \alpha_2)\,\ddot{z}\,T_2,\\
q_1 &= \left(\frac{G}{A}\omega^2\gamma + \frac{\Gamma_1}{A}\right)\sin\omega t, \qquad q_2 = 0,
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
C/A &= \mu c_1(\kappa_1, \alpha_1), \qquad & \overline{C}/\overline{A} &= \mu c_2(\kappa_2, \alpha_2),\\
D/A &= \mu d_1(\kappa_1, \alpha_1), \qquad & \overline{D}/\overline{A} &= \mu d_2(\kappa_2, \alpha_2),\\
E/A &= \mu e_1(\kappa_1, \alpha_1), \qquad & \overline{E}/\overline{A} &= \mu e_2(\kappa_2, \alpha_2),\\
\overline{G}/\overline{A} &= \mu g_2(\kappa_2, \alpha_2).
\end{aligned}$$

We look for an approximate solution of (4.1) in the form

$$T_s = Y_s + Z_s\,, \qquad s = 1, 2, \tag{4.3}$$

where Y_s is the solution for $\mu = 0$, Z_s is the solution of the set of equations

$$\ddot{Z}_s + \omega_{0s}^2 Z_s = \mu F_s(\dot{Z}, Z, t), \qquad s = 1, 2. \tag{4.4}$$

So we have

$$Y_1 = \frac{\dfrac{G}{A}\omega^2\gamma + \dfrac{\Gamma_1}{A}}{\omega_{01}^2 - \omega^2}\sin\omega t, \qquad Y_2 = 0. \tag{4.5}$$

Next we put

$$Z_s = U_s + V_s, \qquad \dot{Z}_s = i\omega_{0s}(U_s - V_s), \quad s = 1, 2, \quad i = \sqrt{-1}\,, \tag{4.6}$$

and Eqs. (4.1) take the form

$$\begin{aligned} \dot{U}_s &= i\omega_{0s}U_s + \mu(1/2i\omega_{0s})F_s[i\omega_{0s}(U - V),\ (U + V),\ t], \\ \dot{V}_s &= -i\omega_{0s}V_s - \mu(1/2i\omega_{0s})F_s[i\omega_{0s}(U - V),\ (U + V),\ t]. \end{aligned} \tag{4.7}$$

If we introduce the following notations:

$$\omega_0 = (N/n)\omega_{01} + (M/n)\omega_{02}, \qquad \nu_s = \omega_{0s}/\omega_0\,,$$

where $\omega_{01}(\kappa_1, \alpha_1)$, $\omega_{02}(\kappa_2, \alpha_2)$ are the natural frequencies of non-prismatic elements and if we introduce dimensionless time $\tau = \omega t$ and take $\omega = \omega_0 + \mu q$ (q is a constant), Eqs. (4.7) take the form

$$\begin{aligned} U_s' &= i\nu_s U_s + \mu[-i(\nu_s/\omega_0)qU_s + (1/2i\omega_{0s}\omega_0)F_s] + \mu^2[\ldots] + \ldots, \\ V_s' &= -i\nu_s V_s - \mu[-i(\nu_s/\omega_0)qV_s + (1/2i\omega_{0s}\omega_0)F_s] + \mu^2[\ldots] + \ldots\,. \end{aligned} \tag{4.8}$$

A superscript $'$ denotes differentiation with respect to τ. Next, the new variables are defined

$$U_s = \alpha_s e^{i\nu_s\tau}, \qquad V_s = \beta_s e^{-i\nu_s\tau}, \qquad s = 1, 2, \tag{4.9}$$

and the equations (4.8) take the following form

$$\begin{aligned} \alpha_s' &= \mu g_s(\tau, \alpha, \beta, \mu) = \mu g_s^{(1)}(\tau, \alpha, \beta) + \mu^2 g_s^{(2)}(\tau, \alpha, \beta) + \ldots\,, \\ \beta_s' &= \mu h_s(\tau, \alpha, \beta, \mu) = \mu h_s^{(1)}(\tau, \alpha, \beta) + \mu^2 h_s^{(2)}(\tau, \alpha, \beta) + \ldots\,, \end{aligned} \tag{4.10}$$

where

$$
(4.11)\qquad
\begin{aligned}
g_1^{(1)} &= -i(\nu_1/\omega_0)q\alpha_1 + (e^{-i\nu_1\tau}/2i\nu_1\omega_0^2)F_1,\\
g_2^{(1)} &= -i(\nu_2/\omega_0)q\alpha_2 + (e^{-i\nu_2\tau}/2i\nu_2\omega_0^2)F_2,\\
h_1^{(1)} &= i(\nu_1/\omega_0)q\beta_1 - (e^{i\nu_1\tau}/2i\nu_1\omega_0^2)F_1,\\
h_2^{(1)} &= i(\nu_2/\omega_0)q\beta_2 - (e^{i\nu_2\tau}/2i\nu_2\omega_0^2)F_2.
\end{aligned}
$$

After these transformations we get the following form of functions F_1, F_2:

$$
\begin{aligned}
F_1 = c_1(\kappa_1,\alpha_1)&(Y_2 + \alpha_2 e^{i\nu_2\tau} + \beta_2 e^{-i\nu_2\tau})(Y_1 + \alpha_1 e^{i\nu_1\tau} + \beta_1 e^{-i\nu_1\tau})\\
&- d_1(\kappa_1,\alpha_1)\left[\dot{Y}_1 + i\omega_{01}(\alpha_1 e^{i\nu_1\tau} - \beta_1 e^{-i\nu_1\tau})\right]\\
&- e_1(\kappa_1,\alpha_1)\left[\dot{Y}_1 + i\omega_{01}(\alpha_1 e^{i\nu_1\tau} - \beta_1 e^{-i\nu_1\tau})\right]\\
&\times\left[Y_1 + (\alpha_1 e^{i\nu_1\tau} + \beta_1 e^{-i\nu_1\tau})\right]^2,
\end{aligned}
$$

$$
(4.12)\qquad
\begin{aligned}
F_2 = -c_2(\kappa_2,\alpha_2)&(Y_1 + \alpha_1 e^{i\nu_1\tau} + \beta_1 e^{-i\nu_1\tau})(Y_2 + \alpha_2 e^{i\nu_2\tau} + \beta_2 e^{-i\nu_2\tau})\\
&- d_2\left[\dot{Y}_2 + i\omega_{02}(\alpha_2 e^{i\nu_2\tau} - \beta_2 e^{-i\nu_2\tau})\right]\\
-e_2\left[\dot{Y}_2 + i\omega_{02}(\alpha_2 e^{i\nu_2\tau} - \beta_2 e^{-i\nu_2\tau})\right]&\left[Y_2 + \alpha_2 e_2^{i\nu_2\tau} + \beta_2 e^{-i\nu_2\tau}\right]^2\\
&- g_2(Y_2 + \alpha_2 e^{i\nu_2\tau} + \beta_2 e^{-i\nu_2\tau})\omega_0^2\gamma\sin\tau,
\end{aligned}
$$

where Y_1, Y_2 are given by (4.5).

