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PARAMETRIC INSTABILITY OF VISCOELASTIC NONLINEAR (ELASTICA) COLUMNS

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The dynamic stability analysis of a uniform, homogeneous, simply supported column, subjected to a periodic axial force, is presented. The viscoelastic behaviour is given in terms of the Boltzmann superposition principle. The equation of motion, derived within the *elastica* and including variations in the column's length, is in the form of a nonlinear integro-differential equation. The stability analysis of this equation is carried out within the Lyapunov exponents concept, which is also used, together with the Fourier power spectrum, in order to examine the possibility of a chaotic situation.

1. INTRODUCTION

The analysis of bars, based on the shape of the elastic curve as found from the exact differential equation, the *elastica*, was performed e.g. by LOVE'S [1] and by TIMOSHENKO and GERE [2]. These investigations neglected the change in the column's length. An energy approach to this problem, and to the problem of dynamical elastica is given by EL NASCHIE [3].

As is well known, bars made of materials such as elastomers, exhibit a nonlinear elastic behaviour, and the change in their length should be considered. The elastica buckling problem for this case was considered by KOUNADIS and MALLIS [4], while the investigation of the parametric instability of a simply supported column was considered recently by CEDERBAUM and MOND [5]. This last work has led to the analytical investigation of a nonlinear Mathieu equation, with special attention to the case when the loading frequency is four times the natural one.

However, in structures made of materials such as polymers, viscoelastic behaviour is observed. This is the reason for this paper which deals with the dynamical stability of a simply supported column, made up from a linear-viscoelastic material, and excited by a parametrical axial load. The analysis is based upon the exact relationship between bending moment and curvature (the elastica), and the effect of the axial shortening of the centerline is also considered.

The analysis yields a nonlinear integro-differential equation of motion. The stability of this equation is studied within the concept of the Lyapunov exponents, which implies that the equation of motion must be in the form of an ordinary differential equation. Within this work, the transformation is carried out by using the Leibnitz rule.

2. Problem formulation

Consider a simply supported straight column of length l and constant cross-sectional area A, subjected to a periodic axial force $P = P_0 \cos(\theta t)$. The material is linear viscoelastic, for which the stress-strain relationship is given by the Boltzmann superposition principle [6]

$$\sigma(t) = R(0)\varepsilon(t) + \int_{0+}^{t} \varepsilon(t') \frac{dE(t-t')}{d(t-t')} dt', \tag{1}$$

where $E(t)$ is the relaxation function, and $E(0) = E_0$ is the initial Young modulus of the material.

The equation of motion of the column is

$$M_{,xx} + PW_{,xx} + J\ddot{W} = 0, \tag{2}$$

where $W = W(x,t)$ is the transverse displacement, $J = \int_A \varrho \, dA$, ϱ is the mass density, comma denotes partial differentiation with respect to x while dot denotes differentiation with respect to t, and the bending moment is given by

$$M(x,t) = -\int_A y\sigma_y \, dA. \tag{3}$$

The bending strain at a distance y from the centroidal plane is

$$\varepsilon_y = \varepsilon_0 - y\varphi_{,x}, \tag{4}$$

where $\varepsilon_0 = \varepsilon_0(t)$ is the strain of the neutral plane and $\varphi = \varphi(x,t)$ is the angle of rotation of the column cross-section. Within the elastica, φ is given by

$$\varphi = \sin^{-1}\left[\frac{W_{,x}}{1+\varepsilon_0}\right]$$

so that

$$\varphi_{,x} = \frac{1}{\cos\varphi}\left[\frac{W_{,xx}}{1+\varepsilon_0}\right]. \tag{5}$$

Let

$$\frac{1}{\cos\varphi} = 1 + \frac{1}{2}\left[\frac{W_{,x}}{1+\varepsilon_0}\right]^2 \tag{6}$$

and

$$\lambda_1(t) = \frac{1}{1+\varepsilon_0(t)}, \qquad \lambda_2(t) = \left[\frac{1}{1+\varepsilon_0(t)}\right]^3. \tag{7}$$

