ENGINEERING TRANSACTIONS • *Engng. Trans.* • **44**, 1, 55–67, 1996
Polish Academy of Sciences • Institute of Fundamental Technological Research

BOUSSINESQ PROBLEM FOR ELASTIC HALF-SPACE REINFORCED WITH FIBRES IN VERTICAL DIRECTION

R. ŚWITKA (BYDGOSZCZ)

Solution of Boussinesq problem has been given in the paper for the particular case of the fibrous composite half-space. The continuous medium has been assumed to consists of the matrix reinforced with vertical elastic fibres. The theory of fibrous composites proposed in [10] is used. The results obtained are compared with the Boussinesq formulae. It has been found that the fibrous phase takes over most of the forces transmitted through the half-space.

1. Introduction

This paper presents a generalization of the known Boussinesq problem to the case of a fibrous composite half-space. Of course, it is not possible in case of a general fibrous composite, because construction of the displacement equations depends on the number of fibre families and orientation (direction) of every family. We will confine our considerations to the case when the continuous medium is reinforced with fibres, so that axial symmetry of the problem will be preserved. It is the case of a fibrous composite with one family of fibres, which are parallel to the z-axis. The z-axis is the axis of a cylindrical coordinate system, being perpendicular to the plane that represents the upper surface of the fibrous composite half-space. The half-space is loaded by a concentrated force P, applied to the origin of the coordinate system, and perpendicular to the boundary plane.

The Boussinesq problem is one of the most basic problems of the mathematical theory of elasticity. Its solution is not only of a great theoretical and cognitive value: it has also essential practical applications in foundation engineering and in contact problems. Joseph Valentine Boussinesq has formulated this problem and announced its solution in 1885 in his paper concerning the application of the potential theory [11]. In modern handbooks it is usually solved by means of various auxiliary functions and using integral transforms. Comprehensive discussion of different solutions was given by W. Nowacki in his monograph [8].

In this paper, the solution is sought in a direct manner, using the Hankel transforms technique.

The papers concerning the mechanics of fibrous composites appeared in the fifties of the present century due to the fact that rubber reinforced with strong nylon cords was used in the tyre production industry. The paper by ADKINS and RIVLIN [1] was the first, and the authors investigated large deformations of elastic materials reinforced with inextensible fibres. This problem was later widely presented in the book of GREEN and ADKINS [5] and in the paper by PIPKIN [9]. The model of a plastic material reinforced with fibres was discussed by MULHERN, ROGERS and SPENCER in [7]. Since that time, many publications have appeared in the field of mechanics of composites. One of important directions of the research has been presented lately by WOŹNIAK [13].

Mechanics of composite materials is in fact the mechanics of heterogeneous and anisotropic bodies. The main problem consists in finding the methods of research suitable for evaluation of the so-called effective [3] modulus of elasticity of the model of homogeneous body. Such an approach to this problem is presented in known monographs [3, 4, 6, 12]. A partial solution of a similar two-dimensional problem is known from the literature, (Flamant's problem) concerning a fibrous composite with vertical fibres [3]; the model is, however, totally different from the one used in this paper.

The paper [10] presents a theory of fibrous composites, which introduces:

- two different phases in the composite material: the matrix and fibrous phase,
- the fibrous phase, due to its nature is discrete, discontinuous,
- both phases are subject to a common field of displacements,
- the theory is based on the laws of thermodynamics.

Finally, a consistent theory has been obtained, which includes constitutive equations, displacement equations and heat conductivity equations. Results of this theory will be used in the present paper.

The displacement equations obtained are, in fact, the equations of anisotropic medium. All the anisotropy constants are defined explicitly and are expressed by material constants of the matrix and fibres, and by the geometry of the fibres. Another advantage of the theory is the possibility of decomposition of the stresses into two parts, one of which is carried by matrix, and a part which is carried by fibres. It is very important for evaluation of the strength of individual components of the composite.

