ENGINEERING TRANSACTIONS • Engng. Trans. • 43, 1–2, 27–44, 1995
Polish Academy of Sciences • Institute of Fundamental Technological Research

PLANE CONTACT OF A CYLINDRICAL OPENING STIFFENED BY A THIN SHELL (*)

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In the present paper the plane contact problem is considered, concerning a circular cylindrical hole stiffened by an elastic circular cylindrical tube (stringer) around its perimeter in a biaxial state of stresses at infinity. For the formulation of the interface conditions, the elastic stringer is considered to behave as a thin shell, and its outer diameter, prior to its insertion into the hole, may be equal or greater than the radius of the hole by a small value of the order of the (infinitesimal) elastic displacements. The solution of this mixed boundary value problem in plane strain conditions is found by numerical integration of a system of a complex singular, and complex regular integral equation describing the boundary and interface conditions of the problem, respectively. The classical method of Kolosov–Muskhelishvili complex potentials $\Phi_0(z)$, $\Psi_0(z)$, in combination with the theory of singular integral equations, is considered in this paper in order to obtain the solution of the mixed boundary value problem stated above.

1. Introduction

Problems of strengthening the bodies by thin-walled elements, such as rectilinear or curvilinear stringers and closed cylindrical shells, belong to the class of contact and mixed boundary value problems of deformable bodies. Solutions of these problems yield the stress-deformation fields at the interfaces of a body, explain the nature of the interaction between its inhomogeneities, and eventually its effectiveness to carry the loads. A problem of this type, constituting also one of the most interesting applications of the mixed boundary value problems in solid mechanics and rock mechanics, is the strengthening of circular cylindrical holes by elastic circular cylindrical tubes. For example, sufficient knowledge of the behavior of the rock mass next to an underground opening (borehole, shaft, tunnel etc.) is important for optimum design, construction and operation of the opening.

The methods based on the complex variable function theory, developed by Muskhelishvili [1, 2], Gakhov [3], and many others, have been applied

(*) Paper presented at 30th Polish Solid Mechanics Conference, Zakopane, September 5–9, 1994.

effectively in the past for solving plane contact problems of the linear elasticity theory. The problem of stregthening of a circular hole in an infinite sheet by a welded elastic ring was considered by SAVIN [4, 5] and others, who made certain simplifying assumptions concerning the interface conditions of the problem. In the present paper, the interaction of an elastic circular cylindrical tube perfectly bonded to a circular cylindrical hole boundary, subject to a biaxial stress field at infinity, the radius of the tube being equal or greater (by the order of elastic displacements) than the hole radius, is investigated under the assumption that the tube behaves like a thin shell. The solution of the mixed boundary value problem is found by applying the classical method of Kolosov-Muskhelishvili complex potentials $\Phi_0(z)$, $\Psi_0(z)$, and the singular integral equations theory.

2. Basic equations of the thin shell theory and problem formulation

Assume that an elastic and isotropic circular cylindrical tube (a stiffener of stringer) of length h and outside radius R_2 is subjected to appropriate forces and pressed into a cylindrical opening of radius R made in an infinite elastic isotropic body; assume that $R_2 > R$, $R_2 - R = \varrho_0$ where ϱ_0 is a small value of the order of the elastic displacements.

Once these forces are removed, the stringer will expand leading to an interaction between the stringer and the material. In the absence of friction

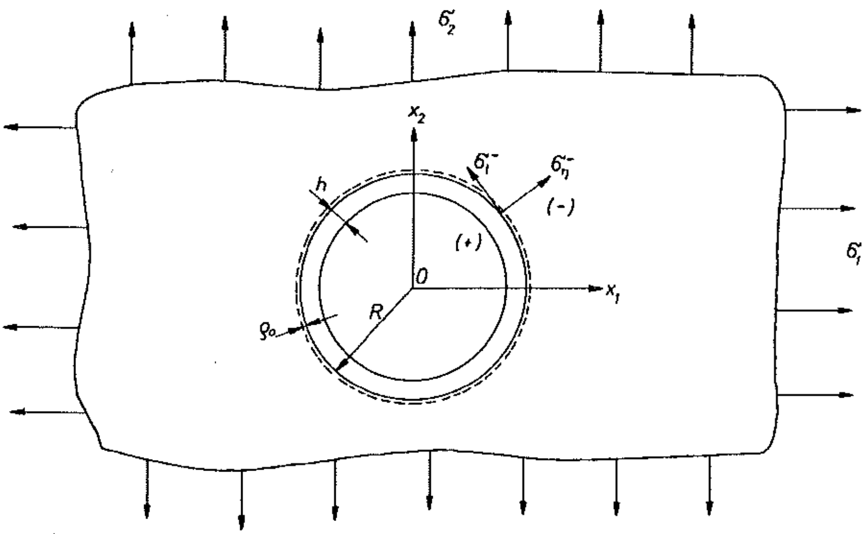

FIG. 1. Schematic diagram of a reinforced circular cylindrical opening.

forces, this interaction will consist of normal forces only. The reinforced body, which is referred to a Cartesian coordinate system x_1Ox_2 with its origin at the centre of the hole, is subjected to normal stresses σ_1 and σ_2 at infinity, as shown in Fig. 1. Also, the part of the plane which lies to the left of the oriented (anti-clockwise) contour of the hole γ will be denoted by S^+, and that located to the right – by S^-.

Prior to the formulation of the boundary conditions of the above problem, the following basic relationships of the thin shell theory are given, based on the classical Kirchhoff–Love hypotheses [6, 7, 8].