Using Eqs. (1), (4) – (7), in Eq. (3), one obtains

$$\begin{aligned} M = {} & IE_0\lambda_1(t)W_{,xx} + \frac{1}{2}IE_0\lambda_2(t)W_{,xx}W_{,x}^2 \\ & + I\int_{0+}^{t}\left[\lambda_1(t')W_{,xx} + \frac{1}{2}\lambda_2(t')W_{,xx}W_{,x}^2\right]\frac{dE(t-t')}{d(t-t')}dt', \end{aligned} \tag{8}$$

where

$$I = \int_A y^2\,dA.$$

Similarly to the loading function, the time-dependent behaviour of the neutral plane strain, ε_0, is assumed to be

$$\varepsilon_0(t) = \bar{\varepsilon}\cos\theta t \tag{9}$$

and since ε_0 is considered to be small, Eqs. (7) can be approximated by taking the first order terms only:

$$\lambda_1(t) = 1 - \varepsilon_0(t), \qquad \lambda_2(t) = 1 - 3\varepsilon_0(t). \tag{10}$$

For a hinged column, the solution function for the transverse displacement is chosen to be

$$W(x,t) = \sin\frac{\pi x}{l}f(t) \equiv \sin(\alpha x)f.$$

By substituting this equation together with Eqs. (8) – (10) into Eq. (2), and applying Galerkin's method, the equation of motion is obtained in the form

$$\begin{aligned} & \ddot{f} + \omega^2[\lambda_1(t) - 2\eta\cos\theta t]f + \omega^2\frac{\alpha^2}{8}\lambda_2(t)f^3 \\ & = -\omega^2\int_{0+}^{t}\left[\lambda_1(t')f(t') + \frac{\alpha^2}{8}\lambda_2(t')f^3(t')\right]\frac{dD(t-t')}{d(t-t')}dt', \end{aligned} \tag{11}$$

where

$$\omega^2 = \frac{E_0I\alpha^4}{J}, \qquad \eta = \frac{P_0}{2E_0I\alpha^2}, \qquad D(t) = \frac{E(t)}{E_0}.$$

Equation (11) is the resulting integro-differential equation, which governs the motion of the column made of a linear viscoelastic material and subjected to an axial periodic loading.

3. Analysis

Let us analyze stability of the unperturbed equilibrium of the linear viscoelastic column. To this end, the integro-differential equation of the perturbed motion, Eq. (11), is to be investigated. This will be carried out within the concept of the Lyapunov exponents.

The Lyapunov exponents serve as a powerful tool for determining whether or not the unperturbed motion is stable. If all the exponents asssociated with a certain equation are negative, then the motion is asymptotically stable. If one of the exponents is positive, then the unperturbed motion is unstable. Thus, in order to identify the nature of the system, it suffices to compute the *largest* Lyapunov exponent. To do this, we use the method of Ref. [7] and the procedure described in [8].

In order to compute the largest Lyapunov exponent, the governing equation (11) has to be transformed into a system of first-order equations. By declaring the variable $x_1 = f(t)$ and x_3 as the integral in Eq. (11), the following set of ordinary integro-differential equations is obtained

$$\begin{aligned}
\dot{x}_1 &= x_2 , \\
\dot{x}_2 &= -\omega^2 \left[1 - (\bar{\varepsilon} + 2\eta)\cos(\theta t)\right] x_1 - \omega^2 \frac{\alpha^2}{8} x_1^3 \\
&\quad + \omega^2 \frac{3\alpha^2}{8} \bar{\varepsilon} \cos(\theta t) x_1^3 - \omega^2 x_3 , \\
\dot{x}_3 &= \frac{\partial}{\partial t} \int_{0+}^{t} \left\{ \left[1 - \bar{\varepsilon}\cos(\theta t')\right] x_1(t') + \frac{\alpha^2}{8} x_1^3(t') \right. \\
&\quad \left. - \frac{3\alpha^2}{8} \bar{\varepsilon} \cos(\theta t') x_1^3(t') \right\} \dot{D}(t - t')\, dt' .
\end{aligned} \tag{12}$$

We consider in the following a viscoelastic material modelled according to the Standard Linear Solid, for which the relaxation function is given by

$$E(t) = A + Be^{-\beta t},$$

so that

$$D(t) = a + be^{-\beta t}, \tag{13}$$

where

$$a = \frac{A}{A+B}, \qquad b = \frac{B}{A+B},$$

and where A, B, and β are appropriate parameters.

