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# ON THE MODELLING OF LASER THERMAL FRACTURING OF HARD ROCK

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The paper deals with the nonstationary problem of elastic half-space heated by laser beam. On the basis of the uncoupled linear thermoelasticity, the distribution of temperature and stresses in the body is obtained. By using the theory of brittle fracture given by Griffith and McClintock and Walsh, the fracture trajectories are presented for three kinds of rocks: granites quartzites and gabbros for a given radius of the heating region and its intensity.

## 1. Introduction

The laser heating of materials is applied in many processes, for instance in thermal fracturing of hard rock [1]. The action of a laser ray is described by a local thermal source, which can be determined by distribution of the heat flow. In practice, two cases of thermal source distribution are investigated: a normal (Gaussian) distribution and a uniform distribution, and as object of heating is taken a half-space. It is shown [2] that the normal distribution of thermal source more accurately describes the action of a laser ray. The problem of finding the temperature field in semi-infinite isotropic body with surface heating by laser beam in the case of the Gaussian heat flow intensity is analyzed in [3].

In this paper the nonstationary temperature distribution and quasi-static stress fields in the half-space heated by normal thermal flux distributed on a circle on its surface is found. It is shown that three zones of tensile, compressive and shearing stresses can be assumed inside the body. The thermoelastic state of stresses can initiate the brittle fracture of material, which may be described by the fracture theory given by Griffith [4] as well as McClintock and Walsh [5]. The fracture paths are obtained for three different types of rocks. The obtained results allow for the optimization of thermal fracturing processes of the rocks.

## 2. Temperature field

The formulation of the axisymmetric problem for laser heating of a semi-infinite body takes the form [6]

$$\frac{\partial^2 T}{\partial \varrho^2}+\frac{1}{\varrho}\frac{\partial T}{\partial \varrho}+\frac{\partial^2 T}{\partial Z^2}=\frac{\partial T}{\partial \mathrm{Fo}}, \qquad 0\le\varrho\le\infty, \quad 0\le Z<\infty, \quad \mathrm{Fo}>0; \tag{2.1}$$

$$T(\varrho,Z,0)=0, \qquad 0\le\varrho<\infty, \quad 0\le Z<\infty; \tag{2.2}$$

$$\frac{\partial T}{\partial Z}-\mathrm{Bi}\,T=-\Lambda e^{-\varrho^2}H(\mathrm{Fo}), \qquad \varrho\ge0, \quad Z=0; \tag{2.3}$$

$$T(\infty,Z,\mathrm{Fo})=T(\varrho,\infty,\mathrm{Fo})=0, \qquad \mathrm{Fo}>0; \tag{2.4}$$

where

$$\varrho=\frac{r}{a}, \qquad Z=\frac{z}{a}, \qquad \mathrm{Fo}=\frac{kt}{a^2}, \qquad \mathrm{Bi}=\frac{ha}{K}, \qquad \Lambda=\frac{qa}{K}, \tag{2.5}$$

and $T$ is temperature, $(r,z)$ denotes the cylindrical coordinates of the system with the origin at the centre of heating, $a$ is the radius of heated circle, $t$ denotes time, $q$ is the maximum density of thermal flux and $H(\cdot)$ denotes the Heaviside function. The constants $K$, $k$ and $h$ denote the coefficients of thermal conductivity, thermal diffusivity and surface temperature conductivity, respectively. The physical significance of boundary condition (2.3) is that the boundary surface dissipates heat by convection according to Newton's law of cooling.

The foregoing equations may be solved by using the integral transform technique [7]. Applying the Hankel transform with respect to radial coordinate $\varrho$, we obtain

$$\frac{\partial^2 \overline{T}}{\partial Z^2}-\xi^2\overline{T}=\frac{\partial \overline{T}}{\partial \mathrm{Fo}}, \tag{2.6}$$

where

$$\overline{T}(\xi,Z,\mathrm{Fo})=\int_0^\infty \varrho T(\varrho,Z,\mathrm{Fo})J_0(\xi\varrho)\,d\varrho. \tag{2.7}$$

