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A NEW METHOD OF FINDING APPROXIMATE SOLUTIONS OF THE HEAT CONDUCTION EQUATION

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The paper presents a new method of determination of approximate solution of one-dimensional boundary value problem for the heat conduction equation with homogeneous mixed boundary conditions. From the numerical analysis of the problem considered for certain time instants and each particle of the layer, and for chosen particles of the layer in the whole time period, it is evident that the new solution approximates well the classical solution of the same problem achieved by the method of seperation of variables. Numerical analysis of the boundary conditions following from the new approach has shown that they become homogeneous for large values of time.

1. Introduction

Linear boundary and initial boundary value problems constitute now an important section of physics, leading to the description of many processes occurring in the nature [1]. Their analysis is of fundamental importance and is used to determine certain interesting physical phenomena. This necessitates, however, a very tedious procedure of solving the initial boundary value problem given. The classical Fourier method, the method of seperation of variables, is used leading to the solution of the decoupled boundary-value problem for the eigenfunctions and the decoupled problem for "time" coefficients of the Fourier series, of the sought solution. In the classical method all the eigenfunctions satisfy the boundary conditions of the problem considered. They also constitute an orthogonal, closed base in $\mathcal{L}^2$ [1]. In the paper it has been shown however, that the solution to the problem can be represented, with arbitrary accuracy, by the Fourier cosine series whose spatial components do not satisfy the boundary conditions given. The time-dependent coefficients are calculated from an infinite coupled set of ordinary differential equations achieved by the method shown in Sections 3 and 4 of the paper.

The new method has been applied to the equation describing the heat conduction subject to mixed boundary conditions (for a limited Biot number). Solving the set mentioned above for 10 components of the Fourier cosine series, a satisfactory approximation of the classical solution for 5 components of the series has been achieved. The new solutions were derived using the package for symbolic integration of ordinary differential equations from *Mathematica.* From numerical analysis of the boundary conditions for the problem we see that they are satisfied for increasing values of time.

The experience gained shows that the approach presented in the paper can succesfully be used for other linear one-dimensional initial boundary value problems. The generalisation of the method to a wider class of linear problems as well as an attempt of its mathematical justification will be presented in future papers on the subject.

2. Integro-Differential-Boundary Equations for the approximate solution

Let us consider the equation

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0 \qquad \text{for} \quad (x,t) \in (0,L) \times (0,T), \tag{1}$$

the boundary conditions for $t \in [0,T]$

$$\begin{aligned} \frac{\partial U}{\partial x} - \mathrm{Bi}\, U &= 0 \qquad \text{for} \quad x = 0, \\ \frac{\partial U}{\partial x} + \mathrm{Bi}\, U &= 0 \qquad \text{for} \quad x = L, \end{aligned} \tag{2}$$

and the initial conditions for $x \in [0,L]$

$$U(x,0) = U_0(x), \tag{3}$$

where Bi stands for the Biot number.

Let us now consider an infinite set of functions $\left\{\cos\left(\pi k \frac{x}{L}\right) =: \Phi_k(x)\right\}_{k=0}^{\infty}$. The set constitutes a complete orthogonal base in $\mathcal{L}^2(0,L)$. Let us multiply equation (1) by $\Phi_k(x)$ and integrate it from 0 to L:

$$\frac{d}{dt}\int_0^L U(x,t)\Phi_k(x)\,dx - \int_0^L \frac{\partial^2 U}{\partial x^2}(x,t)\Phi_k(x)\,dx = 0, \qquad k = 1,1,2,\ldots. \tag{4}$$

Since $\Phi_k'' = -\gamma_k^2 \Phi_k \in \mathcal{C}(0,L)$ ($\gamma_k = k\pi/L$), then the second term after integration by parts can be presented as

$$\Phi_k(x)\frac{\partial U}{\partial x}(x,t)\Big|_0^L - \Phi_k'(x)U(x,t)\Big|_0^L - \gamma_k^2\int_0^L U(x,t)\Phi_k(x)\,dx.$$

Substituting it to (4) we obtain an infinite set of equations with the right-hand side depending on the boundary conditions for the function U,

$$\frac{d}{dt}\int_0^L U(x,t)\Phi_k(x)\,dx + \gamma_k^2 \int_0^L U(x,t)\Phi_k(x)\,dx = U(0,t)[\Phi_k'(0) - \mathrm{Bi}\,\Phi_k(0)] - U(L,t)[\Phi_k'(L) + \mathrm{Bi}\,\Phi_k(L)], \tag{5}$$

which we are going to call Integro-Differential-Boundary Equations (IDBE). In the equations, at the right-hand side, the boundary conditions (2) have been used.

