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POLYNOMIAL BOUNDARY PERTURBATION FOR OPTIMAL PLASTIC DESIGN OF HEADS OF PLANE TENSION MEMBERS

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The paper presents the boundary perturbation method applied to optimal plastic shape design. Perfect plasticity is assumed. The procedure consists of two steps: determination of a class of fully plastic solutions in the limit state, and then the choice of the optimal shape from among those solutions. A trigonometric polynomial with *n* terms is assumed for the general solution. Final results are given for a binomial. Moreover, a further step towards optimization is done by using extrapolation. The results are verified by the ADINA program.

1. Introduction

The method of boundary perturbation is rather seldom applied to optimal shape design. In a series of papers by Kordas and her collaborators, started in 1970, the boundary perturbation method was used to investigate the fully plastic states at the stage of collapse of various perfectly plastic structural elements. KORDAS and ŻYCZKOWSKI [10] considered noncircular shapes of cylinders under pressure; KORDAS [5] discussed pipe-lines of variable diameters; KORDAS and SKRABA [9] analyzed cylinders under pressure with bending; KORDAS [6] presented a general approach to the problem under consideration; KORDAS [7] considered noncircular shapes of disks under pressure; DOLLAR and KORDAS [2] discussed the problem of frame corners subject to bending, tension and shear; KORDAS and POSTRACH [8] analyzed the rotating disks. Full plastification at the stage of collapse is the first step towards optimization, since the material in rigid or elastic zones at the stage of collapse is not properly utilized. In many cases, however, the above condition is not sufficient, and then certain additional optimization proves to be necessary.

Examples of such additional optimization have been presented in the papers by BOCHENEK, KORDAS and ŻYCZKOWSKI [1], and by EGNER, KORDAS and ŻYCZKOWSKI [3]. The second paper is devoted to the optimal design of yoke elements, and namely to plane problems of plasticity with two boundary conditions along each contour; it turns out that they all may be satisfied simultaneously, and optimal shape ensuring the limit load-carrying capacity to reach a maximum can be found. In both problems the circular (annular) shapes were subject to perturbations.

In the present paper, optimal plastic design of heads of plane tension members, shown in Fig. 1 is discussed, result of the paper [4] being generalized. Assume plane strain conditions and suppose that the stress in the tension member is at the yield point. Then the force transmitted is determined. The transversal dimension (thickness) h is large, the external boundary is stress-free and minimal volume of the head is looked for.

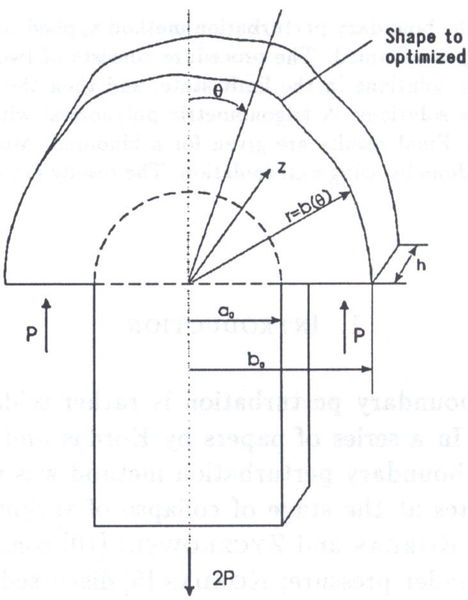

FIG. 1. Shape of a plane head to be optimized.

Two simple solutions of this problem were given by SZCZEPIŃSKI and SZLAGOWSKI [11]. The simplest one is based directly on the solution for a circular cylinder without perturbations. Assume that inside the semi-circle ($0 < r < a$, $-\pi/2 < \theta < \pi/2$), the uniform triaxial tension $\sigma_r = \sigma_\theta = \sigma_z = 2\sigma_0/\sqrt{3}$ is necessary to make the tension member fully plastic; then, considering the cylinder $a < r < b$ to be fully plastic we find $b/a = e$, and the volume equals

$$V = \frac{\pi}{2} b^2 h = 11.61 a^2 h. \tag{1.1}$$

Another solution given in that paper for a rectangular head is based on piece-wise uniform stress fields. Under the assumption of six such stress fields, five of which correspond to fully plastic states, the authors obtained the volume

$$V = 4a^2 h \operatorname{ctg} 22.5^\circ = 9.66a^2 h, \tag{1.2}$$

i.e. by 16.8 percent smaller than the volume given by Eq. (1.1).

W. EGNER, Z. KORDAS and M. ŻYCZKOWSKI obtained in [4] the volume

$$V = 0.835\frac{\pi}{2}e^2 ah = 9.69a^2 h, \tag{1.3}$$

being by 16.5 per cent less than the basic result.

The result (1.3) is not particularly valuable from the point of view of optimal design, since the volume obtained is even slightly larger than (1.2). There are two reasons for that: first, in the series determining the boundary, one term only was retained and it was not sufficient to describe properly the optimal shape. Second, no extrapolation procedure was employed and hence the small parameter in the perturbation method was very strongly limited, namely the condition $|\alpha| \leq 0.2$ was assumed. In the present paper these drawbacks are removed. First, we consider a polynomial perturbation, much more flexible even if two terms only are retained. Second, we use an efficient extrapolation procedure and much larger small parameters are then admissible. Hence, the final result will be much better than (1.3), namely the volume will be much smaller.

2. Assumptions

In this paper the boundary perturbation method will be used to determine the optimal shape of the head. We adopt the following assumptions:

- the material is perfectly plastic and incompressible, subject to the Huber-Mises-Hencky yield condition;
- small strains are assumed throughout the paper;
- Hencky-Ilyushin or Levy-Mises constitutive equations are employed;
- the head is in plane strain conditions.

Then from the law of similarity of deviators we obtain $\sigma_z = \sigma_m$ (mean stress), the stress σ_z may be eliminated and the problem becomes statically pseudo-determined. After this elimination, two equilibrium equations and one yield condition determine three unknown stresses σ_r, σ_θ, $\tau_{r\theta}$. In this case, the yield condition is given by

$$(\sigma_r - \sigma_\theta)^2 + 4\tau_{r\theta}^2 = 4\tau_0^2 . \tag{2.1}$$

Consider now a circular shape of the head as the basic solution. Under the assumption of uniform radial loading along the inner contour $r = a$ and stress-free outer contour $r = b$, we find the stress distribution in the fully plastic state

$$\sigma_r = -\frac{2}{\sqrt{3}}\sigma_0 \ln\frac{r}{b}, \qquad \sigma_\theta = -\frac{2}{\sqrt{3}}\sigma_0\left(1 + \ln\frac{r}{b}\right), \qquad \sigma_z = -\frac{2}{\sqrt{3}}\sigma_0\left(\frac{1}{2} + \ln\frac{r}{b}\right), \qquad \tau_{r\theta} = 0, \tag{2.2}$$

and the strain distribution is as follows

$$\varepsilon_r = \frac{C}{r^2}, \qquad \varepsilon_\theta = -\frac{C}{r^2}, \qquad \varepsilon_z = \tau_{r\theta} = 0, \tag{2.3}$$

where $\sigma_0 = \tau_0\sqrt{3}$ is the yield-point stress at uniaxial tension and C is an arbitrary constant. The limit internal pressure may be found from the boundary condition at the internal contour $r = a$, namely

$$p_a = -\frac{2}{\sqrt{3}}\sigma_0 \ln\frac{a}{b} = \frac{2}{\sqrt{3}}\sigma_0, \tag{2.4}$$

and $b = ea$. The solutions (2.2) and (2.3) with the volume V given by Eq. (1.1), will be regarded as the basic solution for subsequent perturbations and will be labelled by the additional subscript "0".