If the influence of temperature variations is ignored, then the general form of the displacement equations in Cartesian coordinate system is as follows:

$$(1.1)\qquad (1-\mu)\left[\mu_L \nabla^2 u_i + (\lambda_L+\mu_L)e_{,i}\right] + \sum_r \underset{r}{\mu}\,\underset{r}{E}\,\underset{r}{s_i}\,\underset{r}{s_j}\,\underset{r}{s_k}\,\underset{r}{s_l}\,u_{j,kl} + \varrho f_i = \varrho \ddot{u}_i\,.$$

In Eq. (1.1), **u** is a vector of displacement, **s** – unit vector which defines the direction of fibres, r – index placed at the bottom of the generic letter indicates the number of the fibre family, $\underset{r}{\mu}$ is the reinforcement density of the r-th family of fibres, $\mu = \sum_r \underset{r}{\mu}$, $e = u_{k,k}$ is the dilatation, μ_L and λ_L are Lamé's constants for elastic matrix, $\underset{r}{E}$ is the elasticity modulus of the r-th family of fibres, ϱ is the density of fibrous composite [10]

$$(1.2)\qquad \varrho = (1-\mu)\,\underset{m}{\varrho} + \sum_r \underset{r}{\mu}\,\underset{r}{\varrho}\,,$$

where $\underset{m}{\varrho}$ is the density of matrix and ϱ_r is the density of material in the r-th family of fibres. Finally, **f** is the vector of body forces.

The tensor of stress in the fibrous composite is expressed by the formula [10]

$$(1.3)\qquad \tau_{ij} = (1-\mu)\sigma_{ij} + \sum_r \underset{r}{\mu}\,\underset{r}{\sigma}\,\underset{r}{s_i}\,\underset{r}{s_j}\,,$$

where σ_{ij} – stress tensor in the matrix, $\underset{r}{\sigma}$ – stress in the fibre of the r-th family.

2. Problem formulation

In a cylindrical coordinates system, axial symmetry with respect to the z-axis being taken into account, the components of the displacement vector are the functions of variables r and z, where $r = \sqrt{x_1^2 + x_2^2}$, and they are independent of variable φ:

$$u_r = u_r(r,z), \qquad u_z = u_z(r,z), \qquad u_\varphi = 0.$$

The displacement equations (1.1) for an elastic medium, which is reinforced with fibres parallel to the z-axis, assume the following form:

$$(2.1)\qquad \begin{aligned} &\left(\nabla^2 - \frac{1}{r^2}\right) u_r + \kappa \frac{\partial e}{\partial r} = 0, \\ &\nabla^2 u_z + \kappa \frac{\partial e}{\partial z} + \varepsilon \frac{\partial^2 u_z}{\partial z^2} = 0. \end{aligned}$$

In the system of partial differential equations (2.1):

$$
(2.2)\qquad
\begin{aligned}
\nabla^2 &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2},\\
e &= \frac{\partial u_r}{\partial r} + \frac{1}{r}u_r + \frac{\partial u_z}{\partial z},\\
\kappa &= \frac{\lambda_L + \mu_L}{\mu_L} = \frac{1}{1-2\nu},\\
\varepsilon &= 2(1+\nu)\frac{\underset{1}{\mu}}{1-\underset{1}{\mu}}\cdot\frac{\underset{1}{E}}{E}.
\end{aligned}
$$

The body and inertia forces have been neglected.

Stresses in the fibrous composite will be calculated using the formulae

$$
(2.3)\qquad
\begin{aligned}
\tau_{rr} &= (1-\mu)\mu_L\left[(\kappa+1)\frac{\partial u_r}{\partial r} + (\kappa-1)\left(\frac{1}{r}u_r + \frac{\partial u_z}{\partial z}\right)\right],\\
\tau_{\varphi\varphi} &= (1-\mu)\mu_L\left[(\kappa+1)\frac{1}{r}u_r + (\kappa-1)\left(\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z}\right)\right],\\
\tau_{zz} &= (1-\mu)\mu_L\left[(\kappa-1)\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)u_r + (1+\kappa+\varepsilon)\frac{\partial u_z}{\partial z}\right],\\
\tau_{zr} &= (1-\mu)\mu_L\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right),
\end{aligned}
$$

which result from (1.3). It has been taken into account, that

$$
(2.4)\qquad \underset{1}{\sigma} = \underset{1}{E}\,\underset{1}{\varepsilon} = \underset{1}{E}\,\underset{1}{s_k}\,\underset{1}{s_l}\,\varepsilon_{kl} = \underset{1}{E}\,\varepsilon_{zz} = \underset{1}{E}\,\frac{\partial u_z}{\partial z}.
$$

Here, $\underset{1}{\sigma}$ is the stress in the fibres of the first family of fibres (which is the only one here).