The Bogolubov–Krylov transformation is applied to Eqs. (4.10)

$$
(4.13)\qquad
\begin{aligned}
\alpha_s &= A_s + \mu\chi_{\alpha s}^{(1)}(\tau, A_1, A_2, B_1, B_2),\\
\beta_s &= B_s + \mu\chi_{\beta s}^{(1)}(\tau, A_1, A_2, B_1, B_2), \qquad s = 1,2,
\end{aligned}
$$

and the first approximation is

$$
(4.14)\qquad A_s' = \mu G_s(A,B), \qquad B_s' = \mu H_s(A,B), \qquad s = 1,2.
$$

Since we are interested in a steady state of combination resonance, we get the following set of equations for the amplitudes of vibration

$$
(4.15)\qquad G_s(A_1^0, A_2^0, B_1^0, B_2^0) = 0, \qquad H_s(A_1^0, A_2^0, B_1^0, B_2^0) = 0, \quad s = 1,2.
$$

So, the functions g_s and h_s $(s = 1,2)$ take the form

$$
(4.16)\qquad
\begin{aligned}
g_1 &= -i(1/4\omega_0)(q - q_0)A_1 + (2e^{-i\,1/4\,\tau}/i\omega_0^2)F_1,\\
h_1 &= i(1/4\omega_0)(q - q_0)B_1 - (2e^{i\,1/4\,\tau}/i\omega_0^2)F_1,\\
g_2 &= -i(1/2\omega_0)(q + q_0)A_2 + (e^{-i\,1/2\,\tau}/i\omega_0^2)F_2,\\
h_2 &= i(1/2\omega_0)(q + q_0)B_2 - (e^{i\,1/2\,\tau}/i\omega_0^2)F_2,
\end{aligned}
$$

where (cf. [11])

(4.17)
$$
\begin{aligned}
G_1 &= \frac{1}{8\pi}\int_0^{8\pi} g_1^{(1)}(\tau, A_1, A_2, B_1, B_2)\,d\tau,\\
G_2 &= \frac{1}{4\pi}\int_0^{8\pi} g_2^{(1)}(\tau, A_1, A_2, B_1, B_2)\,d\tau,\\
H_1 &= \frac{1}{8\pi}\int_0^{8\pi} h_1^{(1)}(\tau, A_1, A_2, B_1, B_2)\,d\tau,\\
H_2 &= \frac{1}{4\pi}\int_0^{8\pi} h_2^{(1)}(\tau, A_1, A_2, B_1, B_2)\,d\tau.
\end{aligned}
$$

Inserting (4.16) to (4.17) and calculating the integrals, we get for the coefficients

$$A_1(\alpha_i, \kappa_i), \quad B_1(\alpha_i, \kappa_i), \quad A_2(\alpha_i, \kappa_i), \quad B_2(\alpha_i, \kappa_i),$$

the following set of algebraic equations

(4.18)
$$
\begin{aligned}
&-i(1/8\omega_{02})(q-q_0)A_1 + (1/2i\omega_{02}^2)\{c_1A_2B_1 - id_1\omega_{01}A_1\\
&\qquad\qquad +2i\omega_{01}e_1A_1Z^2 - i\omega_{01}e_1A_1^2B_1\} = 0,\\
&i(1/8\omega_{02})(q-q_0)B_1 - (1/2\omega_{02}^2)\{c_1B_2A_1 + id_1\omega_{01}B_1\\
&\qquad\qquad -2i\omega_{01}e_1B_1Z^2 + i\omega_{01}e_1A_1B_1^2\} = 0,\\
&-i(1/4\omega_{02})(q+q_0)A_2 + (1/4i\omega_{02}^2)\{-c_2ZB_2 - id_2\omega_{02}A_2\\
&\qquad\qquad -i\omega_{02}e_2A_2^2B_2 - ig_2B_2\gamma\} = 0,\\
&i(1/4\omega_{02})(q+q_0)B_2 - (1/4i\omega_{02}^2)\{c_2ZA_2 + id_2\omega_{02}B_2\\
&\qquad\qquad +i\omega_{02}e_2A_2B_2^2 + ig_2A_2\gamma\} = 0,
\end{aligned}
$$

where

(4.19)
$$Z = (1/2i)\frac{\dfrac{G}{A}\omega^2\gamma + \dfrac{\Gamma_1}{A}}{\omega_{01}^2 - \omega^2}.$$

After some transformations one gets from $(4.18)_{1,2}$ and $(4.18)_{3,4}$ the amplitude R_1 of element I and amplitude R_2 of element II, respectively, in the steady state of periodic combination resonance:

$$(4.20)\qquad \begin{aligned} R_1^2 &= 4\frac{D}{E}\left\{\frac{A}{D}\sqrt{\left(\frac{C}{A}\right)^2\frac{1}{\omega_{01}^2}X_2(\alpha_i,\kappa_i) - 4\left(\omega_{01}-\frac{\omega}{4}\right)^2} - 1\right\} \\ &\qquad - 2\frac{(G\omega^2\gamma + \Gamma_1)^2}{A^2(\omega_{01}^2-\omega^2)^2}, \\ R_2 &= 2\sqrt{X_2} \\ &= 2\sqrt{\frac{\overline{D}}{\overline{\overline{E}}}}\left\{\frac{\overline{A}}{\overline{\overline{D}}}\sqrt{\left(\frac{\overline{C}\overline{Z}}{i\omega_{02}\overline{A}} + \frac{4\omega_{02}\gamma\overline{G}}{\overline{A}}\right)^2 - 4\left(\frac{\omega}{2}-\omega_{02}\right)^2} - 1\right\}^{1/2}, \end{aligned}$$

where $\overline{Z} = \dfrac{\Gamma_1 + G\omega\gamma}{2iA(\omega_{01}^2-\omega^2)}$.

The amplitudes (4.20) are functions of the shape parameters. From relations holding for the combination resonance, one can infer that element II acts parametrically on element I, and the external loading is subharmonic for element I; hence, in element I two phenomena coexist. External harmonic and kinematic loadings are parametric for element II. The amplitude R_1 as well as the amplitude R_2 has a parametric character. The longitudinal forces S_1, S_2 and external loading play the role of parametric excitation.

5. Parametric optimization

The amplitudes R_1 and R_2 are the objective functions and the parameters defining the shape of rods are control parameters.