1. Equilibrium equations:

$$\begin{gathered}
\frac{1}{A_1A_2}\left(\frac{\partial A_2T_1}{\partial\alpha_1}+\frac{\partial A_1T_{21}}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}T_{12}-\frac{\partial A_2}{\partial\alpha_1}T_2\right)+\frac{N_1}{R_1}+\tau^+-\tau^-=0,\\
\frac{1}{A_1A_2}\left(\frac{\partial A_2T_{12}}{\partial\alpha_1}+\frac{\partial A_1T_2}{\partial\alpha_2}+\frac{\partial A_2}{\partial\alpha_1}T_{21}-\frac{\partial A_1}{\partial\alpha_2}T_1\right)+\frac{N_2}{R_2}+p^+-p^-=0,\\
\frac{1}{A_1A_2}\left(\frac{\partial A_2N_1}{\partial\alpha_1}+\frac{\partial A_1N_2}{\partial\alpha_2}\right)-\frac{T_1}{R_1}-\frac{T_2}{R_2}+q^+-q^-=0,\\
\frac{1}{A_1A_2}\left(\frac{\partial A_2M_1}{\partial\alpha_1}+\frac{\partial A_1M_{21}}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}M_{12}-\frac{\partial A_2}{\partial\alpha_1}M_2\right)-N_1=0,\\
\frac{1}{A_1A_2}\left(\frac{\partial A_1M_2}{\partial\alpha_2}+\frac{\partial A_2M_{12}}{\partial\alpha_1}+\frac{\partial A_2}{\partial\alpha_1}M_{21}-\frac{\partial A_1}{\partial\alpha_2}M_1\right)-N_2=0.
\end{gathered}\tag{2.1}$$

2. Strain-displacement relations:

$$\begin{aligned}
\varepsilon_1&=\frac{1}{A_1}\frac{\partial u_1}{\partial\alpha_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}u_2+\frac{w}{R_1},\\
\varepsilon_2&=\frac{1}{A_2}\frac{\partial u_2}{\partial\alpha_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}u_1+\frac{w}{R_2},\\
\omega&=\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{u_1}{A_1}\right)+\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{u_2}{A_2}\right),\\
\kappa_1&=-\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}-\frac{u_1}{R_1}\right)-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\frac{1}{A_2}\left(\frac{\partial w}{\partial\alpha_2}-\frac{u_2}{R_2}\right),\\
\kappa_2&=-\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}-\frac{u_2}{R_2}\right)-\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{1}{A_1}\left(\frac{\partial w}{\partial\alpha_1}-\frac{u_1}{R_1}\right),\\
\tau&=-\frac{1}{A_1A_2}\left(\frac{\partial^2 w}{\partial\alpha_1\partial\alpha_2}-\frac{1}{A_1}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial w}{\partial\alpha_1}-\frac{1}{A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial w}{\partial\alpha_2}\right)\\
&\quad+\frac{1}{R_1}\left(\frac{1}{A_2}\frac{\partial u_1}{\partial\alpha_2}-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}u_1\right)+\frac{1}{R_2}\left(\frac{1}{A_1}\frac{\partial u_2}{\partial\alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}u_2\right).
\end{aligned}\tag{2.2}$$

3. Relations between forces, moments and strains:

$$
(2.3)\quad
\begin{aligned}
T_1 &= \frac{E_1 h}{1-\nu_1^2}(\varepsilon_1 + \nu_1\varepsilon_2), & T_2 &= \frac{E_1 h}{1-\nu_1^2}(\varepsilon_2 + \nu_1\varepsilon_1),\\
M_1 &= \frac{E_1 h^3}{12(1-\nu_1^2)}(\kappa_1 + \nu_1\kappa_2), & M_2 &= \frac{E_1 h^3}{12(1-\nu_1^2)}(\kappa_2 + \nu_1\kappa_1),\\
T_{12} &= \frac{E_1 h}{2(1+\nu_1)}\left(\omega + \frac{h^2}{6R_2}\tau\right), & T_{21} &= \frac{E_1 h}{2(1+\nu_1)}\left(\omega + \frac{h^2}{6R_1}\tau\right),\\
M_{12} &= M_{21} = H = \frac{E_1 h^3}{12(1+\nu_1)}\tau .
\end{aligned}
$$

4. Compatibility equations:

$$
(2.4)\quad
\begin{aligned}
&\frac{\partial A_1\kappa_1}{\partial\alpha_2} - \kappa_2\frac{\partial A_1}{\partial\alpha_2} - \frac{\partial A_2\tau}{\partial\alpha_1} - \tau\frac{\partial A_2}{\partial\alpha_1} + \frac{\omega}{R_1}\frac{\partial A_2}{\partial\alpha_1}\\
&\qquad - \frac{1}{R_2}\left(\frac{\partial A_1\varepsilon_1}{\partial\alpha_2} - \frac{\partial A_2\omega}{\partial\alpha_1} - \varepsilon_2\frac{\partial A_1}{\partial\alpha_2}\right) = 0,\\
&\frac{\partial A_2\kappa_2}{\partial\alpha_1} - \kappa_1\frac{\partial A_2}{\partial\alpha_1} - \frac{\partial A_1\tau}{\partial\alpha_2} - \tau\frac{\partial A_1}{\partial\alpha_2} + \frac{\omega}{R_2}\frac{\partial A_1}{\partial\alpha_2}\\
&\qquad - \frac{1}{R_1}\left(\frac{\partial A_2\varepsilon_2}{\partial\alpha_1} - \frac{\partial A_1\omega}{\partial\alpha_2} - \varepsilon_1\frac{\partial A_2}{\partial\alpha_1}\right) = 0,\\
&\frac{\kappa_1}{R_2} + \frac{\kappa_2}{R_1} + \frac{1}{A_1A_2}\left\{\frac{\partial}{\partial\alpha_1}\frac{1}{A_1}\left[A_2\frac{\partial\varepsilon_2}{\partial\alpha_1} + \frac{\partial A_2}{\partial\alpha_1}(\varepsilon_2 - \varepsilon_1)\right.\right.\\
&\qquad \left. - \frac{1}{2}A_1\frac{\partial\omega}{\partial\alpha_2} - \frac{\partial A_1}{\partial\alpha_2}\omega\right] + \frac{\partial}{\partial\alpha_2}\frac{1}{A_2}\left[A_1\frac{\partial\varepsilon_1}{\partial\alpha_2}\right.\\
&\qquad \left.\left. + \frac{\partial A_1}{\partial\alpha_2}(\varepsilon_1 - \varepsilon_2) - \frac{1}{2}A_2\frac{\partial\omega}{\partial\alpha_1} - \frac{\partial A_2}{\partial\alpha_1}\omega\right]\right\} = 0,
\end{aligned}
$$