By transforming the conditions (2.2), (2.3) and (2.4) we have:

$$\overline{T}(\xi,Z,0)=0; \tag{2.8}$$

$$\overline{T}(\xi,\infty,\mathrm{Fo})=0; \tag{2.9}$$

$$\frac{\partial \overline{T}}{\partial Z}=-\Lambda\varphi(\xi)+\mathrm{Bi}\overline{T} \qquad \text{when} \quad Z=0. \tag{2.10}$$

The function $\varphi(\xi)$ in Eq. (2.10) is given by [8]:

$$\varphi(\xi) = \int_0^\infty \varrho e^{-\varrho^2} J_0(\xi\varrho)\, d\varrho = 1/2 e^{-\xi^2/4}. \tag{2.11}$$

By using now the Fourier integral transform with a general trigonometric kernel with respect to variable $Z$, from Eqs. (2.6), (2.8)–(2.10) it follows that

$$\frac{d\tilde{\overline{T}}}{d\mathrm{Fo}} + (\zeta^2 + \xi^2)\tilde{\overline{T}} = \left\{\frac{2}{\pi}\right\}^{1/2} \Lambda\zeta\varphi(\xi), \qquad \mathrm{Fo} > 0, \tag{2.12}$$

$$\tilde{\overline{T}}(\xi, \zeta, 0) = 0, \tag{2.13}$$

where

$$\tilde{\overline{T}}(\xi, \zeta, \mathrm{Fo}) = \left\{\frac{2}{\pi}\right\}^{1/2} \int_0^\infty \overline{T}(\xi, Z, \mathrm{Fo}) N(Z, \zeta)\, dZ\,, \tag{2.14}$$

$$N(Z, \zeta) = \zeta\cos(\zeta Z) + \mathrm{Bi}\sin(\zeta Z). \tag{2.15}$$

Solution of the ordinary differential equation (2.12) with condition (2.13) can be written in the form

$$\tilde{\overline{T}}(\xi, \zeta, \mathrm{Fo}) = \Lambda\varphi(\xi)\tilde{\Phi}_0(\xi, \zeta, \mathrm{Fo}), \tag{2.16}$$

where

$$\tilde{\Phi}_0(\xi, \zeta, \mathrm{Fo}) = \left\{\frac{2}{\pi}\right\}^{1/2} \frac{\zeta}{\zeta^2 + \xi^2}\left[1 - e^{-(\zeta^2+\xi^2)\mathrm{Fo}}\right]. \tag{2.17}$$

Applying the inverse Fourier and Hankel transforms:

$$\begin{aligned} \overline{T}(\xi, Z, \mathrm{Fo}) &= \left\{\frac{2}{\pi}\right\}^{1/2} \int_0^\infty \frac{N(Z, \zeta)}{\zeta^2 + \mathrm{Bi}^2}\tilde{\overline{T}}(\xi, \zeta, \mathrm{Fo})\, d\zeta, \\ T(\varrho, Z, \mathrm{Fo}) &= \int_0^\infty \xi\overline{T}(\xi, Z, \mathrm{Fo}) J_0(\xi\varrho)\, d\xi, \end{aligned} \tag{2.18}$$

to the solution (2.14), we obtain the function

$$T(\varrho, Z, \mathrm{Fo}) = \Lambda\int_0^\infty \xi\varphi(\xi)\Phi_0(\xi, Z, \mathrm{Fo}) J_0(\xi\varrho)\, d\xi, \tag{2.19}$$

where

$$\Phi_0(\xi,Z,\mathrm{Fo}) = \frac{1}{2}\left[\frac{e^{-\xi Z}}{\mathrm{Bi}+\xi}\operatorname{erfc}\left(\frac{Z}{2\sqrt{\mathrm{Fo}}}-\xi\sqrt{\mathrm{Fo}}\right) + \frac{e^{\xi Z}}{\mathrm{Bi}-\xi}\operatorname{erfc}\left(\frac{Z}{2\sqrt{\mathrm{Fo}}}+\xi\sqrt{\mathrm{Fo}}\right)\right] - \frac{\mathrm{Bi}\, e^{\mathrm{Bi}\, Z}}{\mathrm{Bi}^2-\xi^2}e^{(\mathrm{Bi}^2-\xi^2)\mathrm{Fo}}\operatorname{erfc}\left(\frac{Z}{2\sqrt{\mathrm{Fo}}}+\mathrm{Bi}\sqrt{\mathrm{Fo}}\right), \tag{2.20}$$

and $\operatorname{erfc}(\cdot) = 1-\operatorname{erf}(\cdot)$, $\operatorname{erf}(\cdot)$ is the error function, and $J_n(\cdot)$ is the Bessel function of the first kind and order $n$.