3. Boundary perturbations

Consider now general cylindrical perturbations of the circular shape of the external boundary of the head. This perturbed shape will be given by $b = b(\theta)$. A more general case of perturbation $b = b(\theta, z)$ was discussed in [4]. Expansions into power series of a certain small parameter α will be introduced. Namely, we write these expansions in the form

$$X = \sum_{i=0}^{\infty} X_i \alpha^i, \tag{3.1}$$

where

$$X = \left[\sigma_r(r,\theta),\ \sigma_\theta(r,\theta),\ \tau_{r\theta}(r,\theta),\ b(\theta)\right]^T. \tag{3.2}$$

The equations of internal equilibrium are linear, hence for all terms of the series (8) they retain their original form

$$\frac{\partial \sigma_{r_i}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta_i}}{\partial \theta} + \frac{\sigma_{r_i} - \sigma_{\theta_i}}{r} = 0, \qquad \frac{\partial \tau_{r\theta_i}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta_i}}{\partial \theta} + 2\frac{\tau_{r\theta_i}}{r} = 0, \tag{3.3}$$

whereas the nonlinear yield condition (2.1), in view of $\tau_{r\theta 0} = 0$, is subject to linearization

$$\sigma_{r_i} - \sigma_{\theta_i} = f_i\left(\sigma_{r_0}, \sigma_{\theta_0}, \dots, \sigma_{r_{i-1}}, \sigma_{\theta_{i-1}}, \tau_{r\theta_{i-1}}\right) \tag{3.4}$$

with $i = 1, 2, \dots$.

For the first two corrections we have

$$f_1 = 0, \qquad f_2 = \frac{1}{\tau_0}\tau_{r\theta_1}^2 . \tag{3.5}$$

Boundary conditions at the free boundary $b = b(\theta)$ with the unit normal n are

$$\begin{aligned} \sigma_r \cos(nr) + \tau_{r\theta}\cos(n\theta) &= 0, \\ \tau_{r\theta}\cos(nr) + \sigma_\theta \cos(n\theta) &= 0, \end{aligned} \tag{3.6}$$

and after expressing the cosine functions in terms of the function $b = b(\theta)$ (KORDAS and ŻYCZKOWSKI [10]), we obtain

$$\begin{aligned} b(\theta)\sigma_r - b'(\theta)\tau_{r\theta} &= 0, \\ b(\theta)\tau_{r\theta} - b'(\theta)\sigma_\theta &= 0. \end{aligned} \tag{3.7}$$

Both these equations determine – in the case of full plastification – one function $b(\theta)$ as well as the boundary values of stresses.

Expansion of Eqs. (3.7) into power series of α is more complicated, since the boundary itself is subject to perturbation. The increment of the independent variable Δr equals

$$\Delta r = \Delta b = \sum_{j=1}^{\infty} b_j(\theta)\alpha^j \tag{3.8}$$

and hence, replacing multiplication of series by a multiple series, we obtain

$$\begin{aligned} \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=0}^{\infty} \alpha^{i+k} \frac{b_i(\theta)}{m!}\frac{\partial^m \sigma_{rk}(r,\theta)}{\partial r^m}\bigg|_{r=b_0} \left[\sum_{j=1}^{\infty} b_j(\theta)\alpha^j\right]^m \\ = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \alpha^{i+k} \frac{b_i'(\theta)}{m!}\frac{\partial^m \tau_{r\theta_k}(r,\theta)}{\partial r^m}\bigg|_{r=b_0} \left[\sum_{j=1}^{\infty} b_j(\theta)\alpha^j\right]^m, \end{aligned} \tag{3.9}$$

$\tau_{r\theta_0} = b_0' = 0$ and similarly for the second Eq. (3.7). In the case under consideration, these expansions (16) are essentially simplified.

4. First-order perturbations

For $j = 1$ the Eqs. (3.3) with the yield condition (3.4) and (3.5) resulting in $\sigma_{r1} = \sigma_{\theta 1}$ may be reduced to one homogeneous hyperbolic equation

$$r^2 \frac{\partial^2 \tau_{r\theta_1}}{\partial r^2} - \frac{\partial^2 \tau_{r\theta_1}}{\partial \theta^2} + 3r \frac{\partial \tau_{r\theta_1}}{\partial r} = 0. \tag{4.1}$$

Solution of this equation may be assumed in the form

$$\tau_{r\theta_1} = \sum_{i=1}^{\infty} f_i(r) \sin \lambda_i \theta, \tag{4.2}$$

where the terms with $\cos \lambda_i \theta$ have been disregarded due to the symmetry requirements.

In the paper [4], just one term of this series was used and hence the results of optimization were not quite satisfactory. Here we consider a trigonometric polynomial with n terms; the final results will be given for a binomial, $n = 2$. Making use of [4] we present the solution in the following form

$$\tau_{r\theta_1} = \frac{1}{r} \sum_{i=1}^{n} (A_i sl_{ri} + B_i cl_{ri}) \sin \lambda_i \theta \tag{4.3}$$

where, for the sake of brevity, we have introduced the notation

$$\begin{aligned} sl_{ri} &= \sin\left(\sqrt{\lambda_i^2 - 1} \ln \frac{r}{b_0}\right), \\ cl_{ri} &= \cos\left(\sqrt{\lambda_i^2 - 1} \ln \frac{r}{b_0}\right), \qquad i = 1, 2, ... n, \end{aligned} \tag{4.4}$$

and λ_i denote the separation constants.

The relevant corrections for normal stresses are

$$\sigma_{r_1} = \sigma_{\theta_1} = \frac{1}{r} \sum_{i=1}^{n} \left[\frac{1}{\lambda_i} (A_i sl_{ri} + B_i cl_{ri}) \cos \lambda_i \theta + \frac{\sqrt{\lambda_i^2 - 1}}{\lambda_i} (A_i cl_{ri} - B_i sl_{ri}) \cos \lambda_i \theta \right] + C_1, \tag{4.5}$$

where C_1 is an integration constant (an additional index i is not needed in this case).

The boundary conditions for the first-order perturbations are found from Eq. (3.9) by equating the coefficients of α at both sides. Substituting Eqs. (2.2) we obtain

$$\frac{2\sigma_0}{\sqrt{3}} b_1(\theta) = -b_0 \sigma_{r_1}|_{r=b_0}, \qquad \frac{2\sigma_0}{\sqrt{3}} b_1'(\theta) = b_0 \tau_{r\theta_1}|_{r=b_0}. \tag{4.6}$$

They determine the unknown constants in Eqs. (4.3) and (4.5) and, first of all, the function $b(\theta)$ corresponding to full plastification of the body. Fulfillment of both Eqs. (4.6) is possible if $A_1 = A_2 = 0$. Finally we obtain

$$\tau_{r\theta_1} = \frac{1}{r}\sum_{i=1}^{n} B_i cl_{ri} \sin\lambda_i\theta, \tag{4.7}$$

$$\sigma_{r_1} = \sigma_{\theta_1} = \frac{1}{r}\sum_{i=1}^{n}\left\{\frac{1}{\lambda_i}B_i cl_{ri}\cos\lambda_i\theta - \frac{\sqrt{\lambda_i^2-1}}{\lambda_i}B_i sl_{ri}\cos\lambda_i\theta\right\} + C_1\,,$$

$$b_1(\theta) = -\frac{\sqrt{3}}{2\sigma_0}\sum_{i=1}^{n}\frac{1}{\lambda_i}B_i\cos\lambda_i\theta - \frac{b_0\sqrt{3}}{2\sigma_0}C_1\,. \tag{4.8}$$

5. Second-order perturbations

Equations (3.3) take now the following form:

$$r^2\frac{\partial^2\tau_{r\theta_2}}{\partial r^2} - \frac{\partial^2\tau_{r\theta_2}}{\partial\theta^2} + 3r\frac{\partial\tau_{r\theta_2}}{\partial r} = r\frac{\partial^2 f_2}{\partial r\partial\theta} + \frac{\partial f_2}{\partial\theta}\,. \tag{5.1}$$

Expanding the right-hand side of Eq. (5.1) we obtain $n(4n-1)$ various terms, since the function f_2, proportional to $\tau^2_{r\theta 1}$ introduces many new terms due to coupling of the individual terms of $\tau^2_{r\theta 1}$. So, we give here just an approximate solution assuming uncoupling: individual terms of the first approximation will be followed by respective terms of the second approximation whereas the cross-products will be omitted. The number of terms $n(4n-1)$ will then be replaced by a much smaller number $3n$. In the case of a binomial, we replace 14 by 6 and we assume that the most important second-order effects will be taken into account.