where α_1, α_2 are the curvilinear coordinates along the middle surface of the shell coinciding with its main curvatures, R_1, R_2 are the radii of curvature of the middle surface of the shell, and A_1, A_2 are the corresponding Lamé parameters; κ_1, κ_2, τ are parameters of curvature and twisting of the middle surface; T, N denote forces per unit length acting in the planes normal to the middle surface of the shell, and forces per unit length acting in the planes of the middle surface, respectively, and M are moments per unit length of the normal sections; u_1, u_2, w denote the displacement components referred to the coordinates α_1, α_2 and the normal to the surface of the shell coordinate α_3, respectively, E_1, ν_1 are the Young's modulus and Poisson's ratio of the shell, respectively, h is the thickness of the shell, and $\tau^\pm$, $p^\pm$, $q^\pm$ denote the components of the external loads on the upper (+) and lower (−) surfaces of the shell.

For the case of a cylindrical thin shell the following relationships hold true

$$
\begin{aligned}
&\alpha_1 = s, \qquad \alpha_2 = z, \qquad \alpha_3 = r = n, \qquad s = R\theta,\\
&A_1 = A_2 = A_3 = 1, \qquad R_1 = R, \qquad R_2 = \infty,\\
&u_2 = \varepsilon_2 = \kappa_2 = \omega = \tau = 0,
\end{aligned}
\tag{2.5}
$$

where s is the arc length of the middle surface of the shell, and R is the radius of curvature of the middle surface of the shell.

Relations (2.3) take the following form if Eqs. (2.5) are taken into account

$$
\begin{aligned}
&T_1 = E_s\varepsilon_1, \qquad T_2 = E_s\nu\varepsilon_1, \qquad M_1 = D_s\kappa_1, \qquad M_2 = D_s\nu\kappa_1,\\
&T_{12} = T_{21} = M_{12} = M_{21} = 0,
\end{aligned}
\tag{2.6}
$$

where

$$
E_s = \frac{E_1 h}{1-\nu_1^2}, \qquad D_s = \frac{E_1 h^3}{12(1-\nu_1^2)}.
$$

Considering the shell stiffness in bending to be negligible ($D_s \approx 0$), the equilibrium equations (2.1) yield

$$
\begin{aligned}
&\frac{1}{R}\frac{\partial T_1(\theta)}{\partial\theta} = \tau(\theta), \qquad -\frac{T_1(\theta)}{R} = q(\theta),\\
&\tau(\theta) = \tau^-(\theta) - \tau^+(\theta), \qquad q(\theta) = q^-(\theta) - q^+(\theta).
\end{aligned}
\tag{2.7}
$$

Next, by taking into account Eqs. (2.7), the boundary conditions for the previously stated mixed-boundary value problem in the plane strain case (length of the hole axis being much larger than any dimension in the x_1Ox_2 plane) are expressed as follows:

(i) The inner surface of the ring is free of external forces

$$
\sigma_n^+ - i\sigma_t^+ = 0, \tag{2.8}
$$

where i is the usual imaginary unit.

Along γ, at the interface between the body and the elastic stringer, the following system of mixed boundary conditions is given:

(ii) According to the above formulated theory of thin shells and for the specific case of a cylindrical thin shell, the normal and shear stress components along the contact zone γ of the two bodies must satisfy the relations

$$
\begin{aligned}
&-\frac{T(\theta)}{R} + (\sigma_n^+ - \sigma_n^-) = 0,\\
&-\frac{1}{R}\frac{dT(\theta)}{d\theta} + (\sigma_t^+ - \sigma_t^-) = 0,
\end{aligned}
\tag{2.9}
$$

where $T(\theta)$ is the circumferential (hoop) force per unit length which acts along the line lying in the middle surface of the cylindrical stringer, and is given by the formula

$$T(\theta) = \frac{E_1 h}{1-\nu_1^2}\varepsilon_\theta^{\mathrm{str}}. \tag{2.10}$$

Here $\varepsilon_\theta^{\mathrm{str}}$ is the circumferential component of strain along the line in the middle surface of the ring, which can be obtained by applying Hooke's law for the case of plane strain

$$\varepsilon_\theta^{\mathrm{str}} = \frac{1}{E_1}\left[(1-\nu_1^2)\sigma_\theta - \nu_1(1+\nu_1)\sigma_r\right]; \tag{2.11}$$

here $\sigma_\theta = \sigma_s$ denotes the circumferential (hoop) stress component, and $\sigma_r = \sigma_n$ is the radial stress component ($r, \theta =$ polar coordinates).

(iii) Finally, along γ the circumferential strains of the two bodies are equal to $\varepsilon_\theta^{\mathrm{str}}$

$$\varepsilon_\theta^{+} = \varepsilon_\theta^{-} = \varepsilon_\theta^{\mathrm{str}}. \tag{2.12}$$

Consideration of boundary conditions (ii) and (iii), together with definitions (2.10) and (2.11), leads to the following relationship that must be satisfied on γ:

$$ER(1-\nu_1^2)(\sigma_n^- - i\sigma_t^-) + E_1 h(1+\nu)\left(1-\frac{d}{dt}\right) \times \left[(1-\nu)(\sigma_n^- + \sigma_s^-) - \sigma_n^-\right] = 0, \tag{2.13}$$

where $t = Re^{i\theta}$ denotes the position of a point on γ, E, ν denote the Young's modulus and Poisson's ratio of the body, respectively, and $\sigma_n = \sigma_n^-$, $\sigma_t = \sigma_t^-$, since from (2.8) it follows that these are the only surviving stresses $(\sigma_n^+ = \sigma_t^+ = 0)$.