Internal resonance (two kinds of nonlinearities). The constraints are: the total mass of the system is constant, i.e. $2M + M_1 + 2M_2 = \mathcal{M} = \text{const}$, and the system is nearly the internal resonance. We look for such optimization parameters α_i, κ_i which satisfy the constraints and minimize the objective function – the amplitude R_2 of the parametrically excited element II. Thus,

$$R_{2\text{opt}} = \min R_2[\kappa_2, \alpha_2(\kappa_2), \kappa_1(\kappa_2), \alpha_1],$$

$$2M + \rho_1 l_1 \alpha_1^2 f_1(\kappa_1) + 2\rho_2 l_2 \alpha_2^2 f_2(\kappa_2) = \text{const}, \qquad \omega_{01} = 2\omega_{02}, \quad \omega \cong \omega_{01}.$$

Combination resonance (without the concentrated masses in articulated joints, nonlinear damping only). The constraints are: the total mass of the system of rods is constant, i.e. $M_1 + 2M_2 = \mathcal{M} = \text{const}$ and the system is nearly the combination resonance. We look for such values of optimization parameters α_i, κ_i which define the shapes and satisfy the constraints and, in addition, minimize the objective function $\mathcal{Z} = |2c_1R_1| + |c_2R_2|$. So

$$\mathcal{Z}_{\text{opt}} = \min \mathcal{Z}[\kappa_2, \alpha_2(\kappa_2), \kappa_1(\kappa_2), \alpha_1],$$

$$\rho_1 l_1 \alpha_1^2 f_1(\kappa_1) + 2\rho_2 l_2 \alpha_2^2 f_2(\kappa_2) = \text{const}\,, \qquad \omega_{02} = 2\omega_{01}\,, \quad \omega \cong 4\omega_{01}\,.$$

We carry on our calculations in the following way:

1. We fix $\kappa_2 \in (-2, 0, 0.8, 0.5, 1)$ and α_1 (parameter connected with internal coupling).

2. For fixed κ_2 we calculate α_2 on the ground of constraint $\mathcal{M} = \text{const}$.

3. The resonance condition $\omega_{02} = 2\omega_{01}$ takes the form

$$c_4(\kappa_2)\kappa_1^4 + c_3(\kappa_2)\kappa_1^3 + c_2(\kappa_2)\kappa_1^2 + c_1(\kappa_2)\kappa_1 + c_0 = 0,$$

and from this we calculate κ_1.

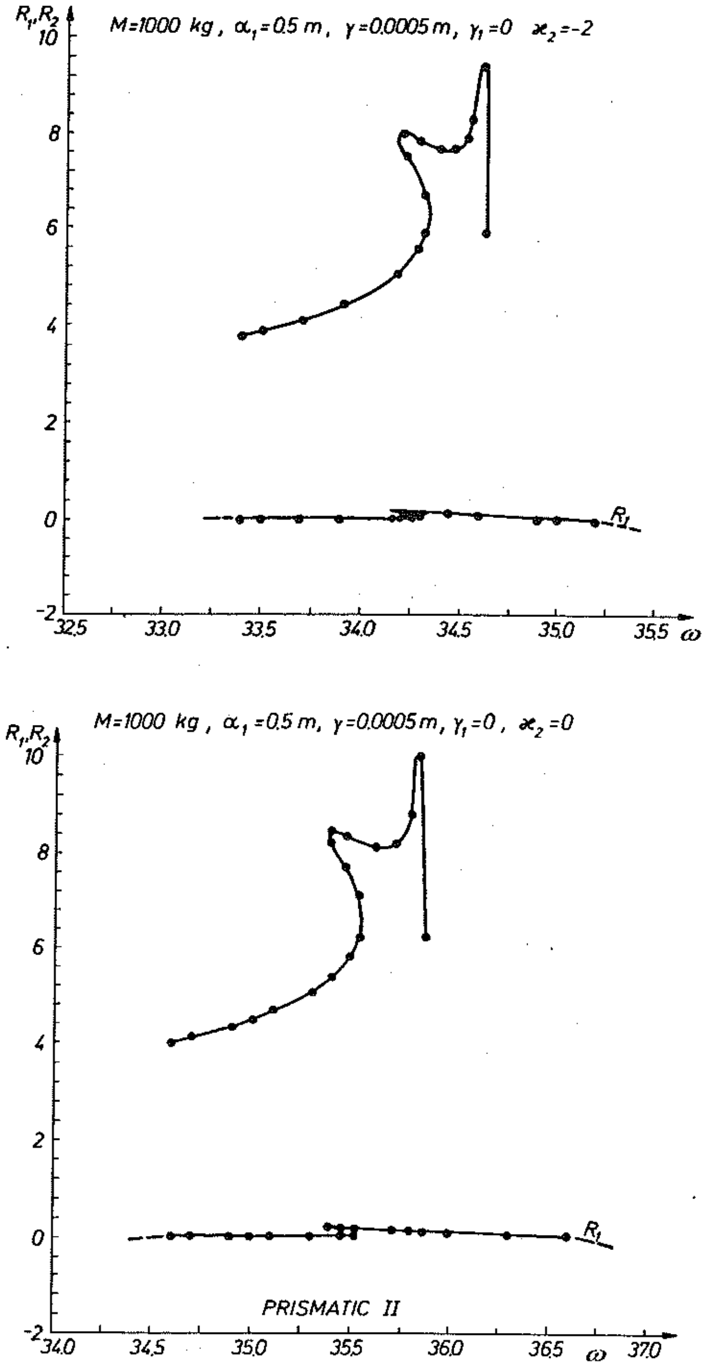
4. Finally we calculate R_1, R_2 and $\mathcal{Z}$ for frequencies ω from the neighbourhood of $4\omega_{01}$.

After numerical calculations the detailed conclusions will be determined.

