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DYNAMIC INSTABILITY OF PLATES MADE OF NONLINEAR VISCOELASTIC MATERIALS

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The dynamic stability analysis of isotropic plates made of a nonlinear viscoelastic material is performed within the concept of the Lyapunov exponents. The material behaviour is modelled according to the Leaderman representation of nonlinear viscoelasticity. The influence of the various parameters involved on the possibility of instability to occur is investigated. It is also shown that in some cases the system is chaotic.

1. Introduction

The dynamic stability of structures subjected to in-plane loads is one of the most interesting problems in the field of structural vibration. When plates are considered, the phenomena can be observed, for example, in bridge dynamics or wing flutter (instability of aircraft in air flow). In the linear case, the behaviour is governed by the Mathieu equation and the stability characterizations are given by the Strutt diagram. Instability meant here is in the sense that the amplitude of the response increases without bounds. The problem was extensively investigated in [1] and further results were given e.g. in [2–3] in a review paper and a monograph, respectively.

When the structure is made of a viscoelastic material, the problem becomes much more complicated since the equation of motion turns out to be an integro-differential one, rather than an ordinary differential equation as in the elastic case. The solution of this problem in the linear case was given in [4] by means of the averaging method, and in [5–7], by using the spring-dashpot representation. The dynamic stability of viscoelastic homogeneous plates, investigated within the concept of the Lyapunov exponents, was performed in [8]. This procedure was used also in [9] to investigate the dynamic stability of viscoelastic laminated plates under shear. In these two studies the Boltzmann superposition principle was incorporated, enabling the modelling of any *linear* viscoelastic material.

However, it is well known that some viscoelastic materials (polymers, for example) are not linear and should be modelled non-linearly in order to give an adequate description of their behaviour. SMART and WILLIAMS [10] made a comparison investigating of the response of polypropylene and polyvinylchloride, obtained by using three single-integral representations of *nonlinear* viscoelasticity: the LEADERMAN model [11], the SCHAPERY model [12] and the BERNSTEIN – KEARSLEY – ZAPAS model [13–14]. Their main conclusion was that the Leaderman model is the most useful representation, as far as prediction and simplicity are concerned. In the present investigation we adopt this result and use the Leaderman model to derive the integro-differential equation of motion, which is nonlinear and with time-dependent coefficients.

The stability analysis of the nonlinear viscoelastic plate is based on the evaluation of associate Lyapunov exponents. If one of the Lyapunov exponents is found to be positive then, according to CHETAEV [15], the unperturbed motion is unstable. Thus, in order to determine the stability condition of the plate, it suffices to compute the *largest* Lyapunov exponent only.

2. PROBLEM FORMULATION

The equation of motion of an isotropic plate subjected to in-plane loads (see e.g. in [16] for the case where $N_{xy} = 0$)

$$M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + N_x w_{,xx} + N_y w_{,yy} + \varrho h \ddot{w} = 0, \tag{1}$$

where N_x and N_y are in-plane loads in the x and y directions, respectively, w is the deflection in the transverse, z, direction, ϱ is the material density and h is the plate thickness. The stress couples, M_{ij} are given by

$$M_{ij} = -\int_{-h/2}^{h/2} z\sigma_{ij}\, dz, \qquad i,j = x,y \tag{2}$$

and σ_{ij} are the stress components. For a nonlinear viscoelastic material the stress-strain constitutive relation is given by (see LEADERMAN [11])

$$\sigma(t) = Q(0)g[\varepsilon(t)] + \int_{0+}^{t} \dot{Q}(t-\tau)g[\varepsilon(\tau)]\, d\tau,$$

where

$$g[\varepsilon(t)] = \varepsilon(t) + \beta\varepsilon(t)^2 + \gamma\varepsilon(t)^3 + \ldots \tag{3}$$

in which for small strain $g(\varepsilon) \to \varepsilon$. β and γ are constants. For the state of plane stress for isotropic plates

$$
\begin{aligned}
Q_{11}(t) &= Q_{22}(t) = \frac{E(t)}{1-\nu(t)^2}, \\
Q_{12}(t) &= \nu(t) Q_{11}(t), \\
Q_{66}(t) &= \frac{1-\nu(t)}{2} Q_{11}(t),
\end{aligned} \tag{4}
$$

where $E(t)$ is a time-dependent relaxation function which at $t = 0$ denotes the initial Young modulus of the material, while $\nu(t)$ is the time-dependent Poisson ratio.