Next, the complex potentials which describe the stress field are defined by

$$\Phi_0(z) = \Phi(z) + \Gamma, \qquad \Psi_0(z) = \Psi(z) + \Gamma', \tag{2.14}$$

where

$$\Phi(z) = \frac{1}{2\pi i}\oint_\gamma \frac{\varphi(\tau)}{\tau - z}\,d\tau, \tag{2.15}$$

and

$$\begin{aligned} \Gamma &= \frac{1}{4}(\sigma_1 + \sigma_2), \\ \Gamma' &= -\frac{1}{2}(\sigma_1 - \sigma_2). \end{aligned} \tag{2.16}$$

Combining relations (2.14) and (2.15) it is found that

$$\Phi_0^-(z) = \frac{1}{2\pi i} \oint\limits_\gamma \frac{\varphi(\tau)}{\tau - z}\, d\tau + \Gamma. \tag{2.17}$$

Then, if the density $\varphi(t)$ is the boundary value of a function analytic in S^- then, according to the Cauchy formula for the infinite domain [3],

$$\begin{aligned} \Phi^-(z) &= -\varphi(z) + \varphi(\infty), \\ \varphi(\infty) &= \Gamma. \end{aligned} \tag{2.18}$$

Taking the limiting values on the contour γ and making use of the Plemelj formulae [1, 2, 3] it is found that

$$\begin{aligned} \Phi(t) &= -\frac{1}{\pi i} \oint\limits_\gamma \frac{\varphi(\tau)}{\tau - t}\, d\tau + 2\Gamma, \\ \varphi(t) &= \Phi(t). \end{aligned} \tag{2.19}$$

Also, from the following boundary conditions on γ [2]

$$\begin{aligned} 2\mu \frac{d}{dt}(-u + iv) &= \Phi(t) - \kappa\overline{\Phi(t)} + \frac{dt}{d\bar{t}}\left(\bar{t}\Phi'(t) + \Psi(t)\right), \\ \sigma_n - i\sigma_t &= \Phi(t) + \overline{\Phi(t)} + \frac{dt}{d\bar{t}}\left(\bar{t}\Phi'(t) + \Psi(t)\right), \end{aligned} \tag{2.20}$$

where $dt/d\bar{t} = -e^{2i\theta}$, the overbar denotes complex conjugate, $\kappa = 3 - 4\nu$ is the Muskhelishvili constant of the matrix material for the plane strain case, and μ is the shear modulus of the matrix material, it can be found that

$$(\kappa + 1)\Phi(t) = (\sigma_n + i\sigma_t) + 2\mu \frac{d}{dt}(u + iv). \tag{2.21}$$

From the well-known relation of transformation of coordinates

$$u + iv = e^{-\theta}(u_r + iu_\theta) \tag{2.22}$$

and after some algebra, the following relation is derived:

$$\frac{d}{dt}(u + iv) = \frac{1}{E}\Big[(1 - \nu^2)\sigma_\theta - \nu(1 + \nu)\sigma_r\Big] - i\frac{\tau_{r\theta}}{\mu} + i\frac{du_\theta}{dr}. \tag{2.23}$$

Combining Eqs. (2.21) and (2.23) we obtain

$$(2.24)\qquad \Phi(t) = \frac{1-\nu}{1+\kappa}(\sigma_n+\sigma_s) - i\frac{\sigma_t}{1+\kappa} + i\frac{2\mu}{1+\kappa}\frac{du_\theta}{dr} = f_1(t)+f_2(t)-if_3(t)+if_4(t),$$

where we have set

$$(2.25)\qquad \begin{aligned} f_1(t) &= \frac{1-\nu}{1+\kappa}\sigma_n(t) = 4\sigma_n(t) = \sigma_r(t),\\ f_2(t) &= \frac{1-\nu}{1+\kappa}\sigma_\theta(t) = 4\sigma_\theta(t),\\ f_3(t) &= \frac{1}{1+\kappa}\sigma_t(t) = \frac{1}{1+\kappa}\tau_{r\theta}(t),\\ f_4(t) &= \frac{2\mu}{1+\kappa}\frac{du_\theta(t)}{dr}. \end{aligned}$$

Furthermore, the initial pressure due to the difference of the stringer-perforation initial radii ($\varrho_0 = R_2 - R$) is given by the following function $p(t)$ under plane strain conditions [9]

$$(2.26)\qquad p(t) = \frac{4\mu_1}{\left(1-\kappa-2\dfrac{\mu_1}{\mu}\right)}\frac{\varrho_0}{R},$$

where μ_1 is the shear modulus of the shell. According to the above formula, relation (2.24) takes the following form:

$$(2.27)\qquad \Phi(t)+f_0(t) = f_0(t)+f_1(t)+f_2(t)-if_3(t)+if_4(t),$$

where $f_0(t) = p(t)$.

Integrating the second of boundary conditions (2.20) with respect to t and solving for $\Psi(z)$, we obtain

$$(2.28)\qquad \Psi(z) = \frac{1}{2\pi i}\oint_\gamma \frac{\sigma_n - i\sigma_t}{\tau - z}d\tau - \frac{1}{2\pi i}\oint_\gamma \frac{\overline{\Phi(\tau)}}{\tau - z}d\overline{\tau} - \frac{1}{2\pi i}\oint_\gamma \frac{\overline{\tau}\Phi(\tau)}{(\tau-z)^2}d\tau + \Gamma'.$$

Returning to the second of boundary conditions (2.20) and substituting the limiting value $\Psi(t)$ which can be found from (2.28) by applying Plemelj's formulae, one finds the following complex singular integral equation that must be satisfied on γ:

$$(2.29)\qquad 2\mathrm{Re}\left[\frac{1}{\pi i}\oint_\gamma \frac{\Phi(\tau)}{\tau - t}d\tau\right] + \frac{dt}{d\overline{t}}\left[\frac{1}{\pi i}\oint_\gamma \frac{\sigma_n - i\sigma_t}{\overline{\tau}-\overline{t}}d\tau - \frac{1}{\pi i}\oint_\gamma \frac{\overline{\Phi(\tau)}}{\tau - t}d\overline{\tau} - \frac{1}{\pi i}\oint_\gamma \frac{\overline{\tau}-\overline{t}}{(\overline{\tau}-\overline{t})^2}\Phi(\tau)\,d\overline{\tau}\right] - (\sigma_n - i\sigma_t) = 2\left[\Gamma + \overline{\Gamma} + \frac{dt}{d\overline{t}}\Gamma'\right],$$

where Re denotes the real part of the expression.