The boundary conditions for the second-order corrections, resulting from the expansions of (3.9), are much more complicated. However, many terms vanish and finally we obtain

$$\begin{aligned}\frac{2\sigma_0}{\sqrt{3}}b_2 + b_0\sigma_{r_2}|_{r=b_0} + b_0 b_1\frac{\partial\sigma_{r_1}}{\partial r}\bigg|_{r=b_0} - \frac{\sigma_0}{b_0\sqrt{3}}b_1^2 - b_1'\tau_{r\theta_1}|_{r=b_0} &= 0,\\ b_0 b_1\frac{\partial\tau_{r\theta_1}}{\partial r}\bigg|_{r=b_0} + b_0\tau_{r\theta_2}|_{r=b_0} - \frac{2\sigma_0}{\sqrt{3}}b_2' - b_1'\sigma_{\theta_1}|_{r=b_0} &= 0.\end{aligned} \tag{5.2}$$

For each term i of such a polynomial with n terms we obtain

$$\tau_{r\theta_{2i}} = -\frac{B_i C_1 \sqrt{3}}{2\sigma_0 r}\sqrt{\lambda_i^2 - 1}\, sl_{r_i} \sin \lambda_i \theta$$
$$-\frac{B_i^2\sqrt{3}}{8\sigma_0 \lambda_i r^2}\left[1 + \sqrt{\lambda_i^2 - 1}\, sl_{2r_i} + \left(2\lambda_i^2 - 1\right) cl_{2r_i}\right] \sin 2\lambda_i \theta ,$$

(5.3)
$$\sigma_{r_{2i}} = -\frac{B_i C_1 \sqrt{3}}{2\sigma_0 \lambda_i r}\left[\sqrt{\lambda_i^2 - 1}\, sl_{r_i} + (\lambda_i^2 - 1) cl_{r_i}\right] \cos \lambda_i \theta$$
$$+\frac{B_i^2\sqrt{3}}{8\sigma_0 \lambda_i^2 r^2}\left[-2\lambda_i^2 + (2\lambda_i^2 - 1)\sqrt{\lambda_i^2 - 1}\, sl_{2r_i} - (3\lambda_1^2 - 1) cl_{2r_i}\right] \cos 2\lambda_i \theta$$
$$+\frac{B_i^2\sqrt{3}}{8\sigma_0 \lambda_i^2 r^2}\left(\lambda_i^2 - \sqrt{\lambda_i^2 - 1}\, sl_{2r_i} + cl_{2r_i}\right) + C_2,$$

(5.4)
$$\sigma_{\theta_{2i}} = -\frac{B_i C_1 \sqrt{3}}{2\sigma_0 \lambda_i r}\left[\sqrt{\lambda_i^2 - 1}\, sl_{r_i} + (\lambda_i^2 - 1) cl_{r_i}\right] \cos \lambda_i \theta$$
$$+\frac{B_i^2\sqrt{3}}{8\sigma_0 \lambda_i^2 r^2}\left[(2\lambda_i^2 - 1)\sqrt{\lambda_i^2 - 1}\, sl_{2r_i} - (\lambda_i^2 - 1) cl_{2r_i}\right] \cos 2\lambda_i \theta$$
$$+\frac{B_i^2\sqrt{3}}{8\sigma_0 \lambda_i^2 r^2}\left[-\lambda_i^2 - \sqrt{\lambda_i^2 - 1}\, sl_{2r_i} - (2\lambda_i^2 - 1) cl_{2r_i}\right] + C_2 ,$$

$$b_2(\theta) = \sum_{i=1}^{n}\left(-\frac{3B_i^2}{16\sigma_0^2 b_0} + \frac{3B_i^2}{16\sigma_0^2 b_0}\cos 2\lambda_i \theta\right) + \frac{3b_0 C_1^2}{8\sigma_0^2} - \frac{b_0 C_2 \sqrt{3}}{2\sigma_0},$$

where C_2 is a new constant (an additional index i is also not needed in this case).

6. The condition of constant transmitted force

Let us now use the condition of constant force transmitted by the cross-section $\theta = \pi/2$, $0 < r < a$, equal to the force acting on the area $\theta = \pi/2$, $a < r < b(\pi/2)$ (half of the force transmitted by the tension member), to determine the parameters C_1 and C_2. The force calculated from the condition of full plastification of the tension member, and the reaction force are given by

(6.1)
$$P = \frac{2}{\sqrt{3}} h a_0 \sigma_0 = -h \int_{a_0}^{b(\pi/2)} \sum_{i=0}^{\infty} \sigma_{\theta_i}(r, \pi/2)\alpha^i \, dr,$$

where $i = 0, 1, 2...$ and the upper limit of integration is given by the power series for b. In what follows, we restrict the polynomial (4.3) to a binomial, $n = 2$.

After integration we demand all the corrections of the reaction force to be zero (coefficients of subsequent powers of α, necessary to satisfy (6.1)) and we obtain

$$
\begin{aligned}
(6.2)\qquad C_1 = -\frac{1}{a_0}\Bigg\{ & \frac{S}{\lambda_1}\cos\left(\lambda_1\frac{\pi}{2}\right)\left[\frac{sl_{a_1}}{\sqrt{\lambda_1^2-1}} + cl_{a_1}\right] \\
& -\frac{C}{\lambda_2}\cos\left(\lambda_2\frac{\pi}{2}\right)\left[\frac{sl_{a_2}}{\sqrt{\lambda_2^2-1}} + cl_{a_2}\right]\Bigg\}, \\
C_2 = C_2(\lambda_1, & \lambda_2, S, C, \overline{\alpha}).
\end{aligned}
$$

Formula for C_2 is very complicated, and will not be given here.

In view of two independent constants B_1 and B_2 appearing in the binomial, we introduce their "intensity" $\sqrt{B_1^2 + B_2^2}$; the shares of B_1 and B_2 in this intensity are denoted by S and C, and the new dimensionless small parameter α will also include this intensity.

$$
(6.3)\qquad S = \frac{B_1}{\sqrt{B_1^2+B_2^2}}, \qquad C = -\frac{B_2}{\sqrt{B_1^2+B_2^2}}, \qquad \overline{\alpha} = \frac{\sqrt{3}\sqrt{B_1^2+B_2^2}}{2\sigma_0 b_0}\alpha .
$$

Then we construct the statically admissible stress fields for the semi-circle $r < a_0$, $-\pi/2 < \theta < \pi/2$. They have to satisfy the conditions of equilibrium, the continuity conditions along the arc $r = a_0$ and along the segment $\theta = \pi/2$, and finally the yield condition. Particular solution of this problem is given in the paper by Egner, Kordas, Życzkowski [4], for a monomial; however, for a binomial a similar procedure may be used. There are no troubles in satisfying the yield condition (2.1) as an inequality, since in the basic solution corresponding to (2.2) we have $\sigma_r = \sigma_\theta = \sigma_z = 2\sigma_0/\sqrt{3}$, and the effective stress is simply equal to zero.