For a homogeneous thin plate, the strain-displacement relations are given by

$$
\begin{aligned}
\varepsilon_x &= -z w_{,xx}, \\
\varepsilon_y &= -z w_{,yy}, \\
\varepsilon_{xy} &= -2z w_{,xy}.
\end{aligned} \tag{5}
$$

The in-plane loadings, which contain constant and periodic terms, are given by

$$
\begin{aligned}
N_x &= N_{xs} + N_{xd}\cos(\theta t), \\
N_y &= N_{ys} + N_{yd}\cos(\theta t),
\end{aligned} \tag{6}
$$

where t is the time and θ is the load frequency.

Using the separation of variables method, the transverse displacement is written in the form

$$
w(x,y,t) = f(t)\varphi(x,y) \tag{7}
$$

which, in the case of a simply supported plate, is given by

$$
w(x,y,t) = f(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \tag{8}
$$

where a and b are the side-lengths of the plate.

Using Eqs. (3)–(5) in Eq. (2), and then together with Eqs. (6), (8), and applying the Galerkin method (see e.g. [1] and [17]), Eq. (1) is rewritten as

$$
\begin{aligned}
&\ddot{f}(t) + \Omega^2\left[1 - 2\eta\cos(\theta t)\right] f(t) + k f^3(t) \\
&\qquad = -\omega^2 \int_{0+}^{t} \dot{D}(t-\tau) f(\tau)\, d\tau - k \int_{0+}^{t} \dot{D}(t-\tau) f^3(\tau)\, d\tau,
\end{aligned} \tag{9}
$$

where $a = b = l$ and

$$\omega^2 = \frac{4IQ_{11}(0)}{\varrho h}\left(\frac{\pi}{\ell}\right)^4, \qquad N = \frac{4\pi^2 IQ_{11}(0)}{\ell^2}, \qquad D(t) = \frac{Q_{11}(t)}{Q_{11}(0)},$$

$$(10) \quad \Omega^2 = \omega^2\left[1 - \frac{N_{xs}+N_{ys}}{N}\right], \qquad \eta = \frac{N_{xd}+N_{yd}}{2[N-(N_{xs}+N_{ys})]},$$

$$k = \frac{27\pi^4 h^2}{640\ell^4}\left[\gamma_{xx}(1+\nu)+4\gamma_{xy}(1-\nu)\right]\omega^2, \qquad I = \frac{h^3}{12}.$$

Here, ω and Ω represent the natural frequency of loaded and unloaded plates, respectively, N is the Euler critical load, η is the excitation parameter and k is the coefficient of nonlinearity.

Equation (9) is the nonlinar integro-differential equation, which governs the motion of the nonlinar viscoelastic plate subjected to in-plane parametric loading.

3. Method of solution

Let us analyze in the stability of the unperturbed equilibrium of the nonlinear viscoelastic plate. To this end the integro-differential equation (9) is investigated. For the treatment of nonlinear differential equations with time-dependent coefficients, Lyapunov introduced the concept of characteristic numbers, the sign of which determines whether or not the unperturbed motion is stable [18]. The negative values of these characteristic numbers are referred to as the Lyapunov exponents.

According to Lyapunov, if all the exponents are negative, then the unperturbed motion is asymptotically stable. In addition, Chetaev [15, 19] showed that if one of the Lyapunov exponents is positive then the unperturbed motion is unstable. Thus, it suffices to compute the *largest* Lyapunov exponent in order to determine the stability of the unperturbed motion of the nonlinear viscoelastic plate in question. To derive the largest Lyapunov exponent of the system we used the procedure given in [20]. To do so, equation (9) must be transformed into a system of first-order equations. By declaring the variable $x_1 = f(t)$ and x_3 as the integral in Eq. (9), the following ordinary integro-differential equations are derived

$$(11) \quad \begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\Omega^2[1-2\eta\cos(\theta t)]x_1 - kx_1^3 - x_3, \\ \dot{x}_3 &= \frac{\partial}{\partial t}\int_{0+}^{t} \dot{D}(t-\tau)\left[\omega^2 x_1(\tau) + kx_1^3(\tau)\right]d\tau. \end{aligned}$$