3. Solution

Let us assume that the solution to the IDBE is a function

$$u : [0,L] \times [0,T) \to R, \tag{6}$$

that is the function u satisfy the equations

$$\frac{d}{dt}\int_0^L u(x,t)\Phi_k(x)\,dx + \gamma_k^2 \int_0^L u(x,t)\Phi_k(x)\,dx = F_k\,, \tag{7}$$

where

$$F_k := u(0,t)[\Phi_k'(0) - \mathrm{Bi}\,\Phi_k(0)] - u(L,t)[\Phi_k'(L) + \mathrm{Bi}\,\Phi_k(L)]. \tag{8}$$

If we now multiply both sides of (7) by $2/L$ and introduce the notation

$$c_k(t) := \frac{2}{L}\int_0^L u(x,t)\Phi_k(x)\,dx,$$

we obtain the following set of equations for c_k :

$$\dot{c}_k + \gamma_k^2 c_k = \frac{2}{L}F_k\,, \qquad k = 0,1,2,\ldots. \tag{9}$$

Since the functions Φ_k are elements of an orthogonal base, then

$$c_k = \frac{2}{L}\int_0^L u(x,t)\Phi_k(x)\,dx$$

are the Fourier coefficients of the solution u to Eq. (7) in the base $\{\Phi_k\}_{k=0}^{\infty}$, and so we can present the solution in the Fourier cosine series form

$$u(x,t) = c_0(t)/2 + \sum_{k=1}^{\infty} c_k(t)\cos(k\,\pi x/L). \tag{10}$$

Application of the equality sign in the above formula is equivalent to the assumption of uniform convergence of the series on $[0, L]$ for $t \in [0, T)$. Such a representation of the solution u enables us to express the functions F_k in terms of the unknown coefficients c_k;

$$F_k = -\mathrm{Bi}\left(\frac{c_0(t)}{2}[1+\cos(k\,\pi)] + \sum_{n=1}^{\infty} c_n(t)[1+\cos(k\,\pi)\cos(n\,\pi)]\right). \tag{11}$$

Inserting (11) into Eq. (9) we obtain the following coupled infinite set of ordinary differential equations for the coefficients of series (10)

$$\begin{gathered}\dot{c}_k + \gamma_k^2 c_k + \frac{2\mathrm{Bi}}{L}\left(\frac{c_0(t)}{2}[1+\cos(k\,\pi)]\right. \\ \left. + \sum_{n=1}^{\infty} c_n(t)[1+\cos(k\,\pi)\cos(n\,\pi)]\right) = 0, \\ c_k(0) = \frac{2}{L}\int_0^L U_0(x)\Phi_k(x)\,dx \qquad k = 0, 1, \ldots .\end{gathered} \tag{12}$$

To evaluate the initial values $c_k(0)$ of coefficients $c_k(t)$ we use the initial conditions $U_0(x)$ of the problem (3) considered.

It is clear that if, instead of the functions $\cos(k\,\pi x/L)$, we substitute the eigenfunctions of the problem, then the set (12) turns into a decoupled set for the Fourier coefficients in the base of the eigenfunctions, known from the method of seperation of variables of the problem.

4. Results

The calculations have been carried out for the initial condition

$$U_0(x) = ax^2 - aLx - aL/\mathrm{Bi},$$

and the data $a = -0.1$, $L = 1$, $\mathrm{Bi} = 0.185$, and for a finite sum of the solution (10)

$$u(x,t) = c_0(t)/2 + \sum_{k=1}^{K} c_k(t)\cos(k\,\pi x/L). \tag{13}$$

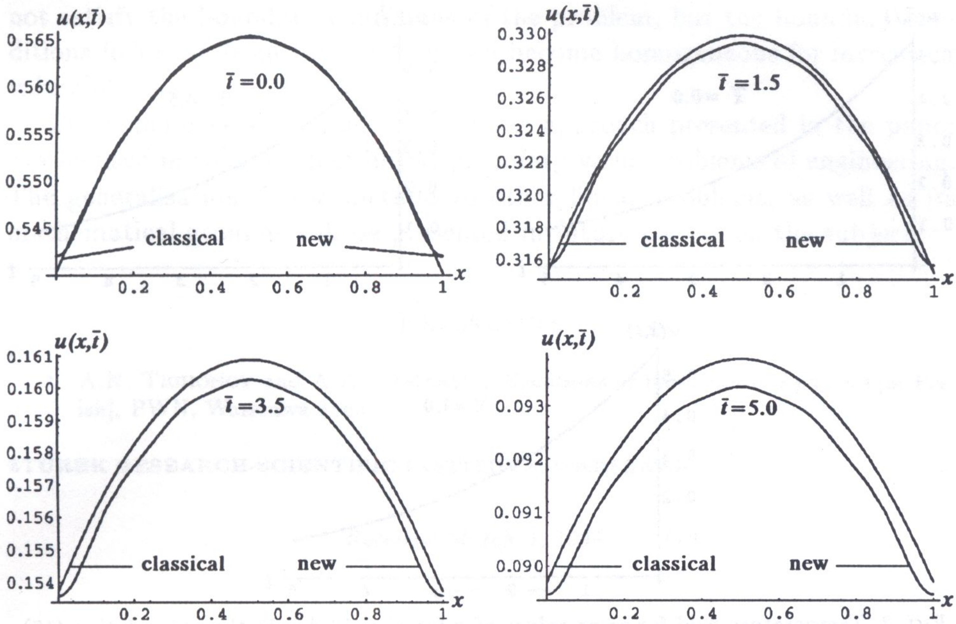

FIG. 1. Temperature field for some values of t for $K = 10$ due to the new solution (13) and for $K = 5$ due to the classical solution (14).