We note that at $\xi = \mathrm{Bi}$ from Eq. (2.20) it follows that

$$\Phi_0(\mathrm{Bi},Z,\mathrm{Fo}) = \frac{1}{4\mathrm{Bi}}e^{-\mathrm{Bi}\, Z}\operatorname{erfc}\left(\frac{Z}{2\sqrt{\mathrm{Fo}}}-\mathrm{Bi}\sqrt{\mathrm{Fo}}\right). \tag{2.21}$$

The function $\Phi_0(\xi,Z,\mathrm{Fo})$ given by (2.20) in absence of the convective cooling, $\mathrm{Bi}=0$, is reduced to the form [9]

$$\Phi_0(\xi,Z,\mathrm{Fo}) = \frac{1}{2\xi}\left[e^{-\xi Z}\operatorname{erfc}\left(\frac{Z}{2\sqrt{\mathrm{Fo}}}-\xi\sqrt{\mathrm{Fo}}\right) - e^{\xi Z}\operatorname{erfc}\left(\frac{Z}{2\sqrt{\mathrm{Fo}}}+\xi\sqrt{\mathrm{Fo}}\right)\right]. \tag{2.22}$$

The stationary temperature at the centre of the heated region $\varrho = 0$, $\mathrm{Fo}\to\infty$ is determined on the basis of Eqs. (2.11), (2.19) and (2.20) by the relation

$$T_{\max}(0,0,\infty) = \frac{\Lambda\sqrt{\pi}}{2} - \frac{\Lambda \mathrm{Bi}}{2}\int_0^\infty \frac{e^{-\xi^2/4}}{\mathrm{Bi}+\xi}d\xi\,. \tag{2.23}$$

From Eq. (2.23) at $\mathrm{Bi}=0$ we obtain the well-known value of maximum temperature in the steady state without the convective exchange [10],

$$T_{\max}(0,0,\infty) = (\sqrt{\pi}/2)\,\Lambda = 0.8862\Lambda. \tag{2.24}$$

## 3. Thermal stresses

Consider now an isotropic thermoelastic half-space free from force loading on its boundary plane. The stresses in the semi-infite body are caused only by the action of the laser beam described in Sec. 2. It is well known that the solution of the uncoupled thermoelastic equations (in the case in which body forces are neglected) may be expressed as a sum of the particular solution of the nonhomogeneous system of equations for displacements (with the temperature field) and the general solution of the homogeneous system (without the temperature) [11, 12]. The particular solution can be found by introducing the thermoelastic displacement potential defined by the relations [11]:

$$u_r^{(1)} = \frac{1}{a}\frac{\partial \psi}{\partial \varrho}, \qquad u_z^{(1)} = \frac{1}{a}\frac{\partial \psi}{\partial Z}. \tag{3.1}$$

The function $\psi$ satisfies the equation

$$\nabla^2 \psi = \beta_t a^2 T, \tag{3.2}$$

where

$$\beta_t = \alpha\frac{1+\nu}{1-\nu} \tag{3.3}$$

and $\alpha$ is the coefficient of thermal expansion, $\nu$ is Poisson's ratio and $T$ is the temperature field obtained in Sec. 2.

The knowledge of potential $\psi$ leads to the stresses according to the following relations [11]:

$$\begin{gathered} \sigma_{rr}^{(1)} = \frac{2\mu}{a^2}\left(\frac{\partial^2\psi}{\partial\varrho^2} - \nabla^2\psi\right), \qquad \sigma_{\theta\theta}^{(1)} = \frac{2\mu}{a^2}\left(\frac{1}{\varrho}\frac{\partial\psi}{\partial\varrho} - \nabla^2\psi\right), \\ \sigma_{zz}^{(1)} = \frac{2\mu}{a^2}\left(\frac{\partial^2\psi}{\partial Z^2} - \nabla^2\psi\right), \qquad \sigma_{rz}^{(1)} = \frac{2\mu}{a^2}\frac{\partial^2\psi}{\partial\varrho\partial Z}, \end{gathered} \tag{3.4}$$

where $\mu$ is the shear modulus.