6. Analysis of results and conclusions

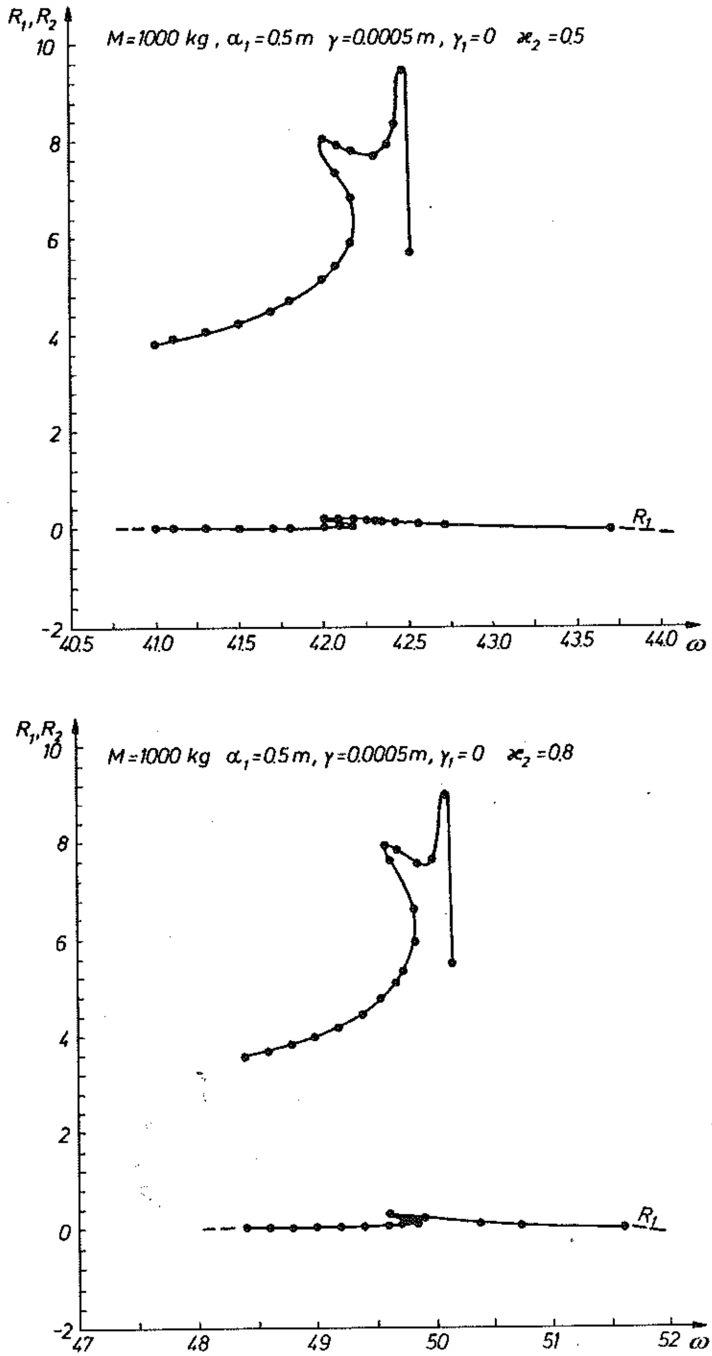
For calculations of amplitudes R_1, R_2 in internal resonance, the following numerical values of parameters are used: $l_1 = 8\,\text{m}$, $l_2 = 14\,\text{m}$, $E_1 = E_2 = 2.2 \cdot 10^{11}\,\text{N/m}^2$, $\rho_1 = \rho_2 = 7.7 \cdot 10^3\,\text{kg/m}^3$, $\eta_{1,2} \in (1 \cdot 10^6 - 1 \cdot 10^8)\ \text{Ns/m}^2$, $k \in (1 \cdot 10^5 - 1 \cdot 10^7)\,\text{kg/s}$, $\alpha_1 = 0.50\,\text{m}$, $M = 1000\,\text{kg}$, $\mathcal{M} = 10\,000\,\text{kg}$. The frequency ω is changed in the small region near ω_{01} (see resonance relations). The value of amplitude of kinematic excitation is $\gamma = 0.0005\,\text{m}$. In the Fig. 3 the amplitudes R_1 and R_2 as functions of ω and κ_2 (the parameter of the shape of element II) are presented. Amplitude of element II has explicitly a nonlinear character. For some values of frequencies three values of the amplitude exist. Not all amplitudes are stable but the stability is not analyzed in this paper. The resonance curve is deflected toward the lower frequencies – it is characteristic when nonlinear inertia dominates. The resonance curve has two maxima (cf. [1] p. 351 and [8]). The value of R_2 (parametrically exciting element) is minimum when the element II is shaped as a cone ($\kappa_2 = 1$).

For calculations of the amplitudes R_1, R_2 and $\mathcal{Z}$ in combination resonance, the following numerical values of parameters are used: $l_1 = 8\,\text{m}$,



R_1, R_2
$M = 1000$ kg, $\alpha_1 = 0.5$ m, $\gamma = 0.0005$ m, $\gamma_1 = 0$ $\varkappa_2 = -2$
10
8
6
4
2
0
-2
32.5
33.0
33.5
34.0
34.5
35.0
35.5
ω
R_1
R_1, R_2
$M = 1000$ kg, $\alpha_1 = 0.5$ m, $\gamma = 0.0005$ m, $\gamma_1 = 0$, $\varkappa_2 = 0$
10
8
6
4
2
0
-2
34.0
34.5
35.0
35.5
36.0
36.5
37.0
ω
R_1
PRISMATIC II

[FIG. 3]


R1,R2
M=1000 kg , α1=0.5 m γ=0.0005 m , γ1=0 æ2 =0.5
10
8
6
4
2
0
-2
R1
40.5
41.0
41.5
42.0
42.5
43.0
43.5
44.0
ω
R1,R2
M=1000 kg α1=0.5 m, γ=0.0005 m, γ1=0 æ2 =0.8
10
8
6
4
2
0
-2
R1
47
48
49
50
51
52
ω

[Fig. 3]