7. Optimal shape design

Now we can formulate the optimization problem. We look for minimal volume V of the head, under the basic constraint $P = \text{const}$. The volume is determined by the integral

(7.1)
$$V = 2h \int\limits_0^{\pi/2} \left(\sum_{i=0}^{\infty} b_i(\theta)\alpha^i \right)^2 d\theta .$$

After calculating the integral we obtain $V = V(\lambda_1, \lambda_2, S, C, \alpha)$, with additional constraint $S^2 + C^2 = 1$ eliminating for example the parameter C. Then we have four free parameters. In the first variant we look for a minimal volume V without any additional constraints, it means with α, λ_1, λ_2, S as the design variables. Numerical optimization yields $\lambda_1 = 2.39$, $\lambda_2 = 3.72$, $S = 0.58$, $\alpha = -0.30$. Minimal value of V amounts to

$$V = 0.71V_0$$

what means that it is by 29 percent smaller than the basic, unperturbed solution (1.1). In the obtained shape we can observe concavities which are ascribed to poor convergence of the series for $|\alpha| = 0.30$. Then, in the second variant we impose additional constraint $|\alpha| \le 0.20$. Optimum solution occurs

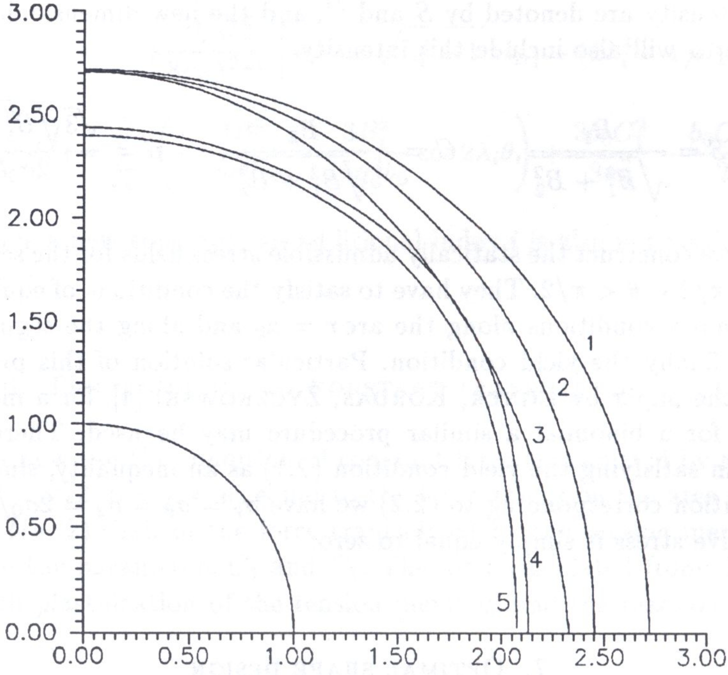

FIG. 2. *1* – unperturbed solution, *2* – solution from paper [1], with constraint $b'(\pi/2) = 0$, $V = 0.90V_0$, *3* – solution from paper [1], without any constraints, $V = 0.84V_0$, *4* – solution without extrapolation ($\lambda_1 = 2.38$, $\lambda_2 = 3.72$, $\alpha = -0.2$), $V = 0.74V_0$, *5* – solution with extrapolation ($\lambda_1 = 2.29$, $\lambda_2 = 3.71$, $\alpha = -0.2$), $V = 0.70V_0$.

for $\lambda_1 = 2.38$, $\lambda_2 = 3.72$, $S = 0.72$, $\alpha = -0.20$, and the minimal volume is equal

$$V = 0.74V_0,$$

i.e. is by 26 percent smaller than the basic one. The solutions are shown in Fig. 2. In this figure, the shape of the head shown by the external curve, presents the unperturbed solution, it means $b = b_0 = a_0 e$. The next two curves present the solution derived in the paper [4]. The first of them (No. 2) shows the solution with constraint $b'(\pi/2) = 0$, and the second one without any constraints. These solutions were obtained by using only one term of the trigonometric series (4.2). In the present solution we have used two terms of that series.

Finally we impose additional constraint $b'(\pi/2) = 0$ (vertical tangent at the point of the contact) and $|\alpha| \leq 0.20$. Then we arrive at the following solution: $\lambda_1 = 2.31$, $\lambda_2 = 3.66$, $S = 0.62$ and $\alpha = -0.193$. The minimal volume:

$$V = 0.75V_0$$

is by 25 percent lower than the basic one.

8. Extrapolation

Convergence of the power series (3.1) in the vicinity of the optimal solution is rather poor, and calculation of the subsequent perturbations seems to be very cumbersome, even under the assumption of uncoupling as it was done in Sec. 5. However, making use of three consecutive terms of a series we may look for an extrapolation leading to a more accurate evaluation of the sum of this series than just the partial sum of its three terms.

Denote by $b_n x^n$ the terms of a series, $n = 0, 1, 2 ...$, and by a_n the terms of the sequence of partial sums of this series, $a_n = \sum_{k=0}^{n} b_k x^k$. We look for an approximation of a_∞. Chances of extrapolation are connected with the model series employed to be as regular as possible. The most known is Aitken's formula based on the geometric series

$$a_\infty = b_0 + \frac{b_1^2 x}{b_1 - b_2 x} = a_0 + \frac{(a_1 - a_0)^2}{2a_1 - a_0 - a_2}. \tag{8.1}$$

Three first terms of this series coincide with $b_0, b_1 x, b_2 x^2$. A certain deficiency of (8.1) is seen in the case $b_1 = b_2 x$, then $a_\infty \to \infty$. We use here another formula, proposed by ŻYCZKOWSKI [12] and based on binomial power series, also with three terms coinciding with b_0, $b_1 x$, $b_2 x^2$, namely

$$a_\infty = b_0 \left[1 + \frac{b_1 x}{m b_0}\right]^m, \qquad m = \frac{b_1^2}{b_1^2 - 2b_0 b_2}. \tag{8.2}$$

In our case, where $b_1 = b_1(\theta)$, $b_2 = b_2(\theta)$, and α is a small parameter, extrapolation (38) takes the form

$$b(\theta) = b_0 \left(1 + \frac{b_1(\theta)}{m(\theta) b_0}\alpha\right)^{m(\theta)}, \qquad m(\theta) = \frac{[b_1(\theta)]^2}{[b_1(\theta)]^2 - 2b_0 b_2(\theta)}. \tag{8.3}$$

This approach was applied first to the last but one solution of the previous paragraph. The result is shown in Fig. 2 (the most internal curve). Here we can observe that this solution yields smaller volume than the previous one. In what follows, we shall apply the extrapolation formula (8.3) to larger values of $|\alpha|$ as well.

9. Verification of the results obtained by FEM program "ADINA"

Finally, the solutions obtained earlier by the boundary perturbation method, are verified using the finite element method. It was done making use of the program ADINA. In order to model the structure, $2D$ solid element was used. This element is in plane strain conditions. The material is perfectly plastic, subjected to the Huber-Mises-Hencky yield condition. Each node of the element has two degrees of freedom (u, v) and moreover, $\varepsilon_{xx} = 0$, $\tau_{xy} = 0$ and $\tau_{xz} = 0$ (axis x is here perpendicular to the lateral surface of the element). The structure is loaded by the forces acting on the tension member. Moreover, along the line $\theta = \pi/2$, $a_0 < r < b(\pi/2)$ a surface without friction is assumed. The basic problem was solved first, it means the circular shape not corrected, $b = a_0 e$ (Fig. 3). It is seen from this figure that the carrying capacity is limited by the tension member (head too large). As the next example, a smaller circular shape was taken with the condition $b = 2a_0$ (Fig. 4). In this case we can see that the carrying capacity is limited by the capacity of the head. Both the presented solutions are not optimal and have drawbacks which result from the circular shape of the head, as well as from wrong proportions between the head volume and volume of the tension member.