In the case of temperature distribution given by (2.19), we obtain the potential $\psi$ in the form

$$\psi(\varrho, Z, \mathrm{Fo}) = \Lambda\beta_t \int_0^\infty \xi\varphi(\xi)\Phi_1(\xi, Z, \mathrm{Fo})J_0(\xi\varrho)\,d\xi, \tag{3.5}$$

where function $\Phi_1$ is defined by

$$\begin{aligned} \Phi_1(\xi, Z, \mathrm{Fo}) = \frac{1}{\mathrm{Bi}^2 - \xi^2}\Bigg\{&\left(\mathrm{Bi}\,\mathrm{Fo} + \frac{Z}{2} + \frac{\mathrm{Bi}}{\mathrm{Bi}^2-\xi^2}\right)G^+(\xi, Z, \mathrm{Fo}) \\ &- \left[\xi\mathrm{Fo} + \frac{\mathrm{Bi}\,Z}{2\xi} - \frac{1}{2\xi} + \frac{\mathrm{Bi}^2}{\xi(\mathrm{Bi}^2-\xi^2)}\right]G^-(\xi, Z, \mathrm{Fo}) - \sqrt{\frac{\mathrm{Fo}}{\pi}}e^{-\left(\xi^2\mathrm{Fo} + \frac{Z^2}{4\mathrm{Fo}}\right)}\Bigg\} \\ &- \frac{\mathrm{Bi}\,e^{\mathrm{Bi}\,Z}}{\mathrm{Bi}^2-\xi^2}e^{(\mathrm{Bi}^2-\xi^2)\mathrm{Fo}}\mathrm{erfc}\left(\frac{Z}{2\sqrt{\mathrm{Fo}}} + \mathrm{Bi}\sqrt{\mathrm{Fo}}\right), \end{aligned} \tag{3.6}$$

$$G^{\pm}(\xi, Z, \mathrm{Fo}) = \frac{1}{2}\left[e^{-\xi Z}\mathrm{erfc}\left(\frac{Z}{2\sqrt{\mathrm{Fo}}} - \xi\sqrt{\mathrm{Fo}}\right) \pm \mathrm{erfc}\left(\frac{Z}{2\sqrt{\mathrm{Fo}}} + \xi\sqrt{\mathrm{Fo}}\right)\right]. \tag{3.7}$$

By substituting the potential $\psi$ presented by (3.5) into Eqs. (3.4) we arrive at the following stresses for $Z > 0$:

$$
\begin{aligned}
\sigma_{rr}^{(1)} &= C\int_0^\infty \varphi(\xi)\Phi_1(\xi,Z,\mathrm{Fo})\left[\frac{\xi^2}{\varrho}J_1(\xi\varrho)-\xi^3 J_0(\xi\varrho)\right]d\xi - 2\mu\beta_t T,\\
\sigma_{\theta\theta}^{(1)} &= -C\int_0^\infty \varphi(\xi)\Phi_1(\xi,Z,\mathrm{Fo})\frac{\xi^2}{\varrho}J_1(\xi\varrho)\,d\xi - 2\mu\beta_t T,\\
\sigma_{zz}^{(1)} &= C\int_0^\infty \varphi(\xi)\Phi_1(\xi,Z,\mathrm{Fo})\xi^3 J_0(\xi\varrho)\,d\xi\,,\\
\sigma_{rz}^{(1)} &= -C\int_0^\infty \varphi(\xi)\Phi_2(\xi,Z,\mathrm{Fo})\xi^2 J_1(\xi\varrho)\,d\xi\,.
\end{aligned}
\tag{3.8}
$$