Substituting Eq. (13) into Eq. (12) modifies only $\dot{x}_3$ which is obtained using the Leibnitz rule

$$\dot{x}_3 = -\beta\left\{x_3 + b\left[(1 - \bar{\varepsilon}\cos(\theta t))x_1 + \frac{\alpha^2}{8}x_1^3 - \frac{3\alpha^2}{8}\bar{\varepsilon}\cos(\theta t)x_1^3\right]\right\}. \tag{14}$$

4. Numerical results and discussion

In this part, the stability of Eq. (11) is analysed with respect to the various parameters involved.

One can easily notice that for the case where $\beta = \bar{\varepsilon} = \alpha = 0$, the well known Mathieu equation is obtained. This equation has been investigated, e.g. by Mc Lachlan [9] and Bolotin [10].

When $\dot{\varepsilon} = \alpha = 0$, $\beta \neq 0$, one obtains the following equation:

$$\ddot{f} + \omega^2[1 - 2\eta\cos\theta t]\,f = -\omega^2\int_{0+}^{t} f(t')\frac{dD(t-t')}{d(t-t')}dt' \tag{15}$$

describing the motion of a column with *small* deflections and linear viscoelastic material.

The stability of this case was analytically investigated by Cederbaum and Mond [11], where the expression for the critical value of the excitation parameter, η_c, at which instability may occur for the case of the Standard Solid model, was found to be

$$\eta_c = \frac{2}{\theta}\left|\dot{D}(0)\right| = \frac{2}{\theta}\beta b \tag{16}$$

(which will be used later on).

For, the case when $\beta = 0$, $\bar{\varepsilon} \neq 0$ and $\alpha \neq 0$ one obtains

$$\ddot{f} + \omega^2[1 - (2\eta + \bar{\varepsilon})\cos\theta t]\,f + \omega^2\frac{\alpha^2}{8}f^3 - \omega^2\frac{3\alpha^2}{8}\bar{\varepsilon}\cos\theta t f^3 = 0, \tag{17}$$

representing a nonlinear form of the Mathieu equation, which was previously investigated by Cederbaum and Mond [5], mainly for the case where $\theta = 4\omega$.

In this paper, we consider the most general case of $\beta \neq 0$, $\bar{\varepsilon} \neq 0$ and $\alpha \neq 0$. Numerical results are obtained by letting $a = 0.1$ and $b = 0.9$ in Eq. (13). We also set $\omega = 1$ and consider the case when $\theta = 2\omega$.

Figure 1 shows the response $f(t)$ for the case of $\alpha = 0$, $\eta = 0.005$, $\beta = 0.01$ (so that $\eta_c = 0.009$) and $\bar{\varepsilon}$ is equal to a) 0.0001, b) 0.008 and c) 0.015. While