As for the material relaxation function, the Standard Linear Solid model

$$E(t) = a + be^{-\alpha t} \tag{12}$$

is considered, where a, b and α are appropriate parameters. Thus, for the material with time-independent Poisson ratio one obtains

$$Q_{11}(t) = \frac{E(t)}{1-\nu^2} = A + Be^{-\alpha t}, \tag{13}$$

so that

$$D(t) = \frac{Q_{11}(t)}{Q_{11}(0)} = \frac{A + Be^{-\alpha t}}{A+B}. \tag{14}$$

Introducing the above model into (11) will affect only the equation for $\dot{x}_3$. Its final form is obtained by differentiation according to the Leibnitz rule

$$\dot{x}_3 = -\alpha\left[x_3 + \frac{B}{A+B}\left(\omega^2 x_1 + kx_1^3\right)\right]. \tag{15}$$

4. Numerical results and discussion

In this section the stability of Eq. (9) is analyzed with respect to the various parameters involved. (The numerical solution of this equation is obtained by the Runge-Kutta method [21]. First, it is observed that for the case when $\alpha = k = 0$ one obtains the well-known Mathieu equation, which was extensively investigated, e.g., by Mc Lachlan [22]. When $k = 0$ and $\alpha \neq 0$, Eq. (9) is in the following form

$$\ddot{f}(t) + \Omega^2[1 - 2\eta\cos(\theta t)]f(t) = -\omega^2 \int_{0+}^{t} \dot{D}(t-\tau)f(\tau)\,d\tau, \tag{16}$$

which describes the motion of a *linear* viscoelastic structure. The stability of this equation was investigated in [8] by using the concept of Lyapunov exponents and later on analytically in [23-24], where the expression for the critical (minimum) value of the excitation parameter, η_c, under which instability may never occur, was obtained. For the case of the Standard Linear Solid model it is

$$\eta_c = \frac{2}{\theta}\left|\dot{D}(0)\right| = \frac{2\alpha B}{\theta(A+B)} \tag{17}$$

and will be used later on. For the case when $\alpha = 0$ and $k \neq 0$, one obtains

$$\ddot{f}(t) + \Omega^2[1 - 2\eta\cos(\theta t)]f(t) + kf(t)^3 = 0, \tag{18}$$

representing a nonlinear version of the Mathieu equation, which was examined in [1].

In the following, we consider the general case when $\alpha \neq 0$ and $k \neq 0$. The numerical results were obtained by using $A = 0.1$ and $B = 0.9$, and where $N_{xs} = N_{ys} = 0$, $\Omega = \omega = 1$ and $\theta = 2\omega$.

Figure 1 shows the response, $f(t)$, as well as the largest Lyapunov exponent, λ_1, derived for the case of $\alpha = 0.001$, $k = 0.01$ and η is equal to a) 0.004, b) 0.009 ($= \eta_c$) and c) 0.5. In Fig. 1 a the system is asymptotically stable, that is λ_1 is negative and the response is approaching zero. In Figs. 1 b and 1 c the system is stable with a limit cycle and $\lambda_1 \to 0$. Yet, in Fig. 1 c the amplitude is much larger than that in Fig. 1 b (when $\eta > \eta_c$, the amplitude can be approximated by $A = 1/\sqrt{k}$ (see e.g., Bolotin [1])).

In Fig. 2 we have $k = 0.01$, $\eta = 0.5$ and the following cases for α are considered: a) 0, b) 0.000001 and c) 0.001. In Figs. 2 a and 2 b λ_1 is positive, indicating instability. For relatively large α (case c), $\lambda_1 \to 0$ and the system is stable.

The response and the largest Lyapunov exponent shown in Fig. 3 concern the cases when $\alpha = 0.000001$, $\eta = 0.5$ and a) $k = 0$, b) $k = 0.00001$ and c) $k = 1$. Figure 3 a represents a linear viscoelastic case with $\eta > \eta_c$ and thus the system is unstable with positive λ_1 and amplitude which grows exponentially. In the nonlinear case, Fig. 3 b, the system is also unstable (positive Lyapunov exponent), but with a finite amplitude. In Fig. 3 c $\lambda \to 0$ so that the system is stable (with a relatively small amplitude).

From the above we may conclude the following:

1. Due to the nonlinear viscoelasticity, the response remains bounded even at instability (contrary to the case of linear viscoelastic material). Moreover, high nonlinearity stabilizes the system, as compared with the unstable case with low nonlinearity, (Fig. 3 c).