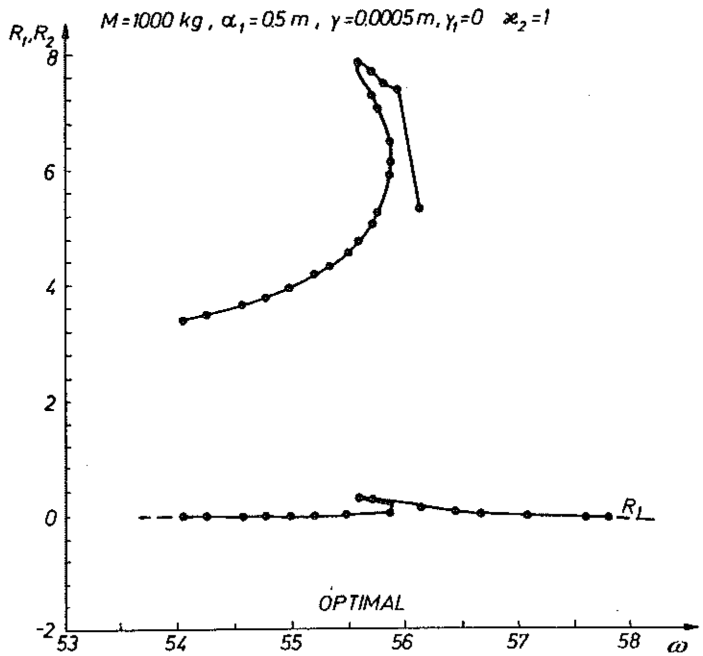
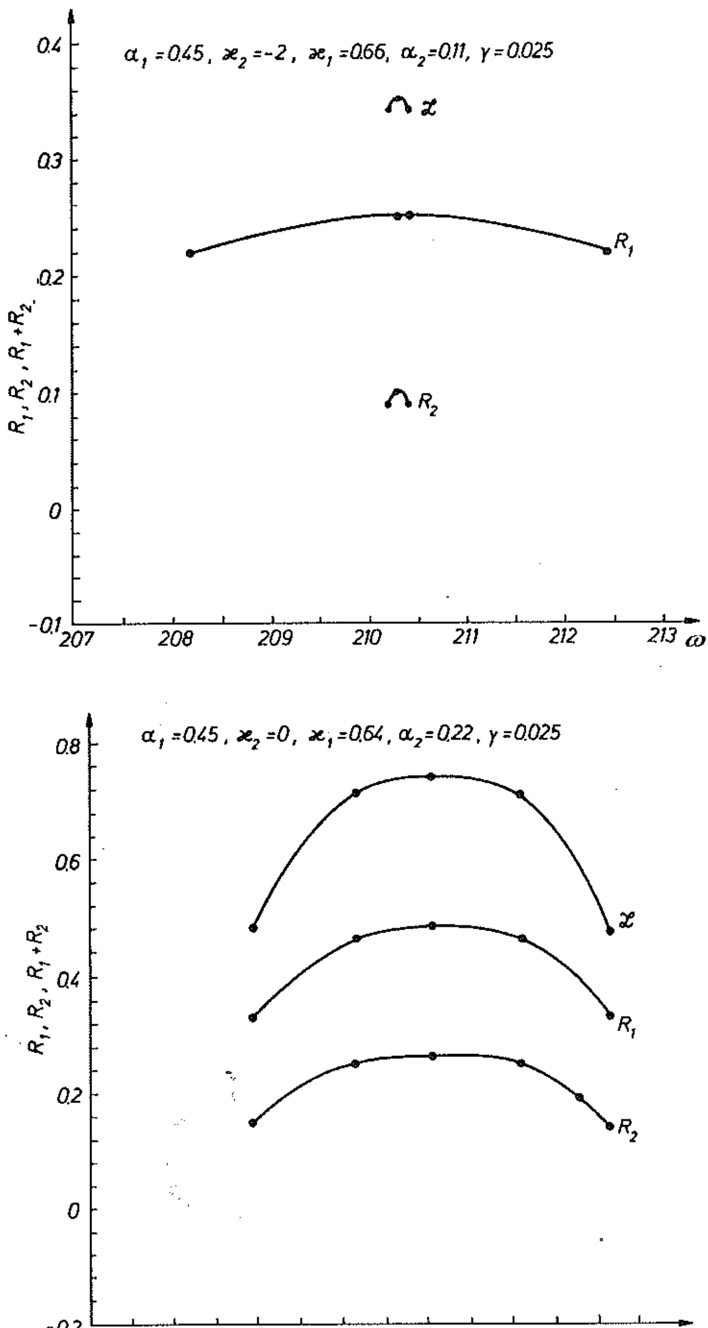


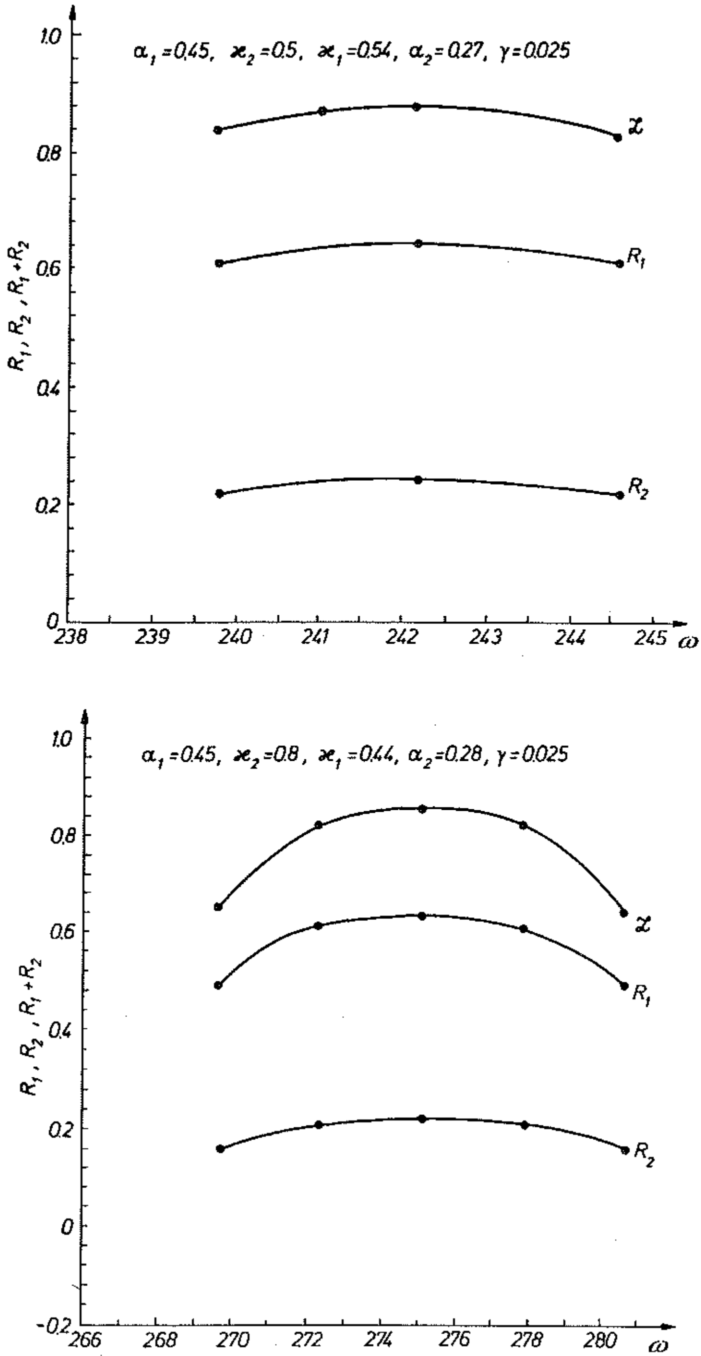
FIG. 3. Internal resonance. Amplitudes R_1, R_2 of non-prismatic elements versus circular frequency ω for different values of parameter κ_2.