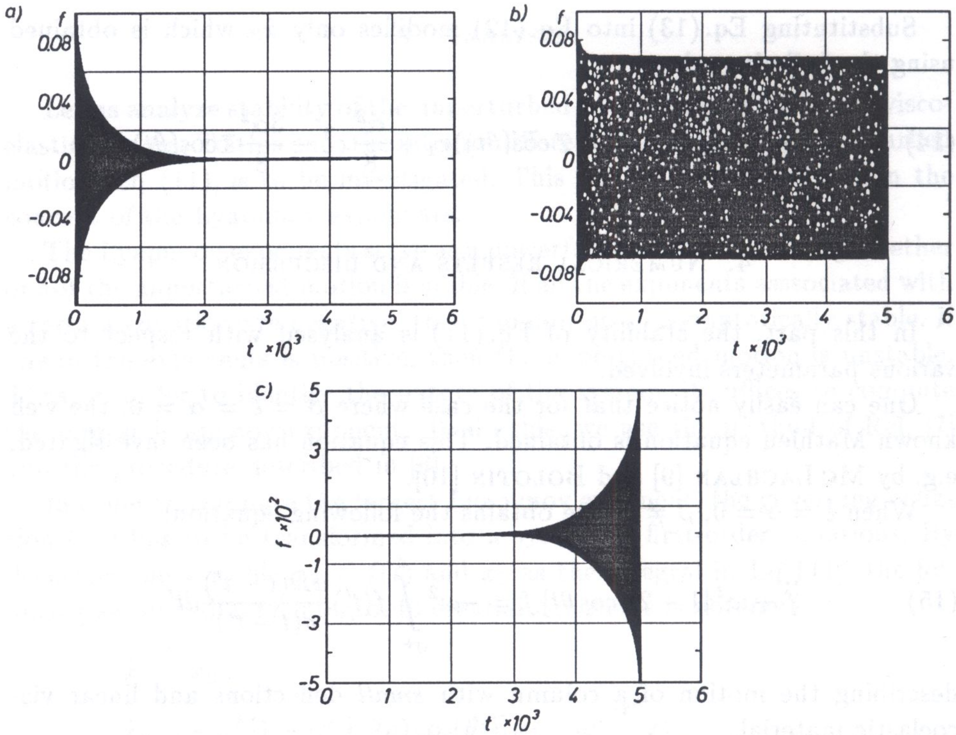

FIG. 1. The response $f(t)$ for $\alpha = 0$, $\eta = 0.005$, $\beta = 0.01$ and $\bar{\varepsilon}$ equal to a) 0.0001, b) 0.008, c) 0.015.

in Fig. 1 a the system is asymptotically stable with response tending to zero, Fig. 1 b presents a stable state with limit cycle. In Fig. 1 c, the system is unstable with response growing exponentially. From the observation of these results, we can conclude that $\bar{\varepsilon}$ has a great influence on the stability of the system in the sense that it destabilizes the system (for $\bar{\varepsilon} > 2(\eta_c - \eta)$, the system is within the instability region even for $\eta < \eta_c$).

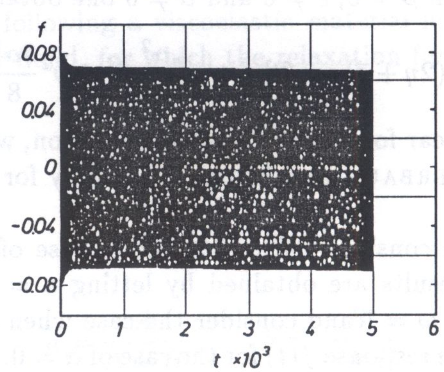

FIG. 2. The response $f(t)$ for $\alpha = 0.1$, $\eta = 0.005$, $\beta = 0.001$ and $\bar{\varepsilon}$ equal to 0.008.