2. The material cofficient, α, has a great influence on the system in the sense that an unstable system may become stable at large values of α (see Fig. 2 c). The above is correct at $\eta > \eta_c$. But α is one of the parameters by which η_c is determined in Eq. (17), in the way that at large α, η_c is increased so that α stabilizes the system in this respect too.

3. At $\eta < \eta_c$, the system is asymptotically stable regardless of the values of α and k.

Finally, it is noted that the Lyapunov exponents are used also as a powerful tool in the study of chaotic motion, and actually, the existence of at

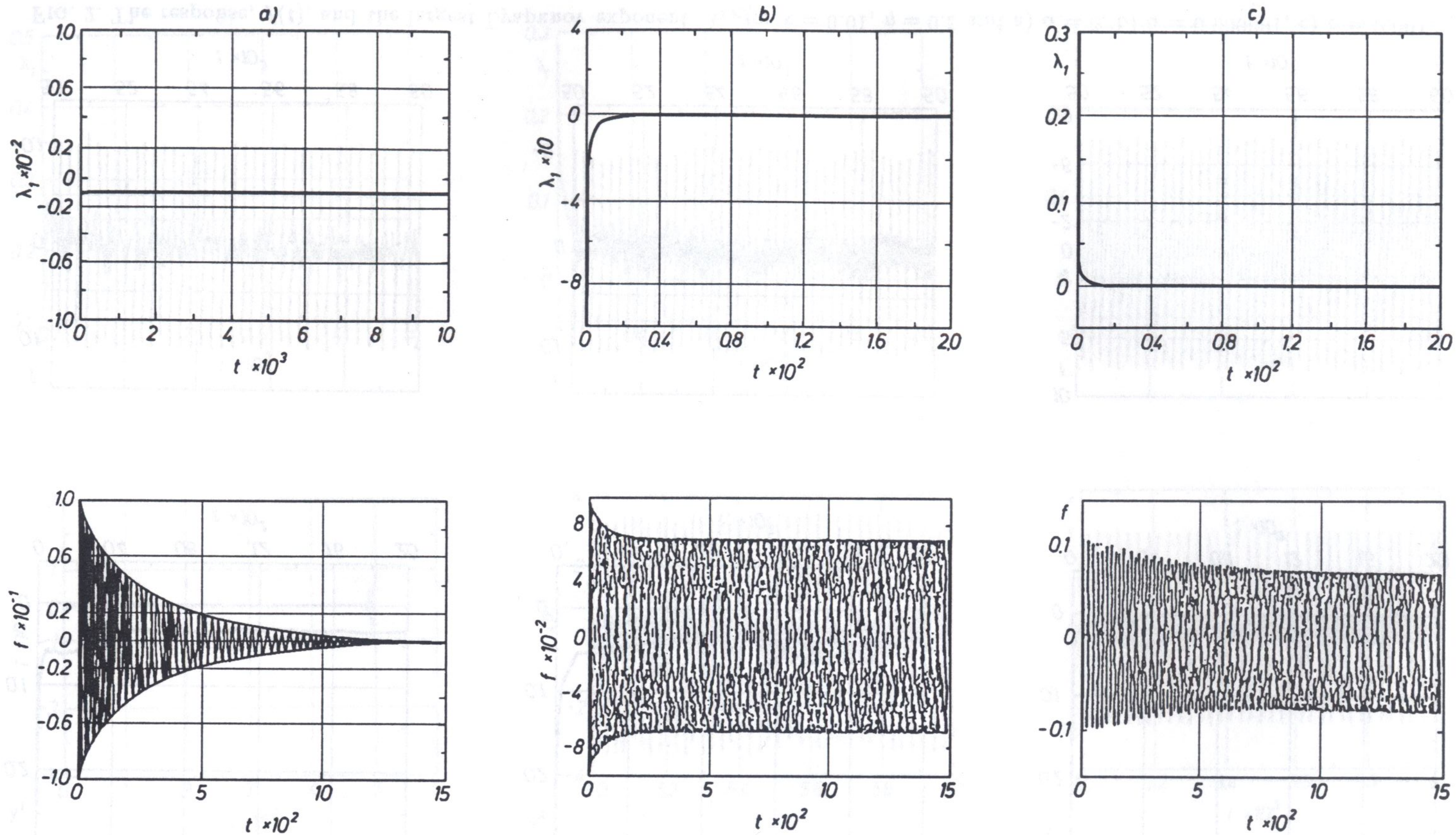

FIG. 1. The response, $f(t)$, and the largest Lyapunov exponent, λ_1, for $\alpha = 0.01$, $k = 0.01$ and a) $\eta = 0.004$, b) $\eta = 0.009$, c) $\eta = 0.5$.