Here

$$
\begin{aligned}
\Phi_2(\xi,Z,\mathrm{Fo}) &= \frac{\partial}{\partial Z}\Phi_1(\xi,Z,\mathrm{Fo})\\
&= \frac{1}{\mathrm{Bi}^2-\xi^2}\Bigg\{\left(\xi^2\mathrm{Fo}+\frac{\mathrm{Bi}\,Z}{2}+\frac{\mathrm{Bi}^2}{\mathrm{Bi}^2-\xi^2}\right)G^+(\xi,Z,\mathrm{Fo})\\
&\quad - \left(\xi\mathrm{Bi}\,\mathrm{Fo}+\frac{\mathrm{Bi}}{2\xi}-\frac{\xi Z}{2}+\frac{\xi\mathrm{Bi}}{\mathrm{Bi}^2-\xi^2}\right)G^-(\xi,Z,\mathrm{Fo})\\
&\quad - \mathrm{Bi}\sqrt{\frac{\mathrm{Fo}}{\pi}}e^{-\left(\xi^2\mathrm{Fo}+\frac{Z^2}{4\mathrm{Fo}}\right)}\Bigg\}\\
&\quad - \frac{\mathrm{Bi}^2 e^{\mathrm{Bi}\,Z}}{\mathrm{Bi}^2-\xi^2}e^{(\mathrm{Bi}^2-\xi^2)\mathrm{Fo}}\operatorname{erfc}\left(\frac{Z}{2\sqrt{\mathrm{Fo}}}+\mathrm{Bi}\sqrt{\mathrm{Fo}}\right);
\end{aligned}
\tag{3.9}
$$

$$
C = 2\mu\beta_t\Lambda\,. \tag{3.10}
$$

It can be observed that for $Z = 0$ it follows that $\sigma_{zz}^{(1)} \neq 0$ and $\sigma_{rz}^{(1)} \neq 0$. In order to satisfy the conditions of the load-free boundary plane, the state of stresses $\sigma_{zz}^{(1)}$, $\sigma_{zr}^{(1)}$, $\sigma_{rr}^{(1)}$, $\sigma_{\theta\theta}^{(1)}$ should be completed by a state $\sigma_{zz}^{(2)}$, $\sigma_{zr}^{(2)}$, $\sigma_{rr}^{(2)}$, $\sigma_{\theta\theta}^{(2)}$ in the elastic half-space such that the following boundary conditions are satisfied for $Z = 0$:

$$
\sigma_{zz}^{(1)}+\sigma_{zz}^{(2)}=0,\qquad \sigma_{rz}^{(1)}+\sigma_{rz}^{(2)}=0 \qquad \text{for}\quad Z=0. \tag{3.11}
$$

The stresses $\sigma_{ij}^{(2)}$, $i,j, \in \{z,r,\theta\}$ can be determined by means of the Love function $L$ [12] which satisfies the biharmonic equation and boundary conditions (3.11).

Thus, we obtain

$$L(\varrho, Z, \mathrm{Fo}) = \frac{2\nu - 1}{2\mu} C a^3 \int_0^\infty \varphi(\xi) e^{-\xi Z} \Big\{ \Phi_1(\xi, 0, \mathrm{Fo}) + \left( \frac{2\nu - 1}{\xi} + Z \right) [\Phi_2(\xi, 0, \mathrm{Fo}) + \xi \Phi_1(\xi, 0, \mathrm{Fo})] \Big\} J_0(\xi\varrho)\, d\xi . \tag{3.12}$$

The stresses corresponding to function $L$ given by (3.12) will be found by the well-known formulae [12] in the form