As a further step toward optimal solution, non-circular shapes were considered, obtained earlier by using the boundary perturbation method. As

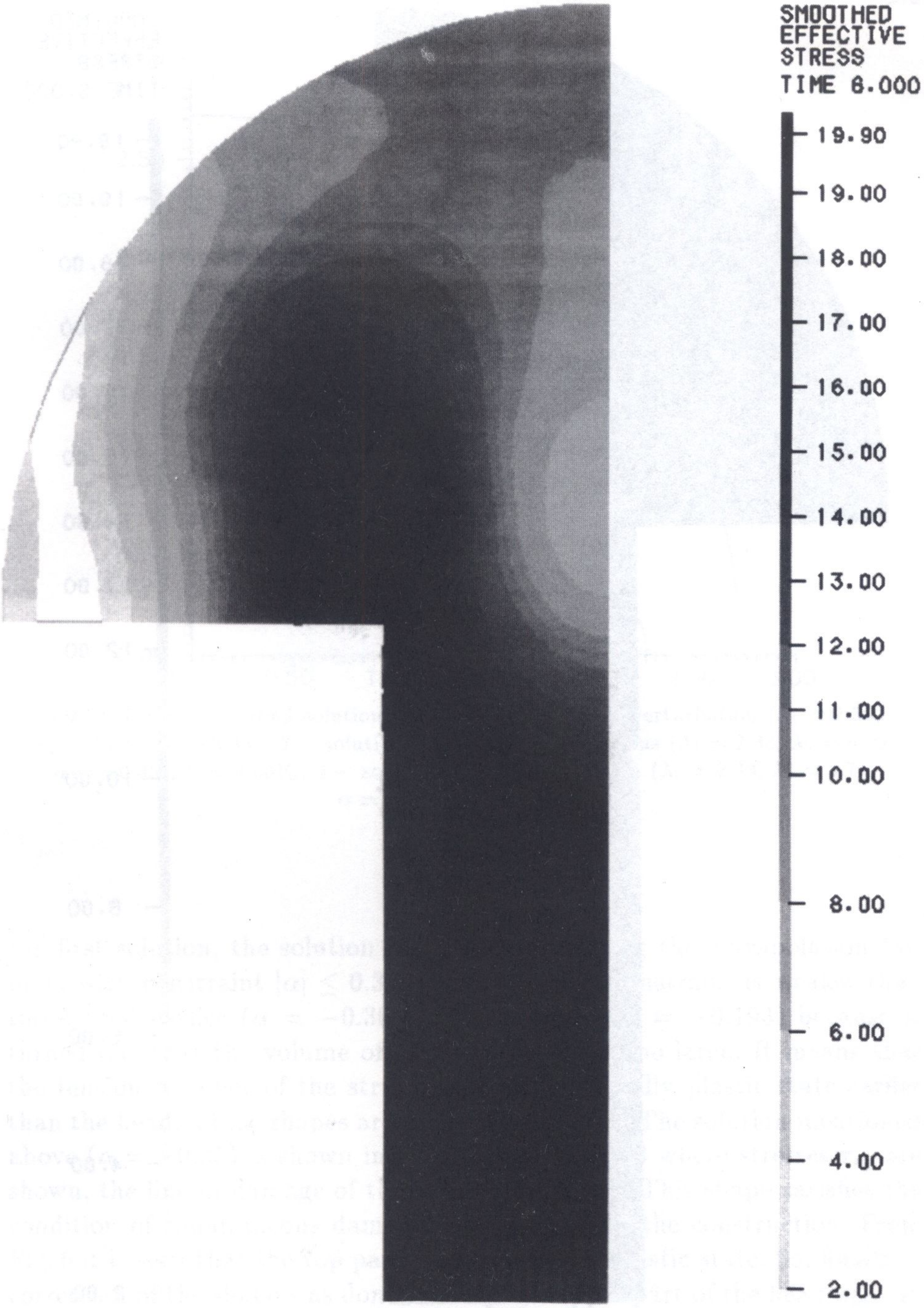

FIG. 3. The circular shape not corrected, $b_0 = a_0 e$, head too large.

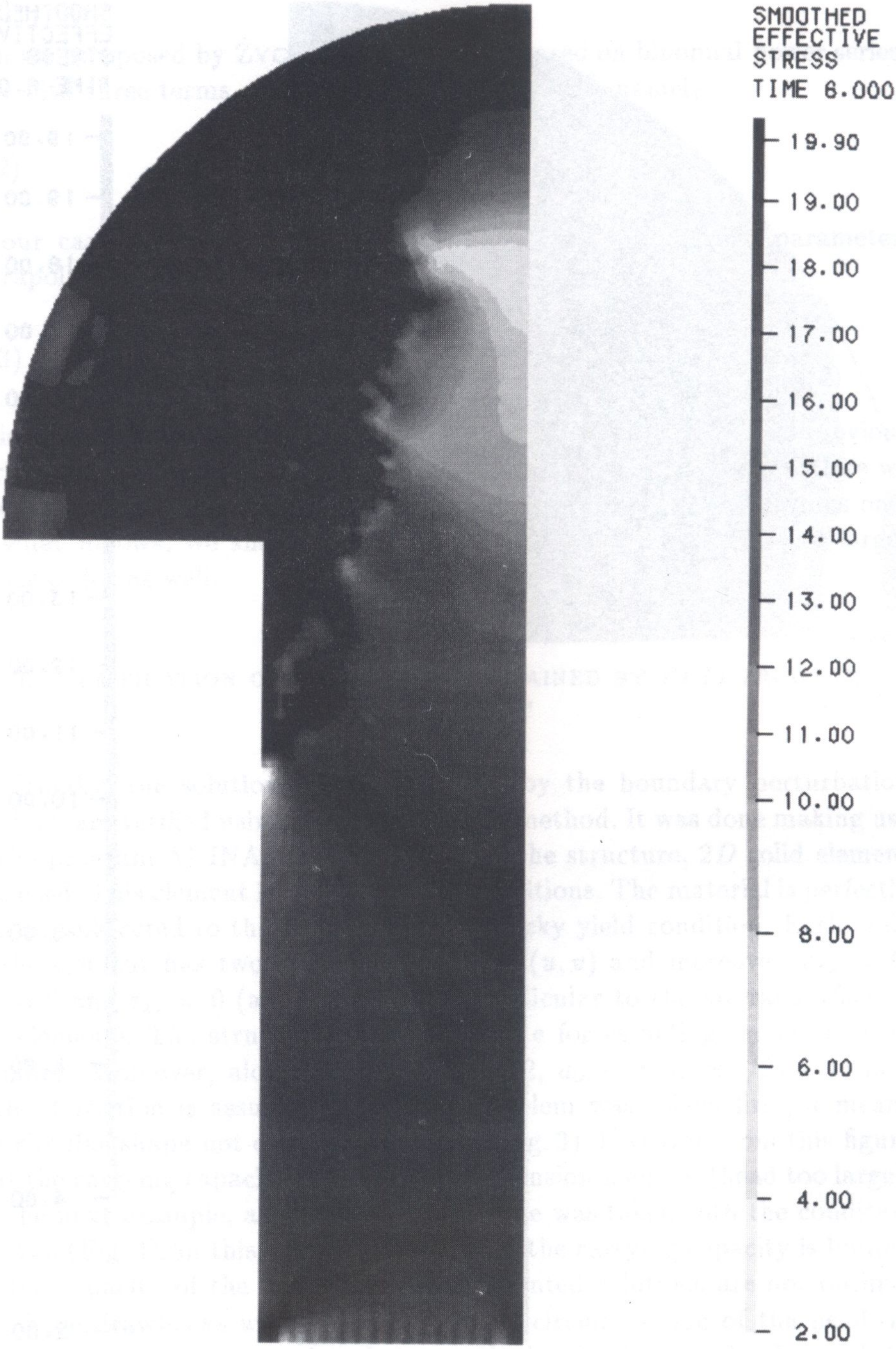

FIG. 4. The circular shape not corrected with $b_0 = 2a_0$, head too small, $V = 0.54V_0$.