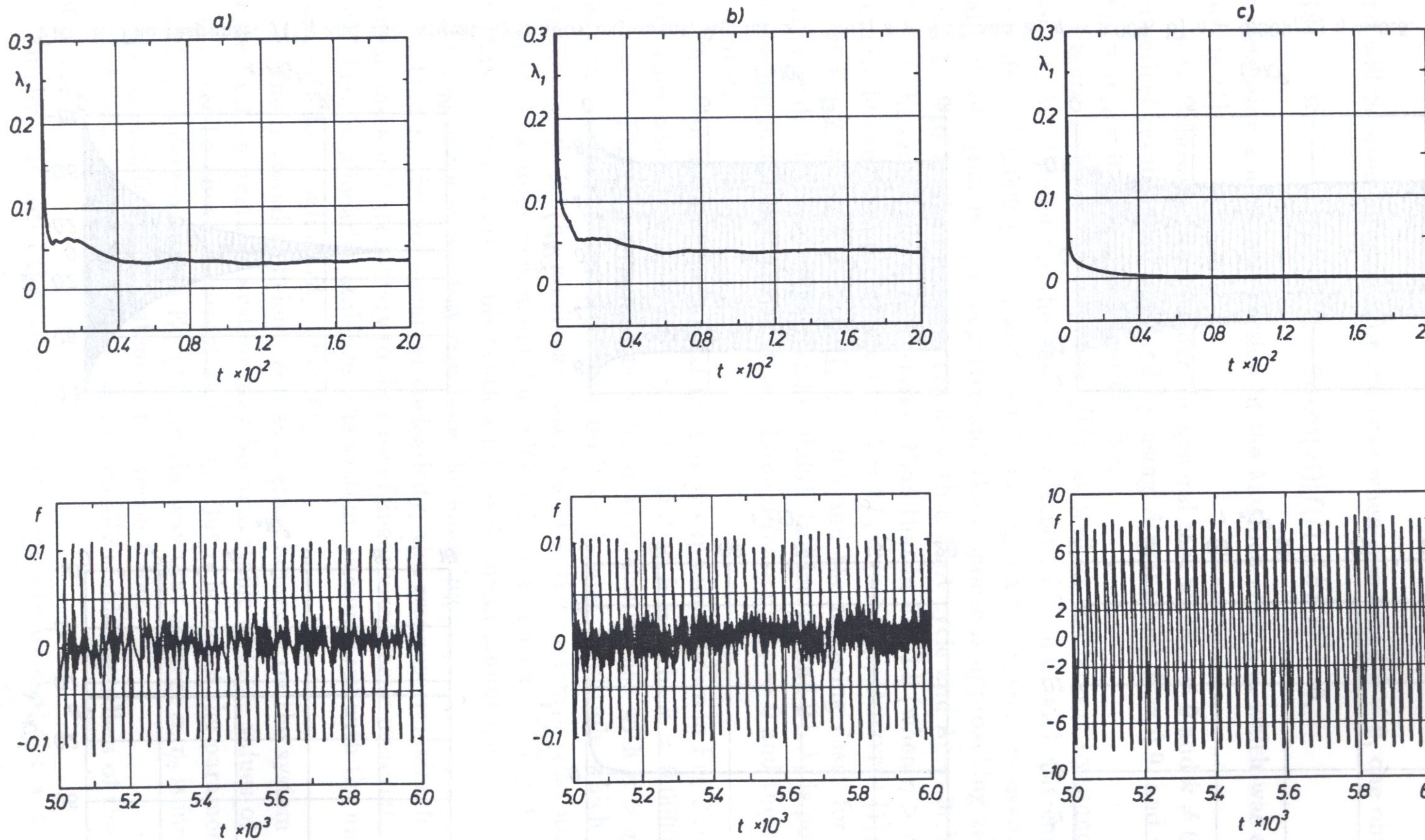

FIG. 2. The response, $f(t)$, and the largest Lyapunov exponent, λ_1, for $k = 0.01$, $\eta = 0.5$ and a) $\alpha = 0$, b) $\alpha = 0.000001$, c) $\alpha = 0.001$.