Finally, substituting in (2.13) and (2.29) the values of the stresses and of the derivatives of the circumferential component of displacement given by (2.25), we end up with the following system of equations that must be satisfied on the stringer-hole interface:

$$
\begin{aligned}
&4\left[ER(1-\nu_1^2)+\nu E_1 h(1+\nu)\right] f_1(t) - 4E_1 h(1-\nu^2) f_2(t) \\
&\qquad = \left[ER(1-\nu_1^2)+\nu E_1 h(1+\nu)\right] p(t), \\
&E_1 h(1+\nu)\frac{d}{d\theta}\left[-4\frac{\nu}{1-\nu}f_1(t)+4f_2(t)\right] - ER\frac{(1-\nu_1^2)}{(1-\nu)(1+\kappa)}f_3(t) = 0,
\end{aligned}
\tag{2.30}
$$

and the complex singular integral equation, considering also the initial pressure due to an oversize in the radius of the stringer:

$$
\begin{aligned}
&\frac{1}{\pi i}\oint_\gamma \frac{f_1(\tau)+f_2(\tau)-if_3(\tau)+if_4(\tau)}{\tau - t}d\tau \\
&\quad -\frac{1}{\pi i}\oint_\gamma \frac{f_1(\tau)+f_2(\tau)-if_3(\tau)+if_4(\tau)}{\overline{\tau}-\overline{t}}d\overline{\tau} \\
&\quad + \left[\frac{1}{\pi i}\oint_\gamma \frac{\sigma_n - i\sigma_t}{\tau - t}d\tau - \frac{\overline{t}}{\pi i}\oint_\gamma \frac{\sigma_n - i\sigma_t}{\tau^2}d\tau - \frac{1}{\pi i}\oint_\gamma \frac{\sigma_n - i\sigma_t}{\tau}d\tau \right.\\
&\quad -\frac{1}{\pi i}\oint_\gamma \frac{f_1(\tau)+f_2(\tau)+if_3(\tau)-if_4(\tau)}{\tau - t}d\tau \\
&\quad +\frac{\overline{t}}{\pi i}\oint_\gamma \frac{f_1(\tau)+f_2(\tau)+if_3(\tau)-if_4(\tau)}{\tau^2}d\tau \\
&\quad +\frac{1}{\pi i}\oint_\gamma \frac{f_1(\tau)+f_2(\tau)+if_3(\tau)-if_4(\tau)}{\tau}d\tau \\
&\quad -\frac{1}{\pi i}\oint_\gamma \frac{f_1(\tau)+f_2(\tau)-if_3(\tau)+if_4(\tau)}{\tau - t}d\tau \\
&\quad \left.+\frac{1}{\pi i}\oint_\gamma \frac{f_1(\tau)+f_2(\tau)-if_3(\tau)+if_4(\tau)}{\tau}d\tau\right] \\
&\quad -(\sigma_n - i\sigma_t) = 2\left(\Gamma + \overline{\Gamma} + \frac{dt}{dt}\Gamma'\right) + p(t).
\end{aligned}
\tag{2.31}
$$

Before proceeding to the numerical solution of the system of Eqs. (2.30) and (2.31), we investigate the limiting case of stress-free hole in the body isotropically stressed at infinity ($\sigma_1 = \sigma_2 = \sigma$). In this case Eqs. (2.30) are satisfied in a trivial manner, and it is valid that $f_1(t) = f_3(t) = 0$ and $f_4(t) = 0$, since $du_\theta/dr = 0$ for an isotropic deformation of the hole. Therefore, Eq. (2.31) can be simplified as follows:

$$\frac{f_2}{\pi i}\oint_\gamma \frac{1}{\tau - t}d\tau - \frac{f_2}{\pi i}\oint_\gamma \frac{1}{\overline{\tau} - \overline{t}}d\overline{\tau} - \frac{f_2}{\pi i}\oint_\gamma \frac{1}{\tau - t}d\tau + \frac{\overline{t} f_2}{\pi i}\oint_\gamma \frac{1}{\tau^2}d\tau + \frac{f_2}{\pi i}\oint_\gamma \frac{1}{\tau}d\tau$$
$$-\frac{f_2}{\pi i}\oint_\gamma \frac{1}{\tau - t}d\tau + \frac{f_2}{\pi i}\oint_\gamma \frac{1}{\tau}d\tau = 4f_2 = 2\left(\Gamma + \overline{\Gamma}\right) = \sigma_1 + \sigma_2 = 2\sigma.$$

Thus, according to the above result and relation $(2.25)_2$, it is found that $\sigma_\theta/\sigma = 2$, which agrees with the stress concentration factor predicted by the classical Kirsch solution.

3. Numerical solution and results

By taking into consideration the following approximate expressions of the functions given in [9, 10], we derive

$$(3.1)\qquad \varphi(t) = \frac{1}{2n+1}\sum_{j=1}^{2n+1} \frac{\sin\dfrac{(2n+1)(\theta_0 - \theta_j)}{2}}{\sin\dfrac{(\theta_0 - \theta_j)}{2}}\left[\varphi_1(t_j) + i\varphi_2(t_j)\right]$$

$$\times\frac{1}{\pi i}\oint_\gamma \frac{\varphi(\tau)}{\tau - t}d\tau = \frac{1}{2n+1}\sum_{j=1}^{2n+1}\left[1 + 2i\frac{\sin\dfrac{n(\theta_0 - \theta_j)}{2}\sin\dfrac{(n+1)(\theta_0 - \theta_j)}{2}}{\sin\dfrac{(\theta_0 - \theta_j)}{2}}\right] \times\left[\varphi_1(t_j) + i\varphi_2(t_j)\right].$$