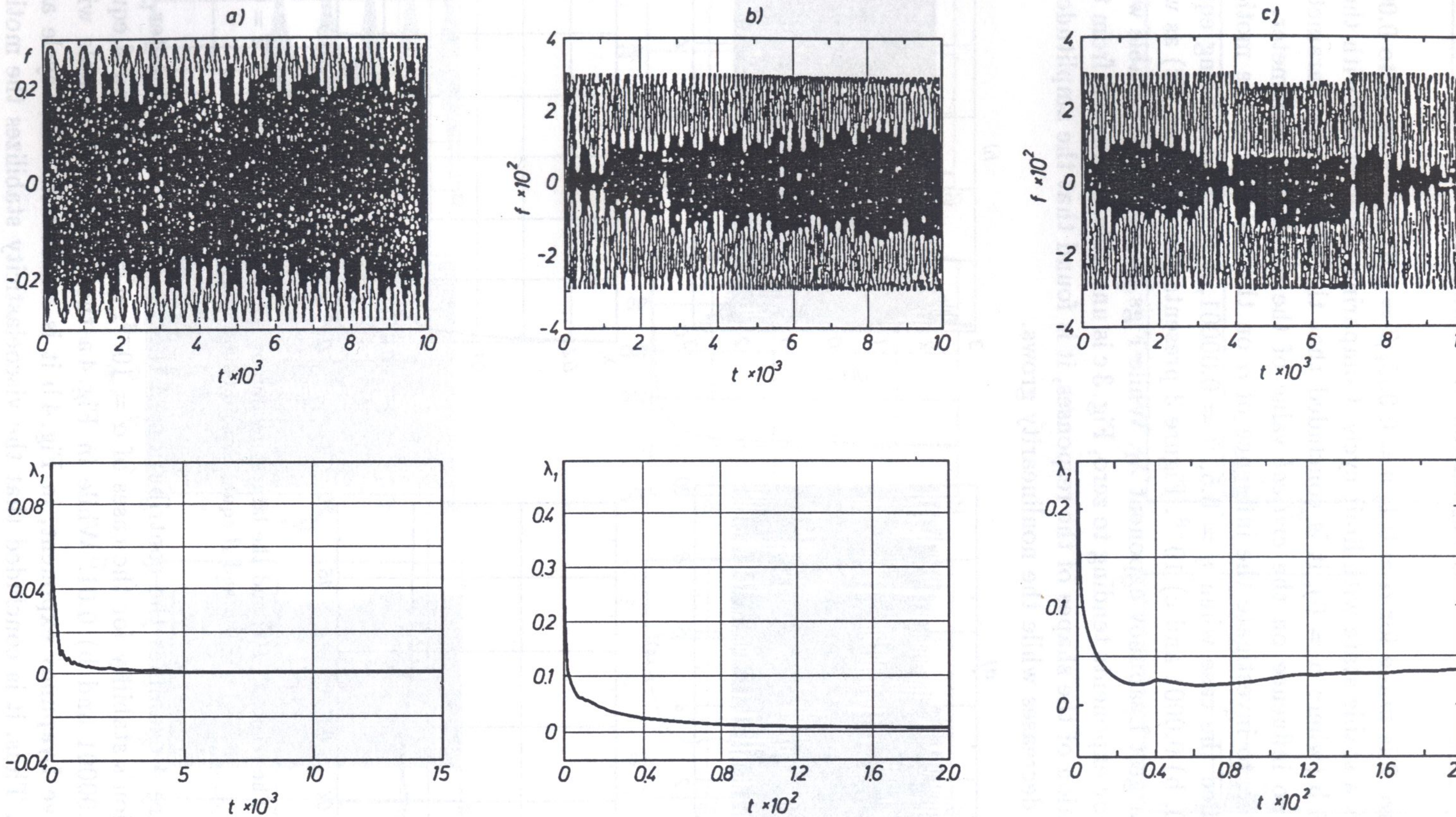

FIG. 3. The response $f(t)$ and the largest Lyapunov exponent λ_1 for $\eta = 0.5$, $\overline{\varepsilon} = 0$, $\beta = 0.00001$ and α equal to a) 10, b) 0.0001, c) 10^{-8}.

Figure 2 is made for $\alpha = 0.1$, $\eta = 0.005$, $\beta = 0.01$ and $\bar{\varepsilon}$ equal to 0.008. It shows a stable state with limit cycle. Comparing these results with those of Fig. 1 b (where $\alpha = 0$), it is concluded that the nonlinearity parameter, α, has no influence on the *critical* value of the excitation parameter, η_c. Moreover, to investigate the influence of α on the stability of the motion, we studied the case when $\eta = 0.5$, $\beta = 0.00001$, $\bar{\varepsilon} = 0$ and α being equal to a) 10, b) 0.0001 and c) 10^{-8}. Figure 3 presents the response $f(t)$ as well as the largest Lyapunov exponent λ_1. While Figs. 3a and 3 b are stable with Lyapunov exponents tending to zero, Fig. 3 c is unstable. Moreover, from the observation of the shapes of the responses, it is found that the amplitude of motion decreases while the nonlinearity grows.