$l_2 = 7.07\,\mathrm{m}$, $E_1 = E_2 = 2.2 \cdot 10^{11}\,\mathrm{N/m^2}$, $\rho_1 = \rho_2 = 7.7 \cdot 10^3\,\mathrm{kg/m^3}$, $\eta_{1,2} \in (1 \cdot 10^6 - 1 \cdot 10^8)\,\mathrm{Ns/m^2}$, $k \in (1 \cdot 10^5 - 1 \cdot 10^7)\,\mathrm{kg/s}$, $\alpha_1 = 0.50\,\mathrm{m}$, $0.45\,\mathrm{m}$, $\mathcal{M} = 10\,000\,\mathrm{kg}$. The frequency ω is changed in the small region near $4\omega_{01}$ (see resonance relations), the value of amplitude of kinematic excitation is $\gamma \in (0.05\,\mathrm{m},\ 0.25\,\mathrm{m})$. On the graphs presented in Figs. 4a, 4b, 5, the amplitudes R_1, R_2 and objective function $\mathcal{Z}$ are shown versus ω and as a function of the parameter κ_2 defining the shape of element II. Both amplitudes R_1 and R_2 have a parametric character. Because of nonlinearity we get non-trivial steady response (the semi-trivial solution is not considered in this paper) with amplitudes R_1 for element I and non-trivial response with amplitude R_2 for element II. Both responses are non-zero in a certain interval of ω, what is the typical feature for parametrically excited systems (the curves $R_1(\omega)$, $R_2(\omega)$ do not reach zero). For both elements the amplitude of kinematic excitation γ and parameter α_1 influence the value of non-zero responses and the interval of frequency. One can see from the Figs. 4a, 4b that the minimal value of objective function, and also the minimal value of the frequency interval are taken for $\kappa_2 = -2$. In the Fig. 5 the minimal values of objective function and of interval of frequency occur for $\kappa_2 = 1$ (the element II is shaped as a cone).



α1 = 0.45, ϰ2 = -2, ϰ1 = 0.66, α2 = 0.11, γ = 0.025
0.4
0.3
0.2
0.1
0
-0.1
R1, R2, R1+R2
207
208
209
210
211
212
213
ω
R1
R2
α1 = 0.45, ϰ2 = 0, ϰ1 = 0.64, α2 = 0.22, γ = 0.025
0.8
0.6
0.4
0.2
0
-0.2
202
204
206
208
210
212
214

[FIG. 4a]



α₁=0.45, æ₂=0.5, æ₁=0.54, α₂=0.27, γ=0.025
R₁, R₂, R₁+R₂
1.0
0.8
0.6
0.4
0.2
0
238
239
240
241
242
243
244
245
ω
R₁
R₂
α₁=0.45, æ₂=0.8, æ₁=0.44, α₂=0.28, γ=0.025
-0.2
266
268
270
272
274
276
278
280

[FIG. 4a]

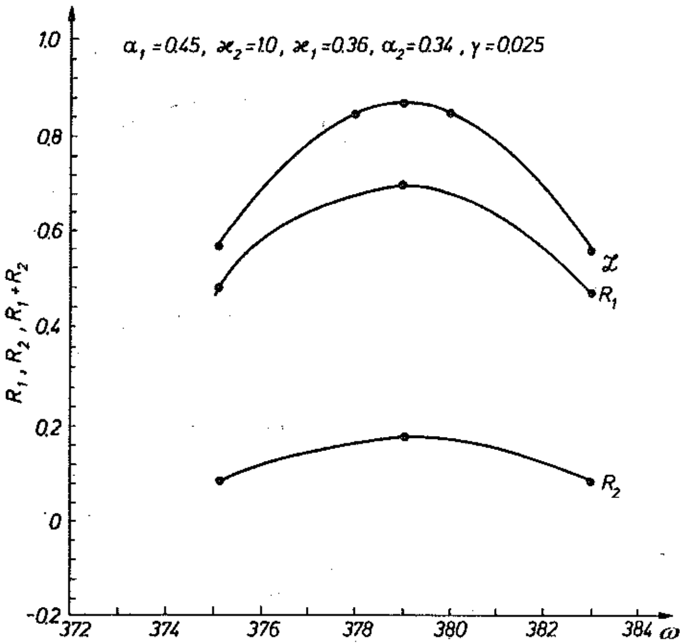
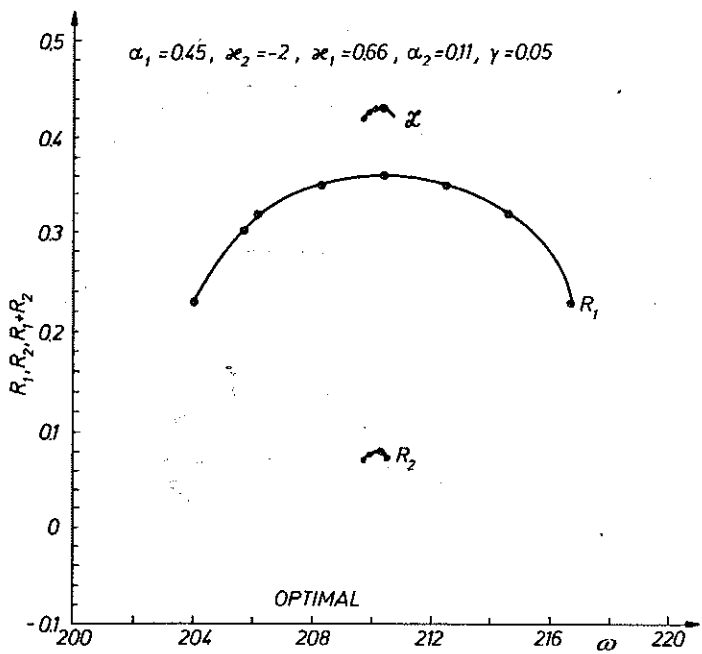
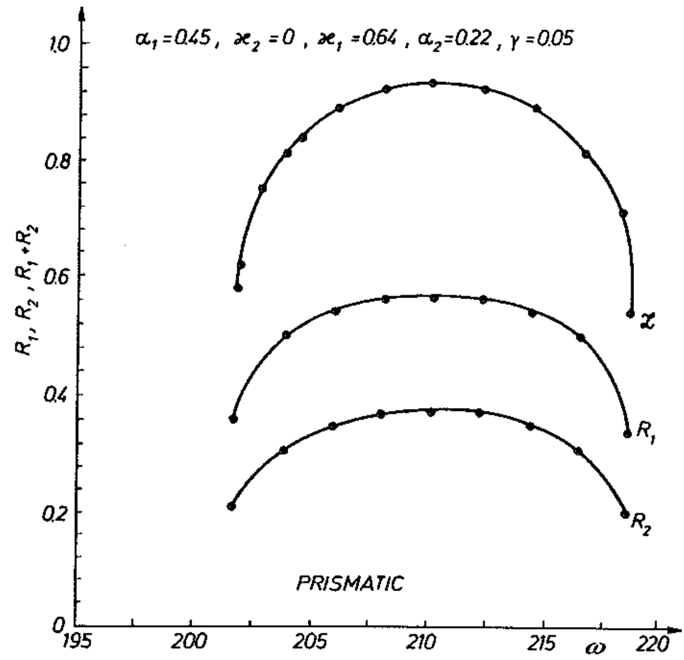
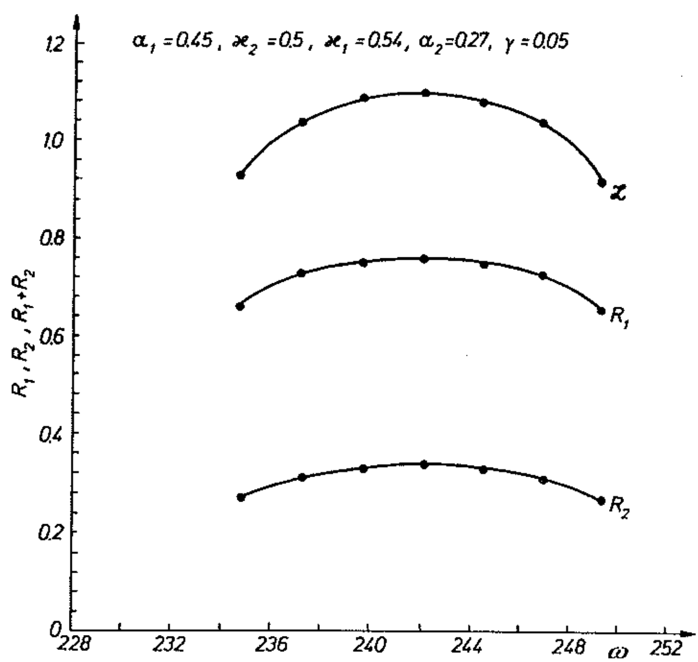