We formulate the analyzed problem in the following way: Define such a displacements vector field, which satisfies Eqs. (2.1) in the region ($0 < r < \infty$; $0 \le z < \infty$), and which on the plane $z = 0$ fulfils the conditions

$$
(2.5)\qquad
\begin{aligned}
\tau_{zz}(r,0) &= -\frac{P\delta(r)}{2\pi r},\\
\tau_{zr}(r,0) &= 0,
\end{aligned}
$$

and disappears at infinity

$$
(2.6)\qquad u_r,\quad u_z \underset{r,z\to\infty}{\longrightarrow} 0.
$$

In $(2.5)_1$ $\delta(r)$ is the Dirac function.

3. Solution of the problem

We will seek the solutions in the form of Hankel integrals

$$
(3.1) \qquad
\begin{aligned}
u_r &= \int_0^\infty \widetilde{u}_r(\alpha, z) J_1(\alpha r)\alpha \, d\alpha \,, \\
u_z &= \int_0^\infty \widetilde{u}_z(\alpha, z) J_0(\alpha r)\alpha \, d\alpha \,,
\end{aligned}
$$

where $J_0(\alpha r)$ and $J_1(\alpha r)$ are the Bessel functions of the first kind and orders zero and one, respectively.

After substitution of (3.1) into (2.1) we obtain

$$
(3.2) \qquad
\begin{aligned}
&\int_0^\infty \left[-\alpha^2 \widetilde{u}_r + \widetilde{u}_r'' - \kappa(\alpha\widetilde{u}_r + \widetilde{u}_z')\alpha\right] J_1(\alpha r)\alpha \, d\alpha = 0, \\
&\int_0^\infty \left[-\alpha^2 \widetilde{u}_z + \widetilde{u}_z'' + \kappa(\alpha\widetilde{u}_r' + \widetilde{u}_z'') + \varepsilon\widetilde{u}_z''\right] J_0(\alpha r)\alpha \, d\alpha = 0.
\end{aligned}
$$

Here $(\;)' = d/dz$. Moreover, the rules of differentiation of the Bessel functions were used:

$$
\begin{aligned}
\frac{d}{dr}[J_0(\alpha r)] &= -\alpha J_1(\alpha r), \\
\frac{d}{dr}[J_1(\alpha r)] &= -\frac{1}{r} J_1(\alpha r) + \alpha J_0(\alpha r).
\end{aligned}
$$

Equations (3.2) will be fulfilled for every r and z, if

$$
(3.3) \qquad
\begin{aligned}
\widetilde{u}_r'' - (1+\kappa)\alpha^2 \widetilde{u}_r - \kappa\alpha\widetilde{u}_z' &= 0, \\
\kappa\alpha\widetilde{u}_r' + (1+\kappa+\varepsilon)\widetilde{u}_z'' - \alpha^2\widetilde{u}_z &= 0.
\end{aligned}
$$

We have obtained the system of ordinary differential equations with unknown transforms $\widetilde{u}_r(\alpha, z)$ and $\widetilde{u}_z(\alpha, z)$.

The solutions should be sought in the following form:

$$
(3.4) \qquad \widetilde{u}_r = A(\alpha)e^{k(\alpha)z}, \qquad \widetilde{u}_z = B(\alpha)e^{k(\alpha)z}.
$$

After substitution of (3.4) into (3.3), we obtain the system of homogeneous algebraic equations

$$
(3.5) \qquad
\begin{aligned}
\left[k^2 - (1+\kappa)\alpha^2\right] A - \kappa\alpha k B &= 0, \\
\kappa\alpha k A + \left[(1+\kappa+\varepsilon)k^2 - \alpha^2\right] B &= 0.
\end{aligned}
$$

The condition of existence of the solutions is

$$\begin{vmatrix} k^2-(1+\kappa)\alpha^2 & -\kappa\alpha k \\ \kappa\alpha k & (1+\kappa+\varepsilon)k^2-\alpha^2 \end{vmatrix} = 0,$$

thus

$$k^4 - \frac{(1+\kappa)(2+\varepsilon)}{1+\kappa+\varepsilon}\alpha^2 k^2 + \frac{1+\kappa}{1+\kappa+\varepsilon}\alpha^4 = 0. \tag{3.6}$$