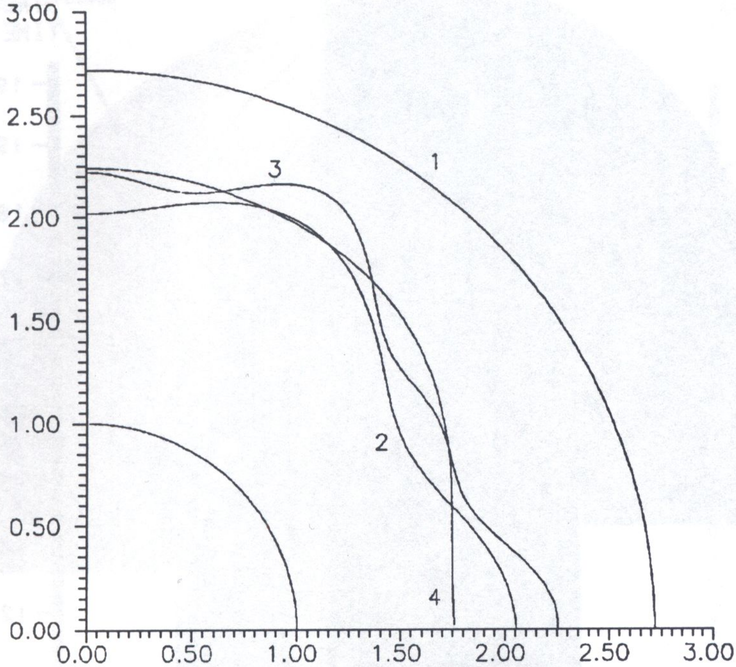

FIG. 5. *1* – unperturbed solution, *2* – solution with first perturbation ($\lambda_1 = 2.38$, $\lambda_2 = 6.09$, $\alpha = -0.35$), *3* – solution with both perturbations ($\lambda_1 = 2.38$, $\lambda_2 = 6.09$, $\alpha = -0.35$), $V = 0.60V_0$, *4* – solution with extrapolation ($\lambda_1 = 2.33$, $\lambda_2 = 3.71$, $\alpha = -0.35$), $V = 0.58V_0$.

the first solution, the solution obtained by applying the extrapolation formula with constraint $|\alpha| \leq 0.35$ was taken. This constraint is weaker than those used earlier ($\alpha = -0.30$, $\alpha = -0.20$ and $\alpha = -0.193$) because it turned out that the volume of the head was still too large. It means that the tension member of the structure will be in a fully plastic state earlier than the head. These shapes are presented in Fig. 5. The solution mentioned above ($\alpha = -0.35$) is shown in Fig. 6, 7, 8. In Fig. 8 where stresses τ_{zy} are shown, the line of damage of the head is presented. This shape satisfies the condition of simultaneous damage in both parts of the construction. From Fig. 6 it is seen that the top part of the head is in elastic state. So, finally, a correction of the shape was done, namely the upper part of the head was cut off by a straight line. The result is shown in Fig. 9, 10. This head is evidently the best from among those considered in the present paper (the elastic zone is the smallest, and the force is kept constant). The head cannot be lower in view of the shearing stresses to be transmitted.

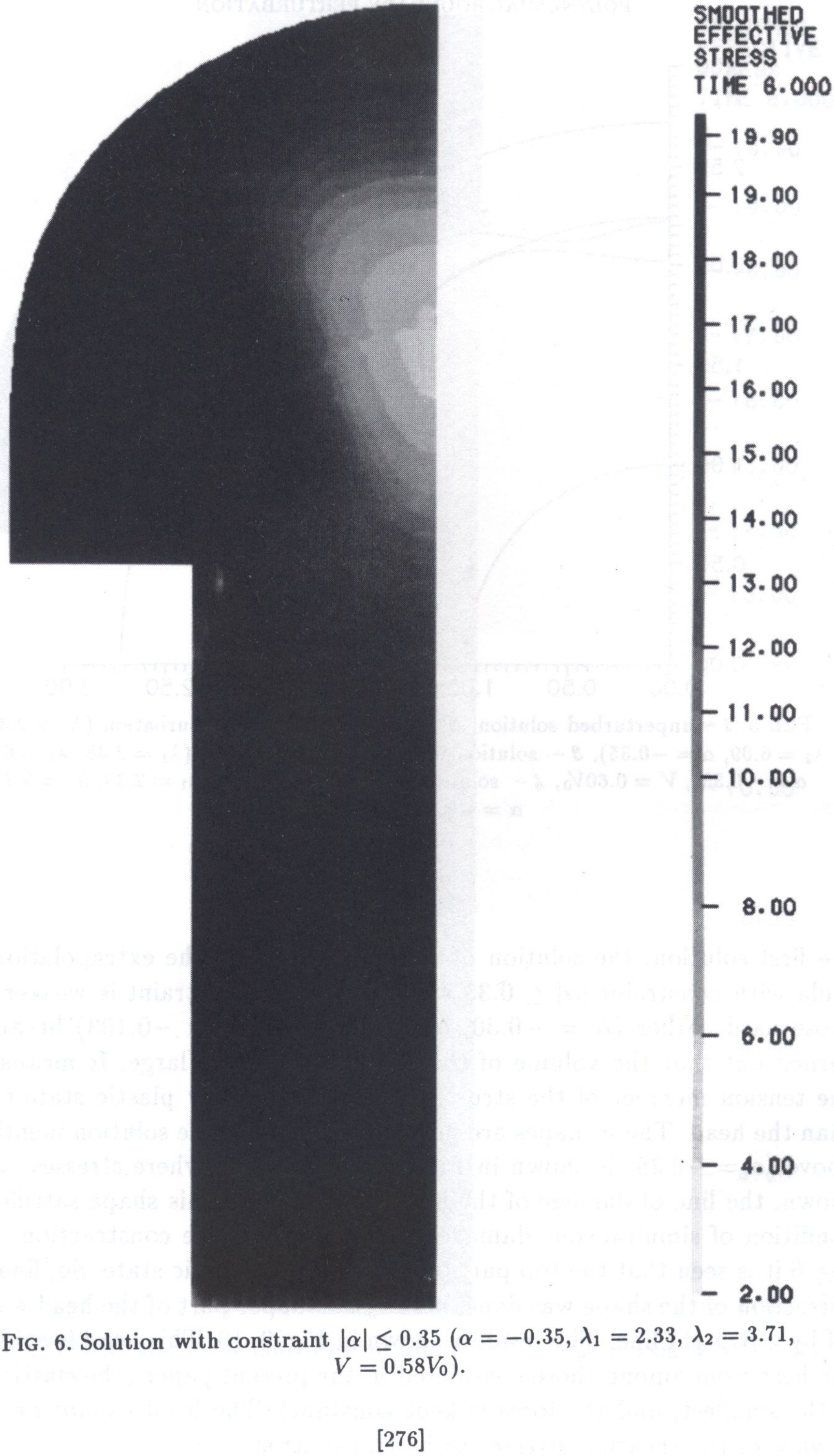

FIG. 6. Solution with constraint $|\alpha| \leq 0.35$ ($\alpha = -0.35$, $\lambda_1 = 2.33$, $\lambda_2 = 3.71$, $V = 0.58V_0$).