Separating the real and imaginary parts of (2.31) and considering also relations (2.30), we derive the following system of linear algebraic equations by applying the approximate relations (3.1):

$$(3.2)\qquad \frac{1}{2n+1}\sum_{j=1}^{2n+1}\left[4ER(1-\nu_1^2) + 4\nu E_1 h(1+\nu)\right]\frac{\sin(2n+1)\dfrac{(\theta_0^{(k)} - \theta_j)}{2}}{\sin\dfrac{(\theta_0^{(k)} - \theta_j)}{2}} f_1(t_j)$$
$$+\frac{1}{2n+1}\sum_{j=1}^{2n+1}\left[-4E_1 h(1-\nu^2)\right]\frac{\sin(2n+1)\dfrac{(\theta_0^{(k)} - \theta_j)}{2}}{\sin\dfrac{(\theta_0^{(k)} - \theta_j)}{2}} f_2(t_j)$$
$$= -\left[ER(1-\nu_1^2) + \nu E_1 h(1+\nu)\right]p(t)$$

and

$$
\begin{aligned}
(3.3)\qquad & \frac{1}{2n+1}\sum_{j=1}^{2n+1}\left[-4\frac{E_1 h\nu(1+\nu)}{1-\nu}\right.\\
&\left.\times\frac{n\cos\dfrac{(2n+1)(\theta_0^{(k)}-\theta_j)}{2}\sin\dfrac{(\theta_0^{(k)}-\theta_j)}{2}-\sin n(\theta_0^{(k)}-\theta_j)}{\sin^2\dfrac{(\theta_0^{(k)}-\theta_j)}{2}}\right]f_1(t_j)\\
&+\frac{1}{2n+1}\sum_{j=1}^{2n+1}\left[4E_1 h(1+\nu)\right.\\
&\left.\times\frac{n\cos\dfrac{(2n+1)(\theta_0^{(k)}-\theta_j)}{2}\sin\dfrac{(\theta_0^{(k)}-\theta_j)}{2}-\sin n(\theta_0^{(k)}-\theta_j)}{\sin^2\dfrac{(\theta_0^{(k)}-\theta_j)}{2}}\right]f_2(t_j)\\
&+\frac{1}{2n+1}\sum_{j=1}^{2n+1}\left[(1+\kappa)ER\frac{(1-\nu_1^2)}{1-\nu}\frac{\sin\dfrac{(2n+1)(\theta_0^{(k)}-\theta_j)}{2}}{\sin\dfrac{(\theta_0^{(k)}-\theta_j)}{2}}\right]f_3(t_j)=0,
\end{aligned}
$$

and for the real part of relation (2.31),

$$
\begin{aligned}
(3.4)\qquad & \frac{1}{2n+1}\sum_{j=1}^{2n+1}\left[-6\cos(\theta_0^{(k)}-\theta_j)-4\frac{\sin\dfrac{(2n+1)(\theta_0^{(k)}-\theta_j)}{2}}{\sin\dfrac{(\theta_0^{(k)}-\theta_j)}{2}}\right]f_1(t_j)\\
&+\frac{1}{2n+1}\sum_{j=1}^{2n+1}\left[4+2\cos(\theta_0^{(k)}-\theta_j)\right]f_2(t_j)\\
&+\frac{1}{2n+1}\sum_{j=1}^{2n+1}\left[2(3+\kappa)\frac{\sin\dfrac{n(\theta_0^{(k)}-\theta_j)}{2}\sin\dfrac{(n+1)(\theta_0^{(k)}-\theta_j)}{2}}{\sin\dfrac{(\theta_0^{(k)}-\theta_j)}{2}}\right.\\
&\left.-2(2+\kappa)\sin(\theta_0^{(k)}-\theta_j)\right]f_3(t_j)
\end{aligned}
$$

(3.4) [cont.]

$$+\frac{1}{2n+1}\sum_{j=1}^{2n+1}\left[-4\frac{\sin\dfrac{n(\theta_0^{(k)}-\theta_j)}{2}\sin\dfrac{(n+1)(\theta_0^{(k)}-\theta_j)}{2}}{\sin\dfrac{(\theta_0^{(k)}-\theta_j)}{2}} + 2\sin(\theta_0^{(k)}-\theta_j)\right] f_4(t_j) = (\sigma_1+\sigma_2)+(\sigma_1-\sigma_2)\cos 2\theta_0^{(k)} + p(t)$$

as well as for the imaginary part of (2.31),

$$\frac{1}{2n+1}\sum_{j=1}^{2n+1}\left[4\frac{\sin\dfrac{n(\theta_0^{(k)}-\theta_j)}{2}\sin\dfrac{(n+1)(\theta_0^{(k)}-\theta_j)}{2}}{\sin\dfrac{(\theta_0^{(k)}-\theta_j)}{2}} - 6\sin(\theta_0^{(k)}-\theta_j)\right] f_1(t) \tag{3.5}$$

$$+\frac{1}{2n+1}\sum_{j=1}^{2n+1}\left[-4\frac{\sin\dfrac{n(\theta_0^{(k)}-\theta_j)}{2}\sin\dfrac{(n+1)(\theta_0^{(k)}-\theta_j)}{2}}{\sin\dfrac{(\theta_0^{(k)}-\theta_j)}{2}} + 2\sin(\theta_0^{(k)}-\theta_j)\right] f_2(t)$$

$$+\frac{1}{2n+1}\sum_{j=1}^{2n+1}\left[2(2+\kappa)\cos(\theta_0^{(k)}-\theta_j)+(1+\kappa) +(1+\kappa)\frac{\sin\dfrac{(2n+1)(\theta_0^{(k)}-\theta_j)}{2}}{\sin\dfrac{(\theta_0^{(k)}-\theta_j)}{2}}\right] f_3(t)$$

$$+\frac{1}{2n+1}\sum_{j=1}^{2n+1}\left[-2\cos(\theta_0^{(k)}-\theta_j)\right] f_4(t_j) = (\sigma_1-\sigma_2)\sin(\theta_0^{(k)}-\theta_j).$$

Here

$\theta_j = \dfrac{2\pi j}{2n+1}$, $j = 1,\dots,2n+1$ are the integration points, and

$\theta_0^{(k)} = \dfrac{(2k-1)\pi}{2n+1}$, $k = 1,\dots,2n+1$ are the collocation points.