If $\varepsilon = 0$, then $k^4 - 2k^2\alpha^2 + \alpha^4 = (k^2-\alpha^2)^2 = 0$ and double roots are obtained. This case leads to the known Boussinesq solutions for the homogeneous elastic half-space. Further we assume that $\varepsilon > 0$, and then we obtain

$$k_{1,3} = \pm\alpha\gamma_1, \qquad k_{2,4} = \pm\alpha\gamma_2, \tag{3.7}$$

where

$$\gamma_{1,2} = \sqrt{\frac{1+\kappa}{1+\kappa+\varepsilon}\left(1+\frac{1}{2}\varepsilon \pm \frac{1}{2}\sqrt{\varepsilon^2+\frac{4\varepsilon\kappa}{1+\kappa}}\right)}. \tag{3.8}$$

Among the solutions $e^{\pm\alpha\gamma_{1,2}z}$, the conditions at infinity (2.6) are fulfilled by the functions with negative exponents only. The relations between constants A and B are obtained e.g. from $(3.5)_1$

$$B = \frac{k^2-(1+\kappa)\alpha^2}{\kappa\alpha k}A. \tag{3.9}$$

Thus we obtain

$$\begin{aligned} u_r &= \int_0^\infty \left(A_1(\alpha)e^{-\gamma_1\alpha z} + A_2(\alpha)e^{-\gamma_2\alpha z}\right) J_1(\alpha r)\alpha\, d\alpha, \\ u_z &= -\int_0^\infty \left[\frac{\gamma_1^2-(1+\kappa)}{\kappa\gamma_1}A_1(\alpha)e^{-\gamma_1\alpha z} \right. \\ &\qquad \left. + \frac{\gamma_2^2-(1+\kappa)}{\kappa\gamma_2}A_2(\alpha)e^{-\gamma_2\alpha z}\right] J_0(\alpha r)\alpha\, d\alpha. \end{aligned} \tag{3.10}$$

The boundary conditions (2.5) should now be used. Since it is known that

$$\frac{P\delta(r)}{2\pi r} = \frac{P}{2\pi}\int_0^\infty J_0(\alpha r)\alpha\, d\alpha,$$

the following system of equation is obtained

$$(3.11)\quad \begin{gathered}\frac{\gamma_1^2(1-\kappa)-(1+\kappa)}{\kappa\gamma_1}A_1(\alpha)+\frac{\gamma_2^2(1-\kappa)-(1+\kappa)}{\kappa\gamma_2}A_2(\alpha)=0,\\ \left[(\kappa-1)+(1+\kappa+\varepsilon)\frac{\gamma_1^2-(1+\kappa)}{\kappa}\right]A_1(\alpha)\\ +\left[(\kappa-1)+(1+\kappa+\varepsilon)\frac{\gamma_2^2-(1+\kappa)}{\kappa}\right]A_2(\alpha)=-\frac{P}{2\pi\alpha(1-\mu)\mu_L},\end{gathered}$$

from which functions $A_1(\alpha)$ and $A_2(\alpha)$ are calculated:

$$(3.12)\quad \begin{aligned}A_1(\alpha)&=\frac{P}{2\pi\alpha(1-\mu)\mu_L K_1(\kappa,\varepsilon)},\\ A_2(\alpha)&=\frac{P}{2\pi\alpha(1-\mu)\mu_L K_2(\kappa,\varepsilon)}.\end{aligned}$$

In formulae (3.12) the notations are used:

$$(3.13)\quad \begin{aligned}K_1(\kappa,\varepsilon)&=\left[(\kappa-1)+(1+\kappa+\varepsilon)\frac{\gamma_1^2-(\kappa+1)}{\kappa}\right]\\ &\quad-\frac{\gamma_2}{\gamma_1}\frac{\gamma_1^2(\kappa-1)+(\kappa+1)}{\gamma_2^2(\kappa-1)+(\kappa+1)}\left[(\kappa-1)+(1+\kappa+\varepsilon)\frac{\gamma_2^2-(\kappa+1)}{\kappa}\right],\\ K_2(\kappa,\varepsilon)&=\frac{\gamma_1}{\gamma_2}\frac{\gamma_2^2(\kappa-1)+(\kappa+1)}{\gamma_1^2(\kappa-1)+(\kappa+1)}\left[(\kappa-1)+(1+\kappa+\varepsilon)\frac{\gamma_1^2-(\kappa+1)}{\kappa}\right]\\ &\quad-\left[(\kappa-1)+(1+\kappa+\varepsilon)\frac{\gamma_2^2-(\kappa+1)}{\kappa}\right].\end{aligned}$$