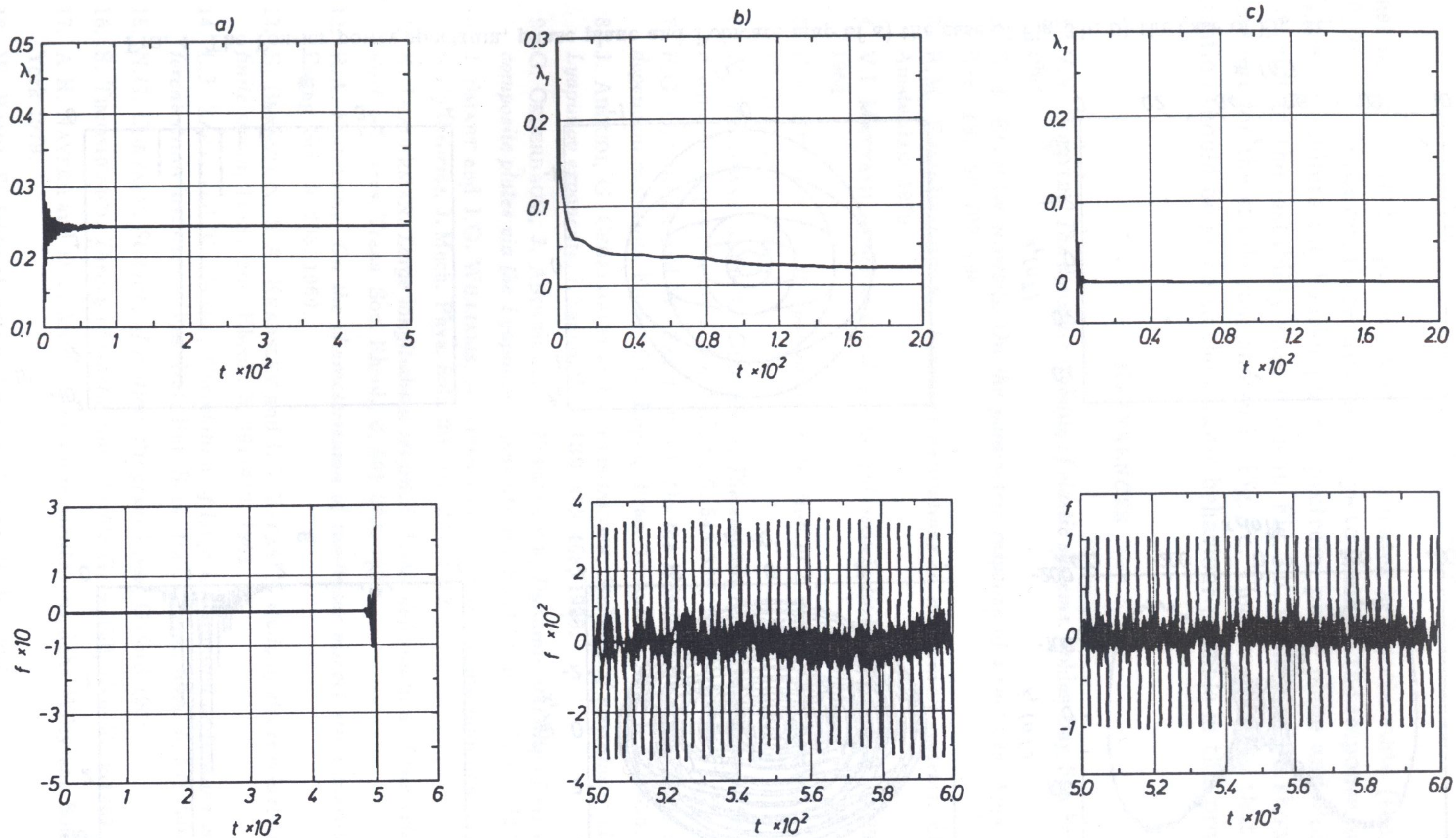

FIG. 3. The response, $f(t)$, and the largest Lyapunov exponent for $\alpha = 0.000001$, $\eta = 0.5$ and a) $k = 0$, b) $k = 0.00001$, c) $k = 1$.