Solving the above established system of linear algebraic equations for $f_1(t_j)$, $f_2(t_j)$, $f_3(t_j)$ and $f_4(t_j)$, we determine the complete distribution of stresses and displacements along the boundary γ. If the number of integration points is increased, the variation of the stresses and displacements will be described with higher accuracy.

In the following we shall compare in a graphical form the results given by SAVIN [5], concerning the influence of the ratio of rigidity of the circular cylindrical tube to the matrix material, containing a hole of radius $R = 1$ and stressed at infinity by a tensile axial stress $\sigma_1 = 1$ MPa, upon the polar components σ_r, σ_θ and $\tau_{r\theta}$ of the stress tensor at several points along the circular contour γ. For simplicity, all dimensions of the problem have been normalized to the length of the radius R of the perforation. It is reasonable to begin this comparative investigation from the extreme case of $E_1/E = 0$, that is, with no effective support of the hole. In this case $\sigma_r = \tau_{r\theta} = 0$ along γ and the circumferential stress σ_θ is given by the classical Kirsch solution. Figure 2 shows that predictions of the proposed method agree very well with the classical Kirsch's solution.

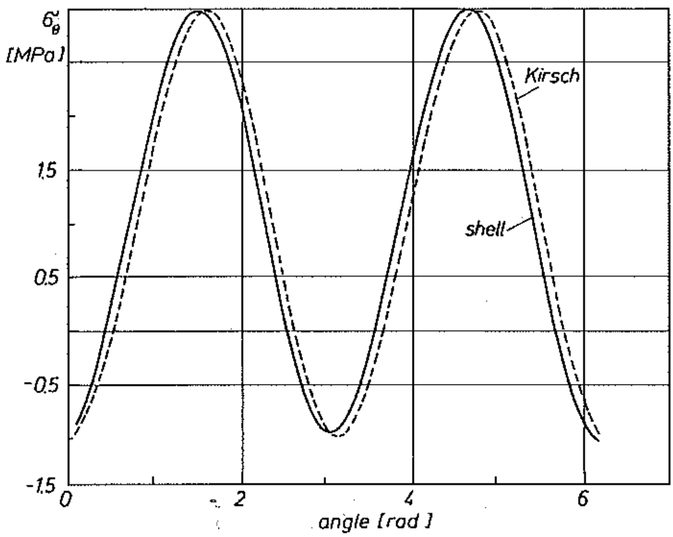

FIG. 2. Angular distribution of the circumferential stress along the interface for $E_1/E = 0$.

In order to evaluate the convergence of the method, Table 1 was constructed containing the angular stress distribution along the interface γ, for $E_1/E = \nu_1/\nu = 1$ and $h = 0.1$, and for a number of integration points n equal to 4, 6 and 8. It is obvious from this table that when the number of integration points increases, the variation of the polar stress components is described with higher accuracy.

Table 1. Angular stress distribution along the common boundary of the shell and the perforation.

θ ($\times\pi$ rad)	σ_r ($\times 10^6$ Nm^{-2})	σ_θ ($\times 10^6$ Nm^{-2})	$\tau_{r\theta}$ ($\times 10^6$ Nm^{-2})
	$n = 4$		
0	−0.01388744	−0.14234650	−0.00236107
±1/9	0.07318417	0.75013810	−0.43572980
±2/9	0.19256280	1.97376800	−0.16643650
±3/9	0.15413280	1.57986100	0.35396130
±4/9	0.02889781	0.29620240	0.27011450
	$n = 6$		
0	−0.02153770	−0.22074100	−0.01893179
±1/13	0.02784573	0.28541860	−0.37904500
±2/13	0.12839200	1.31601900	−0.42900510
±3/13	0.19745690	2.02393400	−0.12696040
±4/13	0.17939760	1.83882500	0.26910780
±5/13	0.09272188	0.95039930	0.42358730
±6/13	0.01343439	0.13770230	0.21164690
	$n = 8$		
0	−0.021523640	−0.22061740	0.00589600
±1/17	0.003799718	0.03894717	−0.27995220
±2/17	0.068100550	0.69803060	−0.41656340
±3/17	0.137224600	1.40655300	−0.33221280
±4/17	0.174469700	1.78831500	−0.07100409
±5/17	0.159827700	1.63823400	0.23018940
±6/17	0.100503700	1.03016400	0.41322080
±7/17	0.027215940	0.27896340	0.38135190
±8/17	−0.021806580	−0.22351770	0.14992020

Next, three different ratios of the tube-to-matrix material elastic moduli are considered for $E_1/E = 0.5$ (soft support), $E_1/E = 1.0$ (support material is the same as in the matrix), and $E_1/E = 10.0$ (stiff stringer). In all the cases examined above the ratio ν_1/ν was equal to 1, h was equal to 0.1 and the number of integration points $n = 16$. Figure 3 shows the angular distribution of the circumferential stress σ_θ at the interface predicted by the shell theory and Savin's method for the case of $E_1/E = 0.5$. As it can be seen, the results obtained by the two methods compare very well, the same holds also true for the radial stresses, while it was found that the predictions of the two methods concerning the shear stresses are different. In Fig. 4 the predictions for σ_θ of the two methods are compared for $E_1/E = 1.0$. As it can be observed from Figs. 3, 4, both the methods predict the decrease of σ_θ (and the consequent

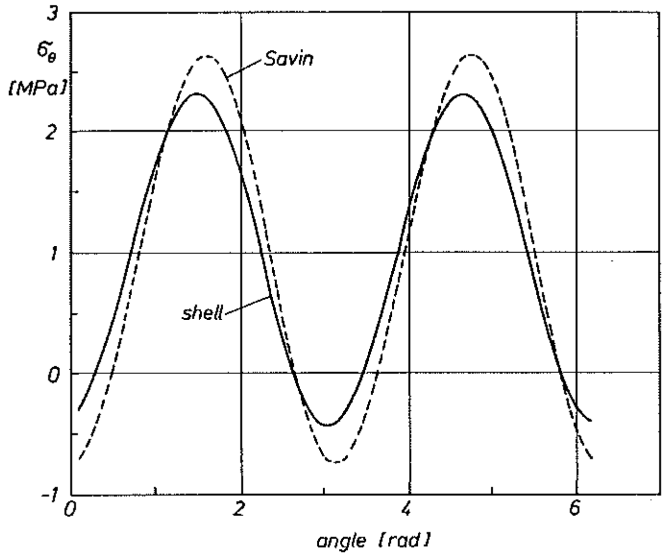

FIG. 3. Angular distribution of the circumferential stress along the interface for $E_1/E = 0.5$.