The improper integrals which appear in the formulae (3.10) are evaluated using the formulae given in [2]. The final results for the displacements can be written in the following form:

$$(3.14)\quad \begin{aligned}u_r&=\frac{Pr}{2\pi(1-\mu)\mu_L}\left[\frac{1}{K_2(\kappa,\varepsilon)R_2(R_2+\gamma_2 z)}\right.\\ &\qquad\left.-\frac{1}{K_1(\kappa,\varepsilon)R_1(R_1+\gamma_1 z)}\right],\\ u_z&=\frac{P}{2\pi\kappa(1-\mu)\mu_L}\left[\frac{\gamma_1^2-(\kappa+1)}{\gamma_1 K_1(\kappa,\varepsilon)R_1}-\frac{\gamma_2^2-(\kappa+1)}{\gamma_2 K_2(\kappa,\varepsilon)R_2}\right],\end{aligned}$$

where

$$(3.15)\quad R_1=\sqrt{r^2+\gamma_1^2 z^2},\qquad R_2=\sqrt{r^2+\gamma_2^2 z^2},\qquad r=\sqrt{x_1^2+x_2^2}.$$

The formula for the vertical displacement of the plane bounding the half-space is of particular importance, and it can be written as follows:

$$u_z(r,0) = u_z^{(B)}(r,0) \cdot K(\kappa,\varepsilon). \tag{3.16}$$

In formula (3.16)

$$u_z^{(B)}(r,0) = \frac{P(1-\nu^2)}{\pi E r}, \tag{3.17}$$

where ν is Poisson's ratio of the matrix, $u_z^{(B)}$ is the displacement according to the Boussinesq theory, whereas

$$K(\kappa,\varepsilon) = \frac{1}{(1-\nu)(1-\mu)\kappa}\left[\frac{\gamma_1^2-(\kappa+1)}{\gamma_1 K_1(\kappa,\varepsilon)} - \frac{\gamma_2^2-(\kappa+1)}{\gamma_2 K_2(\kappa,\varepsilon)}\right] \tag{3.18}$$

is the correction coefficient which reflects the effect of reinforcement of the half-space with fibres, and which is the function of parameters κ and ε. Coefficient K can also be presented as a function of ν and ε, because $\kappa = 1/(1-2\nu)$ (see (2.2)). After additional transformations we obtain

$$K(\nu,\varepsilon) = \frac{\gamma_1+\gamma_2}{(1-\mu)[2+(1-\nu)\varepsilon]}\sqrt{1+\frac{1}{2}\cdot\frac{1-2\nu}{1-\nu}\cdot\varepsilon}\,. \tag{3.19}$$

It is easy to determine that $K(\nu,0) = 1$, and in the particular case $\nu = 0$, $K(0,0) = 1$. The diagram of $K = K(\nu,\varepsilon)$ is shown in Fig. 1.

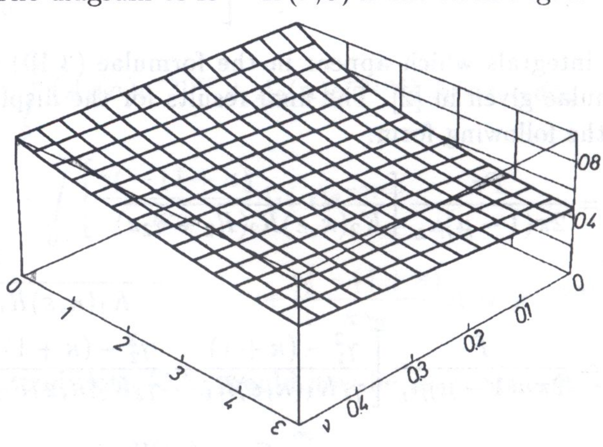

FIG. 1. Diagram of coefficient K as a function of Poisson's ratio ν of the matrix and of the fibre characteristics ε.