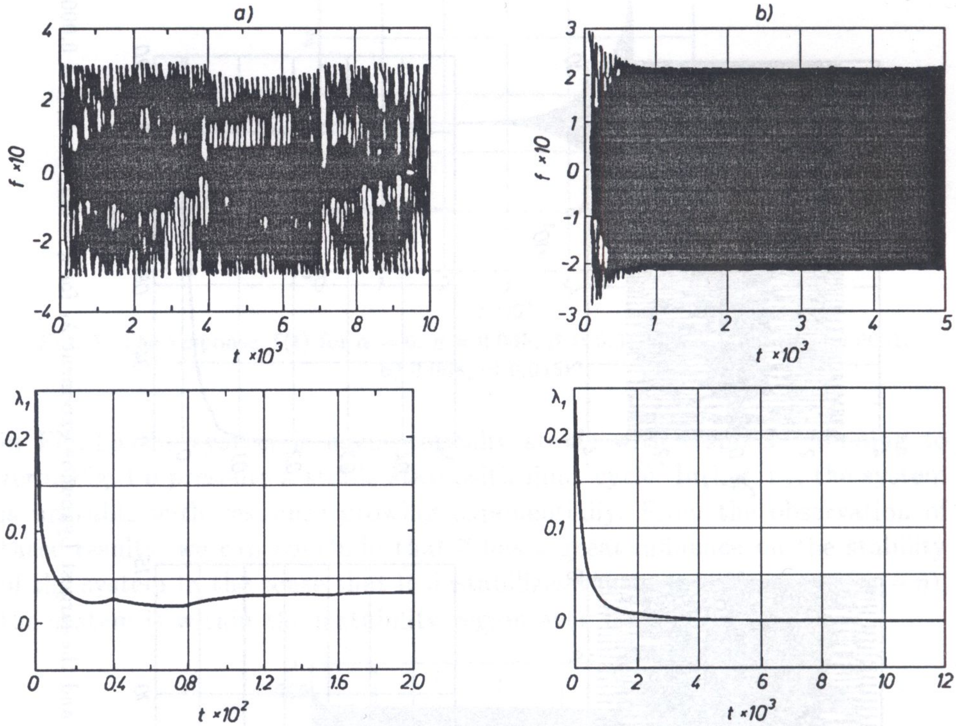

FIG. 4. The response $f(t)$ and the largest Lyapunov exponent λ_1 for $\alpha = 10^{-8}$, $\bar{\varepsilon} = 0$, $\eta = 0.5$ and β equal to a) 0.00001 and b) 0.01.

Figure 4 examines the contribution of the viscoelasticity parameter, β, on motion's stability for the cases of $\alpha = 10^{-8}$, $\eta = 0.5$, $\bar{\varepsilon} = 0$ and β equal to a) 0.00001 and b) 0.01. While in Fig. 4 a the motion is unstable with a positive Lyapunov exponent, in Fig. 4 b it is stable with limit cycle and $\lambda_1 \to 0$. Thus, it is concluded that the viscoelasticity stabilizes the motion of a nonlinear system even at high values of η (much higher than η_c).