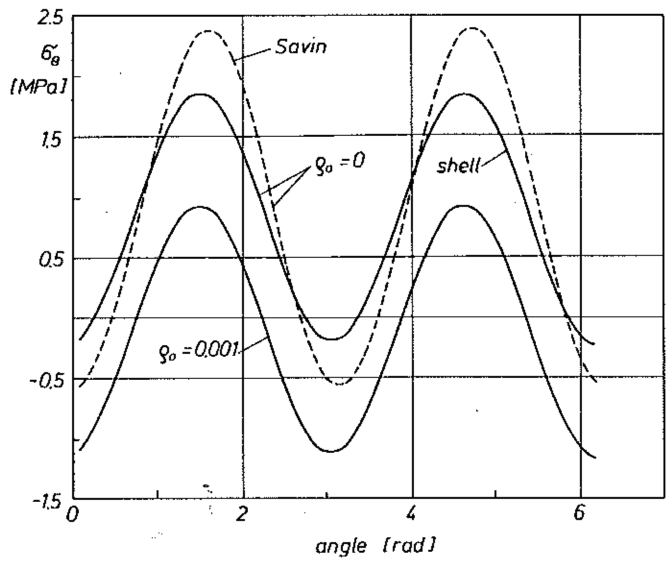

FIG. 4. Angular distribution of the circumferential stress along the interface for $E_1/E = 1.0$ and for two values of $\varrho_0 = R_2 - R$.

increase of σ_r and $\tau_{r\theta}$) when the ratio E_1/E increases. In Fig. 4 the angular distribution of the circumferential stress for $\varrho_0 = R_2 - R = 0.001$ is also given. In this figure it is shown that: (a) Savin's method overpredicts (by a small amount) the magnitude of σ_θ for the angle θ close to $\pi/2$ and $3\pi/2$, and (b) – the initial pressure exerted on the perforation due to an oversized stringer radius leads to a better support of the opening. It was also found that for the same elastic moduli, the shear stresses predicted by the two methods agree better when compared with the previous case of the softer support.

Finally, in Fig. 5 the angular distribution of σ_θ for $E_1/E = 10.0$ is presented. In this case both methods predict approximately the same shear stresses, while Savin's method overestimates the magnitude of the radial and tangential stresses in the direction perpendicular to the direction of application of the remote load, and underestimates the magnitude of both these components of the stress tensor in the direction parallel to the direction of the external normal stress (σ_1).

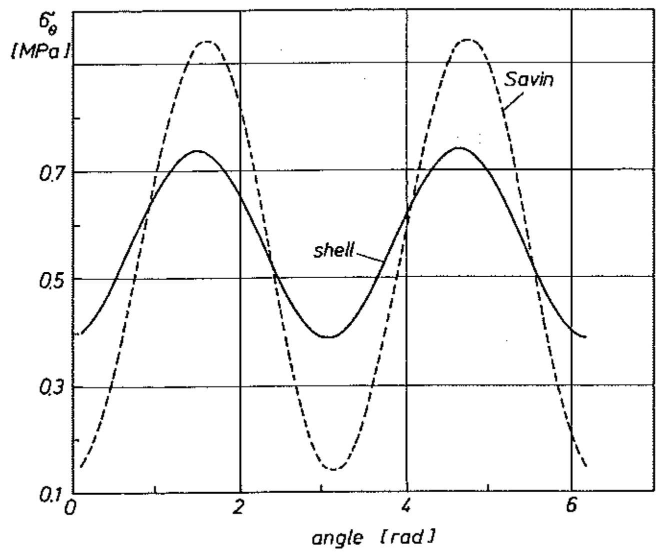

FIG. 5. Angular distribution of the circumferential stress along the interface for $E_1/E = 10.0$.

From the above comparison it is concluded that, although the results given by Savin and the present method are quantitatively comparable for the rigidity of the stringer being not very large compared to the matrix, they give quantitatively different results for the magnitudes of the radial and circumferential stresses for a more rigid support. More specifically, as the ratio E_1/E increases from zero, the difference between the maximum values of σ_θ predicted by the two methods (which is most critical for the design of the support of a perforation) increases monotonically, with the shell theory to predict always the lower value. This may be attributed to the consideration of the shell effect in supporting the hole.

4. Concluding remarks

The method developed in this paper yields the solution of the plane strain elastic problem of a circular cylindrical opening stiffened along its perimeter by a thin shell, which may be oversized prior to its insertion into the opening. The solution of the problem was achieved by forming a system of two regular and two singular integral equations, which are further reduced to a system of linear algebraic equations. The numerical solution of this system gives the angular stress and displacement distribution along the contact boundary of the stringer and the perforation. A comparison of the method with Kirsch's and Savin's solutions for various ratios of the elastic moduli of the stringer to the matrix material was presented, and the convergence of the method was evaluated. From this comparison it was concluded that, although the results given by Savin [4, 5] and the present method were quantitatively comparable in cases of the rigidity of the stringer being not very large compared to the matrix, they give different quantitative results for the magnitudes of the radial and circumferential stresses for much more rigid support. This may be attributed to the consideration of the shell effect in supporting the hole.

The mixed boundary value problem of interaction of the hole support (stringer) with internal cracks in the matrix material is currently under development, and the results will be presented in a future article. It is worth noticing that the above problem has not been solved yet even for the case of simple interface conditions (continuity of stresses and displacements).

Acknowledgements

This work is supported by INTAS of the European Union under the contract INTAS–93–2600. This support is kindly acknowledged.

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Received November 4, 1994.