4. Stresses

Returning now to the formula (2.3), it is possible to calculate the components of the stress tensor in a cylindrical coordinates system:

$$\tau_{zz} = -\frac{P}{2\pi}\cdot\frac{\gamma_1\gamma_2 z}{\gamma_1-\gamma_2}(R_2^{-3}-R_1^{-3}),$$

$$\tau_{zr} = -\frac{P}{2\pi}\cdot\frac{\gamma_1\gamma_2 r}{\gamma_1-\gamma_2}(R_2^{-3}-R_1^{-3}),$$

(4.1)
$$\tau_{rr} = \frac{P}{2\pi}\left\{\frac{\gamma_1\gamma_2 z}{\gamma_1-\gamma_2}(\gamma_2^2R_2^{-3}-\gamma_1^2R_1^{-3}) - 2\left[\frac{1}{K_2(\kappa,\varepsilon)R_2(R_2+\gamma_2 z)} - \frac{1}{K_1(\kappa,\varepsilon)R_1(R_1+\gamma_1 z)}\right]\right\},$$

$$\tau_{\varphi\varphi} = \frac{P}{2\pi}\left\{\frac{\kappa-1}{\kappa}\left[\frac{\gamma_2^2-1}{K_2(\kappa,\varepsilon)}\cdot\frac{\gamma_2 z}{R_2^3} - \frac{\gamma_1^2-1}{K_1(\kappa,\varepsilon)}\cdot\frac{\gamma_1 z}{R_1^3}\right] + 2\left[\frac{1}{K_2(\kappa,\varepsilon)R_2(R_2+\gamma_2 z)} - \frac{1}{K_1(\kappa,\varepsilon)R_1(R_1+\gamma_1 z)}\right]\right\},$$

$$\gamma_1 > \gamma_2\,.$$

A certain basic property of the Boussinesq solution is known for an isotropic half-space; namely, the direction of the resultant stress at any point of every horizontal plane ($z = \text{const} > 0$) passes through the force application point ($z = 0$, $r = 0$). If we compare formulae $(4.1)_1$ and $(4.1)_2$, we obtain

(4.2)
$$\frac{\tau_{zz}}{\tau_{zr}} = \frac{z}{r},$$

thus the above property is preserved in a fibrous composite half-space.

The stresses τ_{zz} can be decomposed into the stresses σ_{zz} within the matrix and the stresses in fibres $\underset{1}{\sigma}$:

(4.3)
$$\sigma_{zz} = -\frac{P}{2\pi}\cdot\frac{z}{(1-\mu)\kappa}\left\{\frac{[3\kappa+1-(\kappa+1)\gamma_2^2]\gamma_2}{K_2(\kappa,\varepsilon)R_2^3} - \frac{[3\kappa+1-(\kappa+1)\gamma_1^2]\gamma_1}{K_1(\kappa,\varepsilon)R_1^3}\right\},$$

(4.4)
$$\underset{1}{\sigma} = -\frac{P}{2\pi}\cdot\frac{\varepsilon z}{\underset{1}{\mu}\,\kappa}\left[\frac{(\kappa+1-\gamma_2^2)\gamma_2}{K_2(\kappa,\varepsilon)R_2^3} - \frac{(\kappa+1-\gamma_1^2)\gamma_1}{K_1(\kappa,\varepsilon)R_1^3}\right]$$

while, according to (1.3),

(4.5)
$$\tau_{zz} = \left(1-\underset{1}{\mu}\right)\sigma_{zz} + \underset{1}{\mu}\,\underset{1}{\sigma}\,.$$

It will be interesting to compare the results obtained with the Boussinesq formulae.

The following assumptions were made for the comparative calculations:

- Poisson's ratio of matrix $\nu = 0.2$,
- fibrous phase parameter $\varepsilon = 2$,
- fibrous phase density $\underset{1}{\mu} = 0.01$.

The results have been shown in the figures as the diagrams of stresses being the functions of variable r for any $z = \text{const}$. In Fig. 2, the diagram of $\sigma_{zz}^{(B)}$ is presented (curve *1* according to Boussinesq) and τ_{zz} (fibrous composite – curve *2*). The difference is shown by higher stress concentration in the neighbourhood of $r = 0$, which is justified by the fact that the great part of stresses is taken over by the fibrous phase. The stress relaxation