FIG. 4a. Combination resonance. Amplitudes and objective functions $\mathcal{Z}$ versus frequency ω and for different values of parameter κ_2. $\gamma = 0.025\,\mathrm{m}$, $\alpha_1 = 0.45\,\mathrm{m}$.



[FIG. 4b]


$\alpha_1=0.45$, $\varkappa_2=0$, $\varkappa_1=0.64$, $\alpha_2=0.22$, $\gamma=0.05$
R_1, R_2, R_1+R_2
1.0
0.8
0.6
0.4
0.2
0
PRISMATIC
195
200
205
210
215
ω
220
$\mathcal{X}$
R_1
R_2


$\alpha_1=0.45$, $\varkappa_2=0.5$, $\varkappa_1=0.54$, $\alpha_2=0.27$, $\gamma=0.05$
R_1, R_2, R_1+R_2
1.2
1.0
0.8
0.6
0.4
0.2
0
228
232
236
240
244
248
ω
252
$\mathcal{X}$
R_1
R_2

[Fig. 4b]

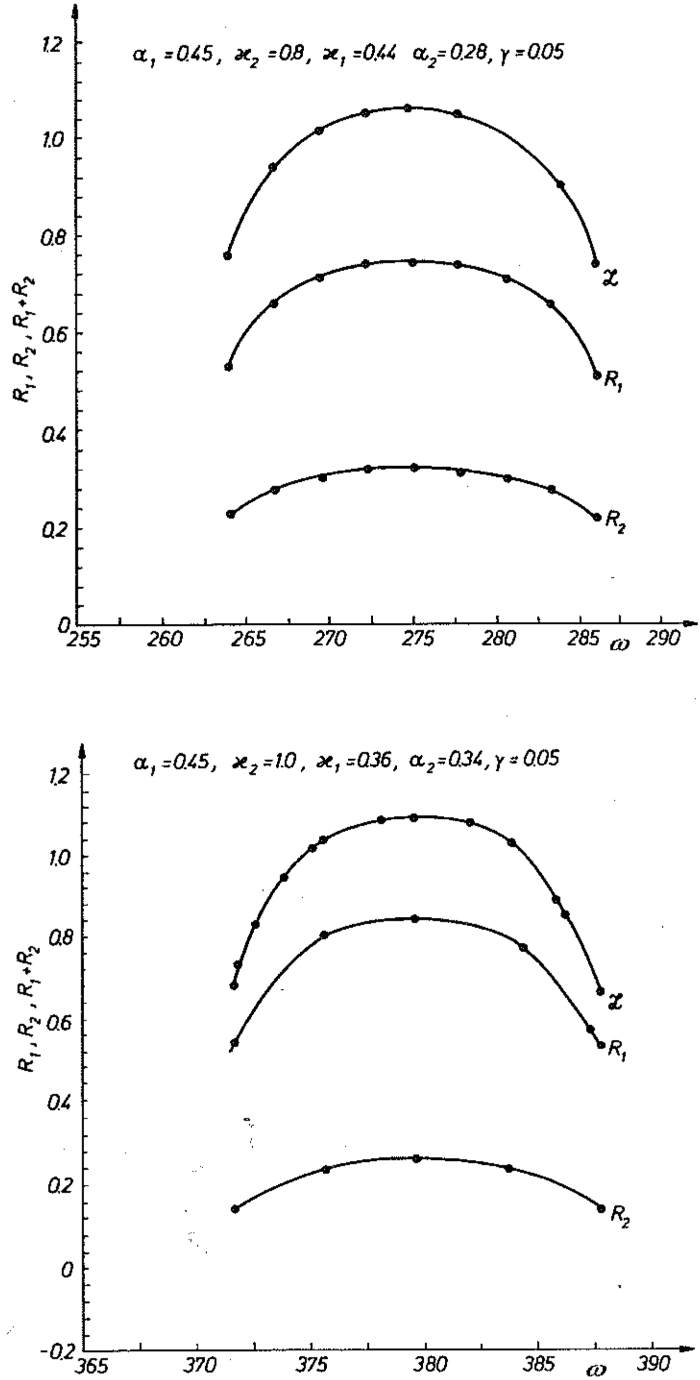
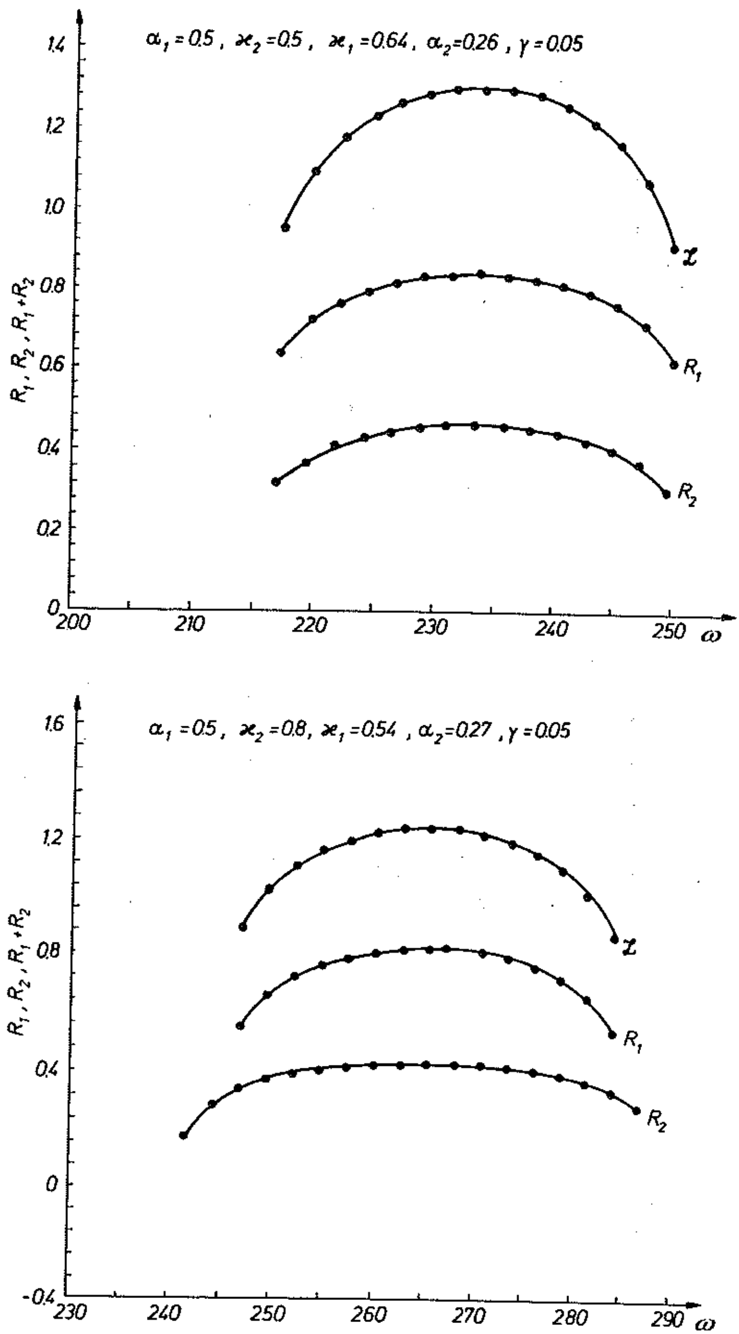