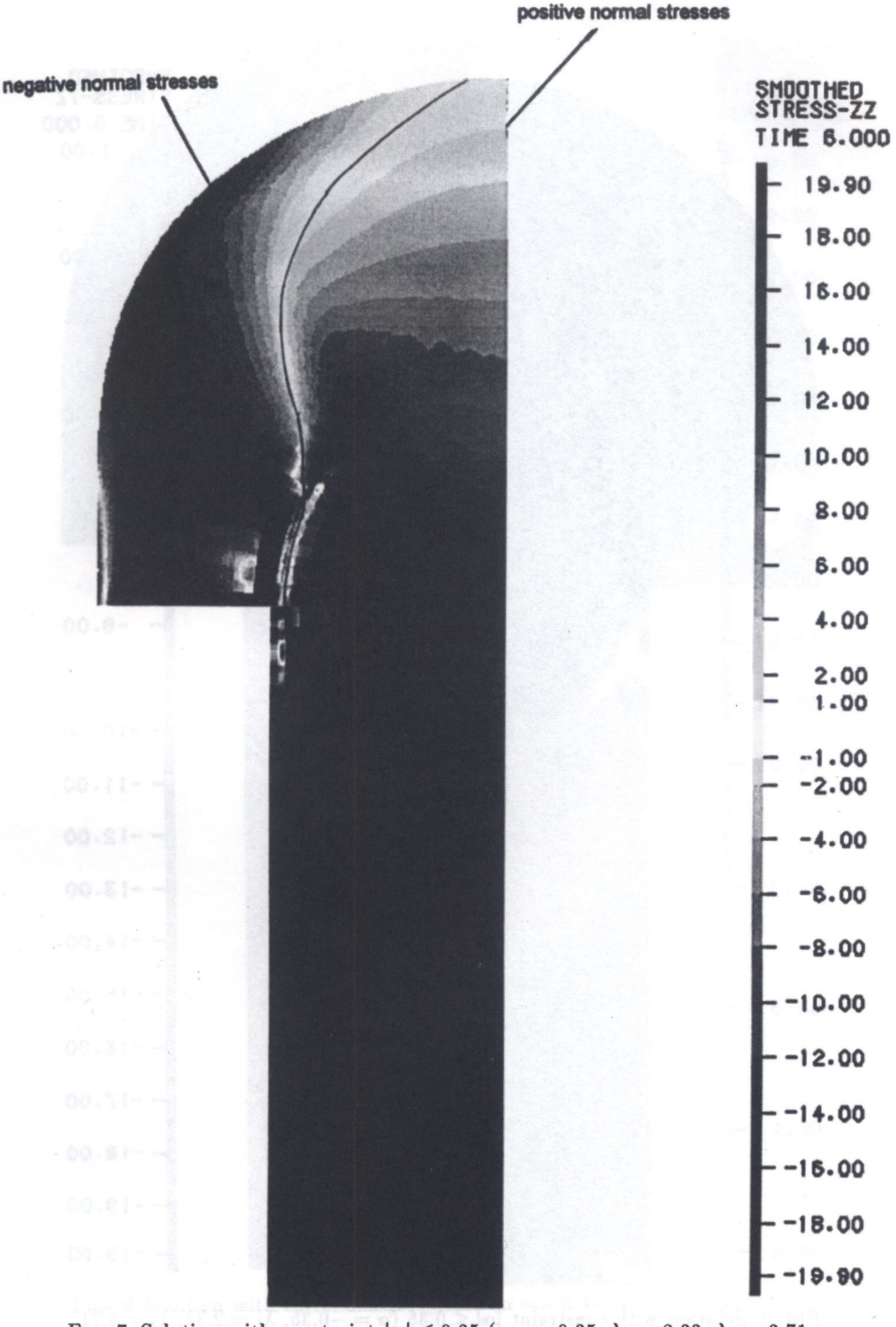

FIG. 7. Solution with constraint $|\alpha| \leq 0.35$ ($\alpha = -0.35$, $\lambda_1 = 2.33$, $\lambda_2 = 3.71$, $V = 0.58V_0$).

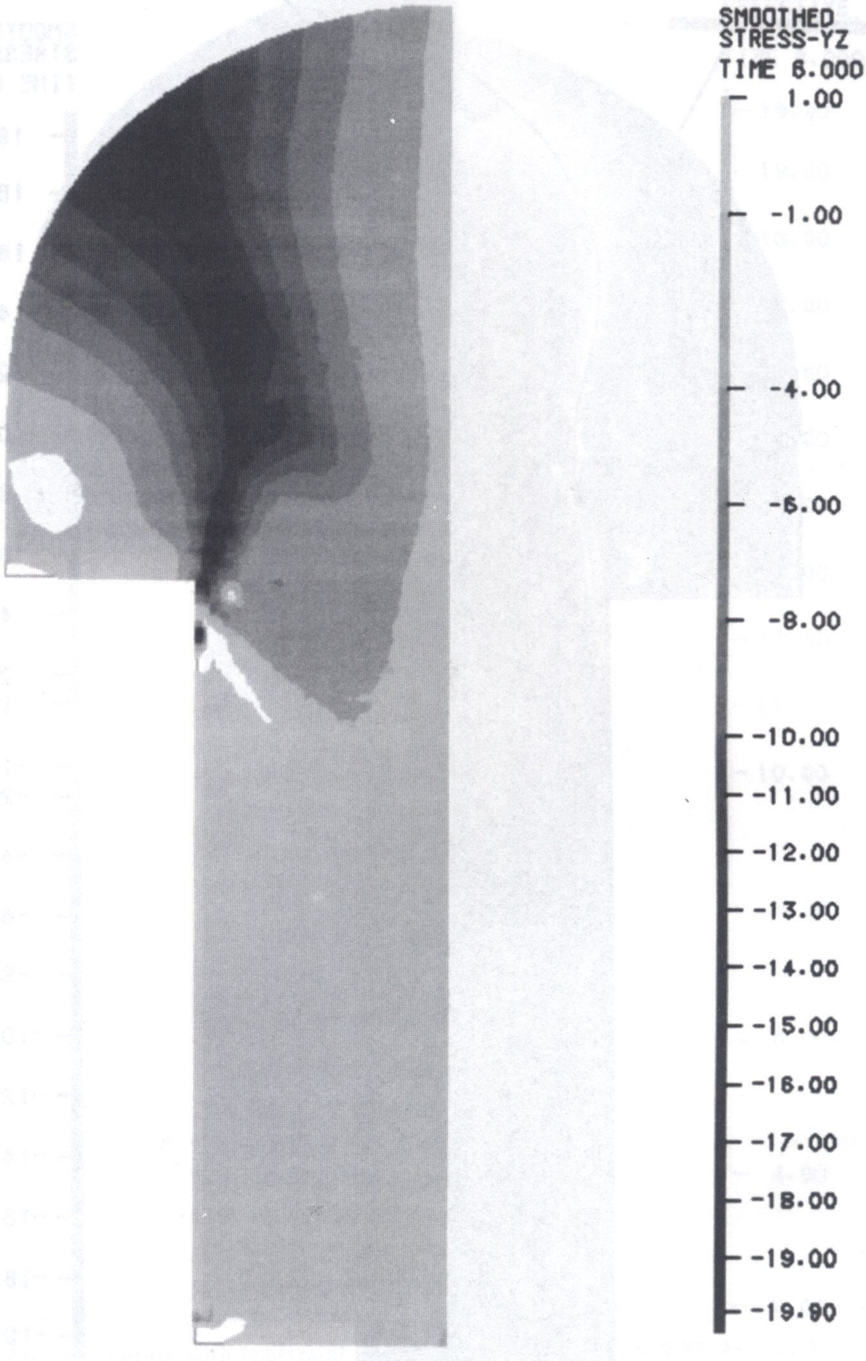

FIG. 8. Solution with constraint $|\alpha| \leq 0.35$ ($\alpha = -0.35$, $\lambda_1 = 2.33$, $\lambda_2 = 3.71$, $V = 0.58V_0$).