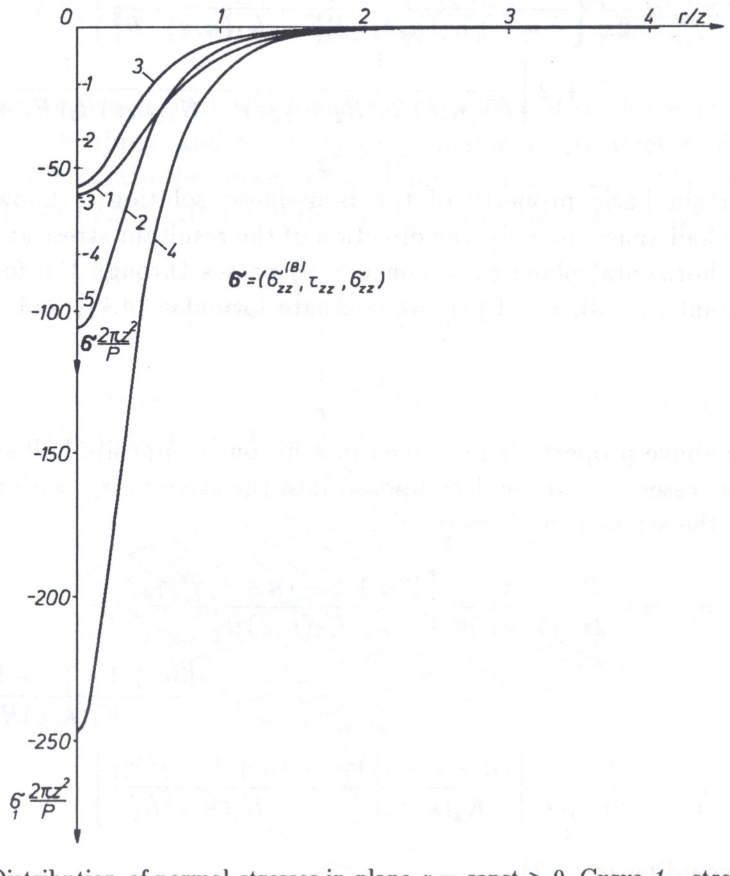

FIG. 2. Distribution of normal stresses in plane $z = \text{const} > 0$. Curve *1* – stresses $\sigma_{zz}^{(B)}$ (according to Boussinesq), curve *2* – stresses τ_{zz} (fibrous composite), curve *3* – stresses σ_{zz} (matrix of fibrous composite), curve *4* – stresses $\underset{1}{\sigma}$ in the fibres.

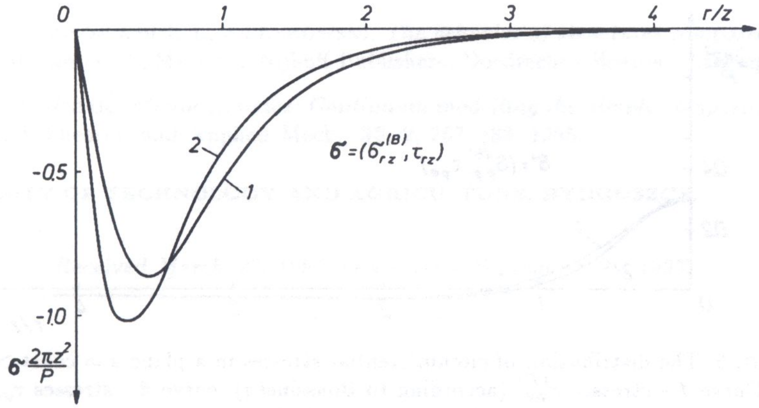

FIG. 3. The distribution of shearing stresses in a plane $z = \text{const} > 0$. Curve *1* – stresses $\sigma_{rz}^{(B)}$ (according to Boussinesq), curve *2* – stresses τ_{rz} (fibrous composite).

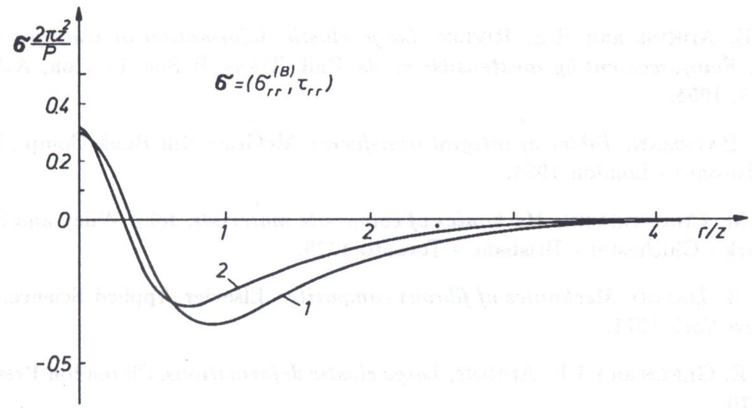

FIG. 4. The distribution of radial stresses in a plane $z = \text{const} > 0$. Curve *1* – stresses $\sigma_{rr}^{(B)}$ (according to Boussinesq), curve *2* – stresses τ_{rr} (fibrous composite).