FIG. 4. Combination resonance. Amplitudes and objective functions Z versus frequency ω and for different values of parameter κ_2. $\gamma = 0.05\,\text{m}$, $\alpha_1 = 0.45\,\text{m}$.



α1 = 0.5 , ϰ2 = 0.5 , ϰ1 = 0.64 , α2 = 0.26 , γ = 0.05
R1 , R2 , R1 + R2
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
200
210
220
230
240
250
ω
Σ
R1
R2
α1 = 0.5 , ϰ2 = 0.8 , ϰ1 = 0.54 , α2 = 0.27 , γ = 0.05
R1 , R2 , R1 + R2
1.6
1.2
0.8
0.4
0
-0.4
230
240
250
260
270
280
290
ω
Σ
R1
R2

[Fig. 5]

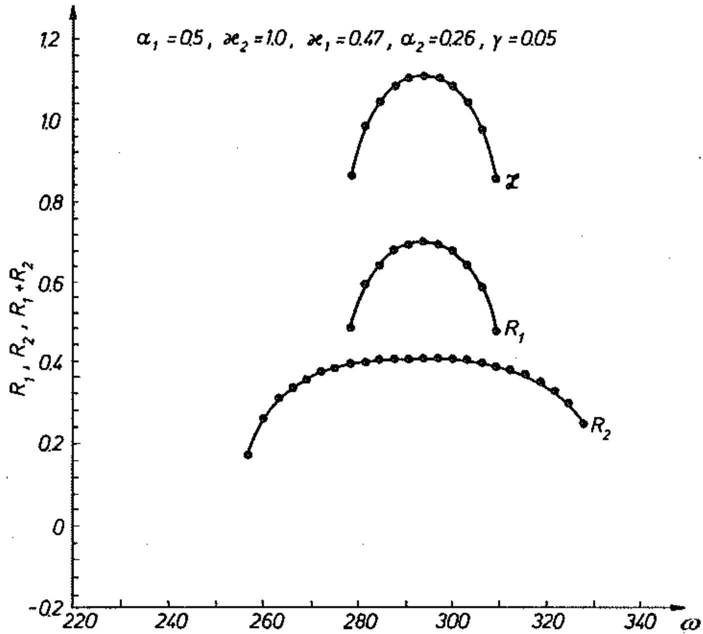


FIG. 5. Combination resonance. Amplitudes and objective function Z versus frequency ω and for different values of parameter κ_2. $\gamma = 0.05\,\mathrm{m}$, $\alpha_1 = 0.5\,\mathrm{m}$.

The paper is a contribution to the study connected with the optimization of the system of rods at the internal and periodic combination resonance of autoparametric nature. The influence of the shapes of rods on some objective functions connected with amplitudes R_1, R_2 are analysed. A suitable selection of the shape parameters of elements may lead to considerable reduction of the frequency interval in which the resonance occurs and may lead to a considerable reduction or total elimination of both or one of the vibrations. The results presented here are of a rather qualitative character and reflect the importance of optimization at autoparametric resonance, especially in kinematically exciting systems. The obtained results may have a practical significance. Due to the coupling of elements, the considered system of rods serves as a simple model of more realistic systems with autoparametric resonance. The beam systems are elements of numerous structures and machines. The fundamental problem is to choose the appropriate model of the described object (structures, buildings, mechanical devices, mechanisms). The model should include some important properties of the object in the particular situation and in particular phenomena. The harmonic load is due to the action of machines or other devices in engine rooms, and kinematic extitation is due to paraseismic phenomena. The coupling forces play a significant role in the considered autoparametric resonances. The autoparametric phenomena can play an essential role in processes of destruction of the object described because weak excitation (e.g. kinamatic) may cause large effects due to the autoparametric resonance. For a real system, when there is an appropriate tuning of frequencies, large values of amplitudes can occur and become dangerous. The analysis presented in this paper may be helpful in considerable reduction or total elimination of these dangerous effects.

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