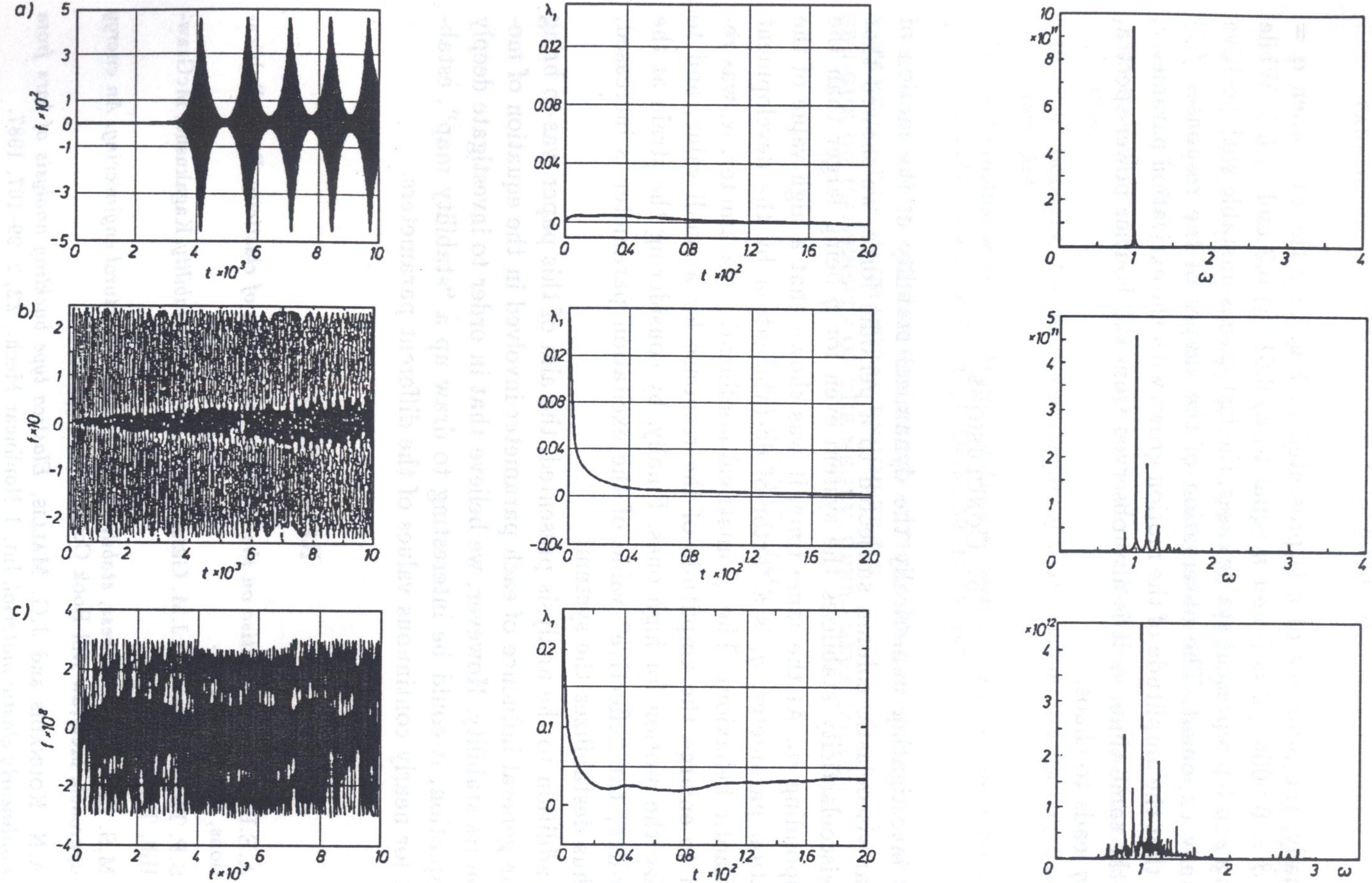

FIG. 5. The response $f(t)$, the largest Lyapunov exponent λ_1 and the Fourier power spectrum for α^{-8}, $\bar{\varepsilon} = 0$, $\beta = 0.00001$ and η equal to a) 0.01, b) 0.3, c) 0.5.

Finally, the influence of η is presented in Fig. 5 for the case when $\alpha = 10^{-8}$, $\beta = 0.00001$, $\bar{\varepsilon} = 0$ and η equal to a) 0.01, b) 0.3 and c) 0.5. While Figs. 5 a and 5 b represent stable cases, the last case is unstable with positive Lyapunov exponent. The observation of the shapes of the responses $f(t)$ shows that the amplitude of the motion grows with the excitation parameter, η. At the same time, as it is also observed from the Fourier power spectra, large η leads to chaos.

5. Conclusions

By investigating numerically the dynamical stability of the motion of a linear viscoelastic column subjected to a periodic force, we showed that high viscoelasticity stabilizes the system even for η being larger than the corresponding η_c. At the same time, it was shown that a high value of the excitation parameter, η, is a factor of destabilization by the development of a chaotic behaviour. The geometrical nonlinearity parameter, α, was revealed to enlarge the amplitude of the response for a small value, and to stabilize the motion for high ones. Finally, by considering the strain at the center-line, the "effective" value of the excitation parameter is increased, and thus destabilizes the system.

In addition to the analysis presented, the aim of this paper was to bring out the general influence of each parameter involved in the equation of motion on its stability. However, we believe that in order to investigate deeply this equation, it could be interesting to draw up a "stability map", established for nearly continuous values of the different parameters.

References

1. A.E.H. Love, *A treatise on the mathematical theory of elasticity*, Dover Publications, New York 1926.
2. S.P. Timoshenko and J.M. Gere, *Theory of elastic stability*, Kagakusha, McGraw-Hill, Tokyo 1961.
3. M.S. El Naschie, *Stress, stability and chaos in structural engineering. An energy approach*, McGraw-Hill Book Company, London 1990.
4. A.N. Kounadis and J.G. Mallis, *Elastica type buckling analysis of bars from nonlinearly elastic material*, Int. J. Nonlinear Mech., **22**, 2, 99–107, 1987.
5. G. Cederbaum and M. Mond, *Instability and chaos in the elastica type problem of parametrically excited columns*, J. Sound Vib., 176, 4, 475–486, 1994.
6. R.M. Christensen, *Theory of viscoelasticity*, Academic Press, New York 1971.

7. A. WOLF, J.B. SWIFT, H.L. SWINNEY and J.A. VASTANO, *Determining Lyapunov exponents from time series*, Physica, **16D**, 285–317, 1985.

8. I. GOLDHIRSCH, A.L. SULEM and S.A. ORSZAG, *Stability and Lyapunov stability of dynamical systems: a differential approach and numerical method*, Physica, **27D**, 311–337, 1987.

9. N.W. MCLACHLAN, *Theory and application of Mathieu functions*, Dover, New York 1964.

10. V.V. BOLOTIN, *The dynamical stability of elastic systems*, Halden-Dag, San Francisco 1964.

11. G. CEDERBAUM and M. MOND, *Stability properties of a vicoelastic column under a periodic force*, ASME J. Appl. Mech., **59**, 16–19, 1992.

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