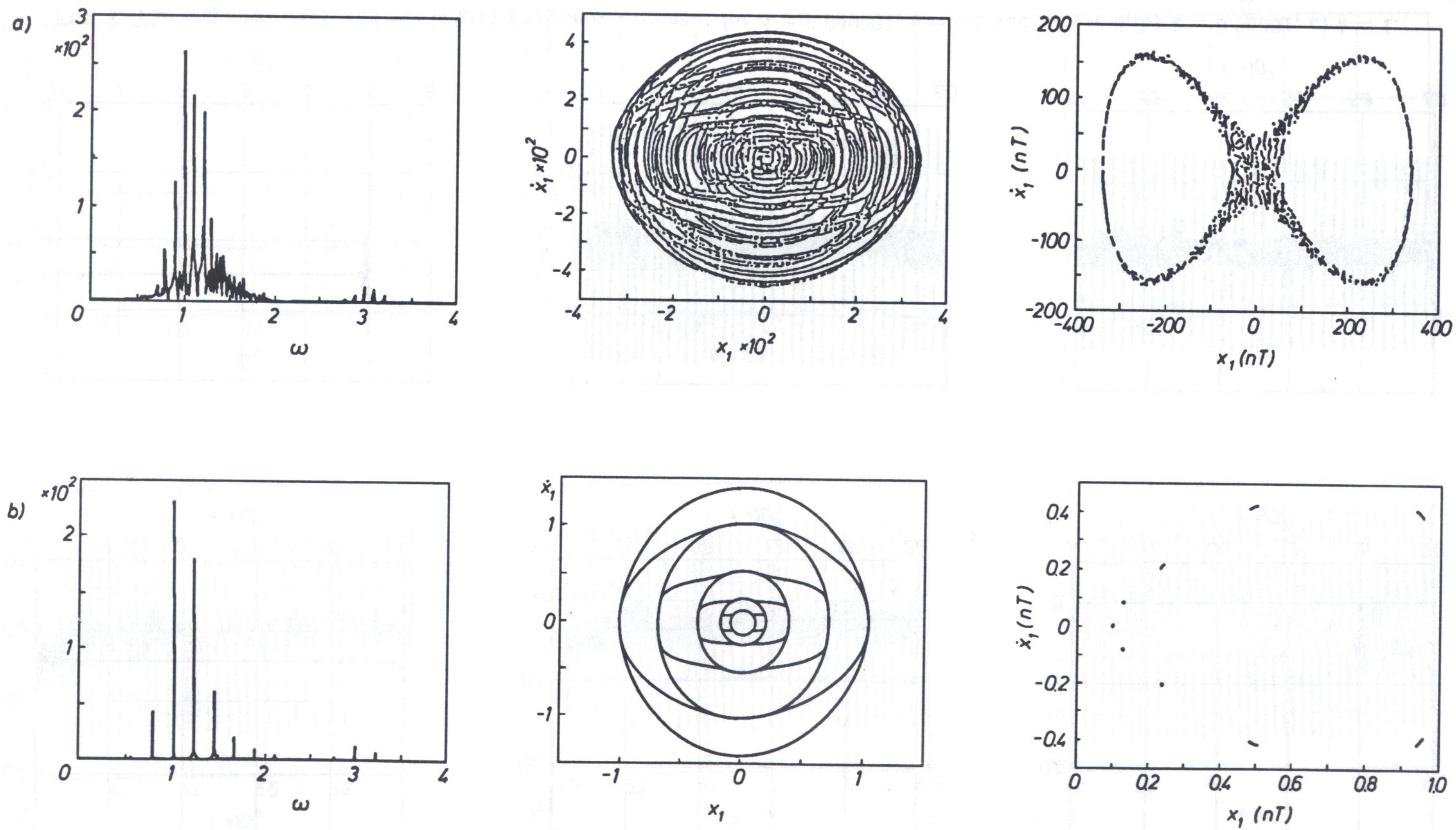

FIG. 4. The Fourier power spectrum, phase plane and Poincaré map of a) the case of Fig. 3 b, b) the case of Fig. 3 c.

least one positive Lyapunov exponent indicates a chaotic state (see, e.g., [20, 25–26]). However, there are other ways to examine the response nature. Figure 4 a exhibits the Fourier power spectrum, phase plane and Poincaré map plot of the instability case given in Fig. 3 b, while Fig. 4 b shows the same but for the stable case shown in Fig. 3 c. Thus, we believe that more attention should be given to the chaotic behaviour possible in this problem.

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