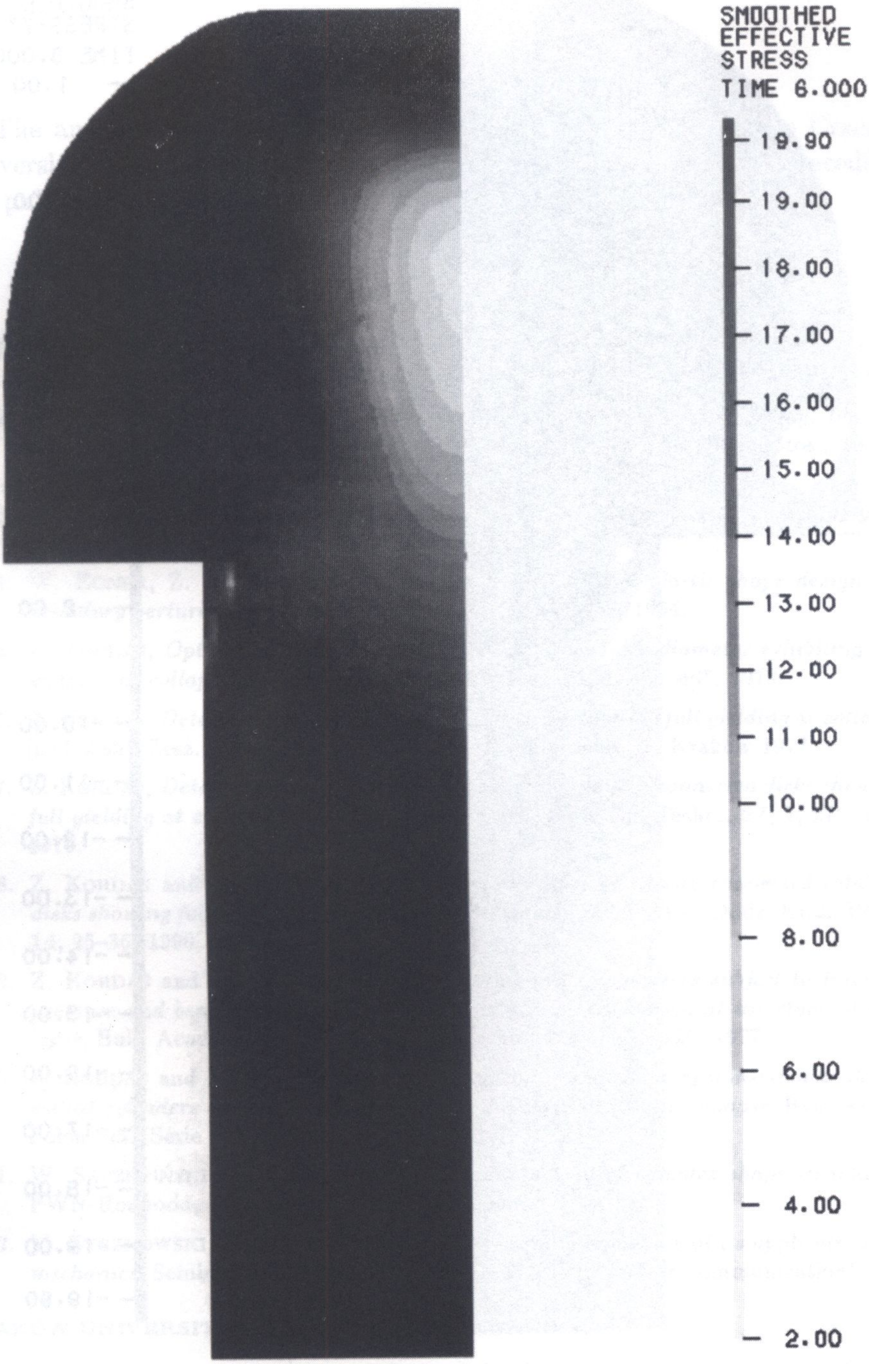

FIG. 9. Solution with correction $|\alpha| \leq 0.35$ ($\alpha = -0.35$, $\lambda_1 = 2.33$, $\lambda_2 = 3.71$, $V = 0.56V_0$).

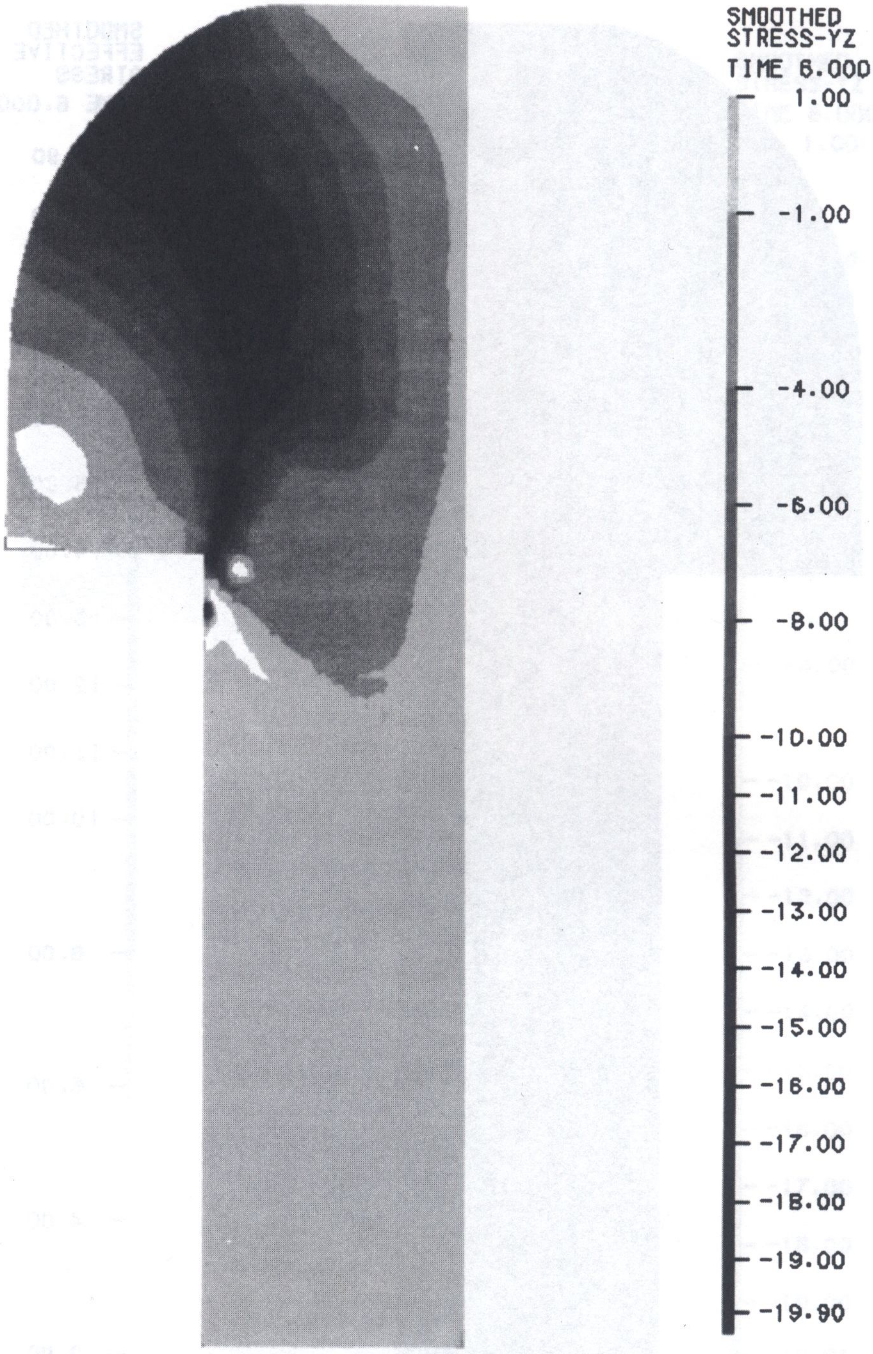

FIG. 10. Solution with correction $|\alpha| \leq 0.35$ ($\alpha = -0.35$, $\lambda_1 = 2.33$, $\lambda_2 = 3.71$, $V = 0.56V_0$).

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