The solutions evaluated for some time instants for every particle of the layer are presented in Fig. 1, but the solutions for chosen particles of the layer in the given time period are shown in Fig. 2. The new results have been compared to the classical solutions of the problem (1)–(3),

$$u_c(x,t) = \sum_{k=1}^{K} a_k \exp(-\beta_k^2 t)\phi_k(x), \tag{14}$$

where

$$\phi_k(x) = \beta_k \cos(\beta_k x) + \mathrm{Bi}\,\sin(\beta_k x)$$

are the eigenfunctions of the problem with the eigenvalues calculated from the equation

$$\mathrm{ctg}(\beta L) = \frac{\beta^2 - \mathrm{Bi}^2}{2\beta\,\mathrm{Bi}},$$

and

$$a_k = \int_0^L U_0(x)\phi_k(x)\,dx / \|\phi_k\|^2.$$

The numerical analysis of the boundary equations (2) following from the new method is illustrated by Fig. 3.

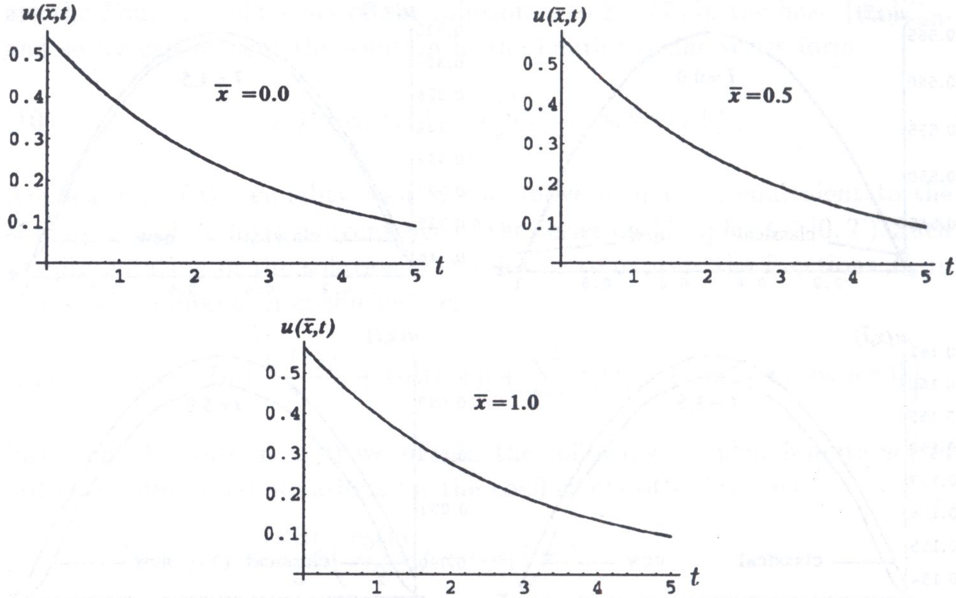

FIG. 2. Temperature field for some values of x for $K = 10$ due to the new solution (13) and for $K = 5$ due to the classical solution (14) (they cannot be distinguished).

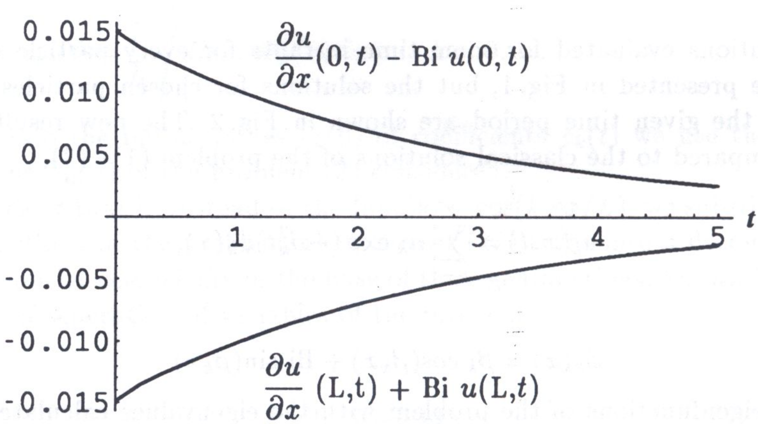

FIG. 3. Boundary error for $K = 10$ of the new solution (13).

5. REMARKS

From the results achieved we see that the solution of the heat conduction equation for homogeneous mixed boundary conditions derived from the Fourier cosine series, approximates well the solution of the same problem derived by the method of seperation of variables. The cosine functions do not satisfy the boundary conditions of the problem, but the boundary conditions following from the new solution become homogeneous for increasing values of time.

The experience gained shows that the approach presented in the paper can be used in solving other initial boundary value problems of engineering. The generalisation of the method to other linear problems, as well as its mathematical grounds will be presented in future papers on the subject.

References

1. A.N. Tichonov and A.A. Samarski, *Equations of mathematical physics* [in Polish], PWN, Warszawa 1963.

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