in the matrix is visible (curve *3*, which presents the diagram of stresses in matrix σ_{zz}). The diagram of stresses in the fibres $\underset{1}{\sigma}$ (curve *4*) indicates their decisive importance close to the point $r = 0$. Figure 3 presents the diagrams of shearing stresses $\sigma_{rz}^{(B)}$ (according to Boussinesq – curve *1*) and τ_{rz} (fibrous composite – curve *2*). Analogous comparisons are presented for radial stresses (Fig. 4) and for circumferential stresses (Fig. 5). It is shown that the influence of the fibrous phase on the stress distribution can be considerable, though the general character of the solution is similar.

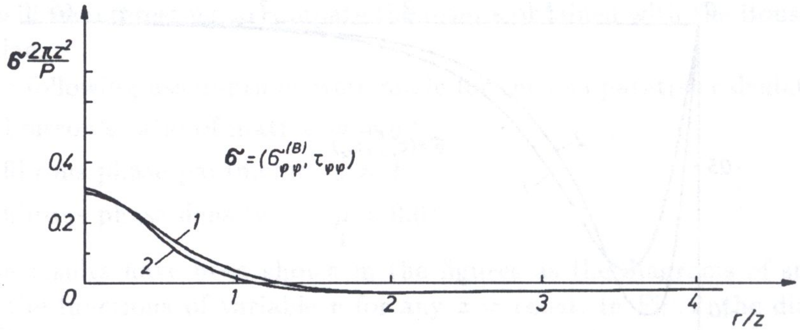

Fig. 5. The distribution of circumferential stresses in a plane $z = \text{const} > 0$. Curve *1* – stresses $\sigma^{(B)}_{\varphi\varphi}$ (according to Boussinesq), curve *2* – stresses $\tau_{\varphi\varphi}$ (fibrous composite).

References

1. J.E. Adkins and R.S. Rivlin, *Large elastic deformation of isotropic materials, X. Reinforcement by inextensible cords*, Phil. Trans. R. Soc. London, **A.248**, 201–223, 1955.

2. H. Bateman, *Tables of integral transforms*, McGraw-Hill Book Comp., New York - Toronto - London 1954.

3. R.M. Christensen, *Mechanics of composite materials*, John Wiley and Sons, New York - Chichester - Brisbane - Toronto 1979.

4. M.H. Datoo, *Mechanics of fibrous composites*, Elsevier Applied Science, London - New York 1971.

5. A.E. Green and J.E. Adkins, *Large elastic deformations*, Clarendon Press, Oxford 1970.

6. R.M. Jones, *Mechanics of composite materials*, McGraw-Hill Book Comp., New York 1975.

7. J.F. Mulhern, T.G. Rogers and A.J.M. Spencer, *A continuum model for fibre-reinforced plastic materials*, Proc. Roy. Soc., **A.301**, 473–492, 1967.

8. W. Nowacki, *Theory of elasticity* [in Polish], PWN, Warszawa 1970.

9. A.C. Pipkin, *Finite deformations of ideal fiber-reinforced composites*, [in:] Composite Materials, vol. 2, G.P. Sendeckyj [Ed.], 251–308, Academic Press, New York and London 1974.

10. R. Świtka, *Thermoelasticity of fiber composites*, J. Theoret. and Applied Mech., **33**, 2, 385–400, 1995.

11. S.P. Timoshenko, *History of strength of materials*, McGraw-Hill Book Comp., New York - Toronto - London 1953.

12. J.R. VINSON and E.L. SIERAKOWSKI, *The behavior of structures composed of composite materials*, Martinus Nijhoff Publishers, Dordrecht - Boston - Lancaster 1986.

13. C. WOŹNIAK, *Microdynamics: Continuum modelling the simple composite materials*, J. Theoret. and Applied Mech., **33**, 2, 267–289, 1995.

UNIVERSITY OF TECHNOLOGY AND AGRICULTURE, BYDGOSZCZ.

Received March 27, 1995; new version September 25, 1995.