$$\begin{aligned}
\sigma_{rr}^{(2)} &= -C \int_0^\infty \varphi(\xi)\xi^2 e^{-\xi Z} \Big\{ [(1 - \xi Z)\xi\Phi_1(\xi, 0, \mathrm{Fo}) \\
&\quad + (2 - \xi Z)\Phi_2(\xi, 0, \mathrm{Fo})]\, J_0(\xi\varrho) + [(2\nu - 1 + \xi Z)\xi\Phi_1(\xi, 0, \mathrm{Fo}) \\
&\quad + (2\nu - 2 + \xi Z)\Phi_2(\xi, 0, \mathrm{Fo})]\, \frac{J_1(\xi\varrho)}{\xi\varrho} \Big\} d\xi , \\
\sigma_{\theta\theta}^{(2)} &= -C \int_0^\infty \varphi(\xi)\xi^2 e^{-\xi Z} \Big\{ [2\nu\xi\Phi_1(\xi, 0, \mathrm{Fo}) + 2\nu\Phi_2(\xi, 0, \mathrm{Fo})]\, J_0(\xi\varrho) \\
&\quad - [(2\nu - 1 + \xi Z)\xi\Phi_1(\xi, 0, \mathrm{Fo}) + (2\nu - 2 + \xi Z)\Phi_2(\xi, 0, \mathrm{Fo})]\, \frac{J_1(\xi\varrho)}{\xi\varrho} \Big\} d\xi , \\
\sigma_{zz}^{(2)} &= -C \int_0^\infty \varphi(\xi)\xi^2 e^{-\xi Z} \Big[ (1 + \xi Z)\xi\Phi_1(\xi, 0, \mathrm{Fo}) \\
&\quad + \xi Z \Phi_2(\xi, 0, \mathrm{Fo}) \Big] J_0(\xi\varrho)\, d\xi , \\
\sigma_{rz}^{(2)} &= -C \int_0^\infty \varphi(\xi)\xi^2 e^{-\xi Z} \Big[ \xi^2 Z \Phi_1(\xi, 0, \mathrm{Fo}) \\
&\quad - (1 - \xi Z)\Phi_2(\xi, 0, \mathrm{Fo}) \Big] J_1(\xi\varrho)\, d\xi .
\end{aligned} \tag{3.13}$$

The final solution describing the state of stresses due to the laser heating of a half-space is obtained by superposing $\sigma_{ij}^{(1)}$, $ij \in \{z, r, \theta\}$ given by (3.8) and $\sigma_{ij}^{(2)}$, $ij \in \{z, r, \theta\}$ given by (3.13). After appropriate calculations we arrive at the following relations:

$$\sigma_{ij} = C \sigma_{ij}^{*}, \tag{3.14}$$

where

$$\sigma_{ij}^{*} = \int_0^\infty \varphi(\xi) S_{ij}(\xi, \varrho, \mathrm{Fo})\, d\xi - T^{*} \varepsilon_{ij} ; \tag{3.15}$$

$$
\begin{aligned}
S_{rr}(\xi,\varrho,Z,\mathrm{Fo}) &= \Phi_1(\xi,Z,\mathrm{Fo})\left[\frac{\xi^2}{\varrho}J_1(\xi\varrho) - \xi^3 J_0(\xi\varrho)\right] \\
&- \xi^2 e^{-\xi Z}\Big\{\left[(1-\xi Z)\xi\Phi_1(\xi,0,\mathrm{Fo}) + (2-\xi Z)\Phi_2(\xi,0,\mathrm{Fo})\right] J_0(\xi\varrho) \\
&+ \left[(2\nu - 1 + \xi Z)\xi\Phi_1(\xi,0,\mathrm{Fo}) + (2\nu - 2 + \xi Z)\Phi_2(\xi,0,\mathrm{Fo})\right]\frac{J_1(\xi\varrho)}{\xi\varrho}\Big\}, \\
S_{\theta\theta}(\xi,\varrho,Z,\mathrm{Fo}) &= -\Phi_1(\xi,Z,\mathrm{Fo})\frac{\xi^2}{\varrho}J_1(\xi\varrho) - \xi^2 e^{-\xi Z}\Big\{\Big[2\nu\xi\Phi_1(\xi,0,\mathrm{Fo}) \\
&+ 2\nu\Phi_2(\xi,0,\mathrm{Fo})\Big] J_0(\xi\varrho) - \Big[(2\nu - 1 + \xi Z)\xi\Phi_1(\xi,0,\mathrm{Fo}) \\
&+ (2\nu - 2 + \xi Z)\Phi_2(\xi,0,\mathrm{Fo})\Big]\frac{J_1(\xi\varrho}{\xi\varrho}\Big\}, \\
S_{zz}(\xi,\varrho,Z,\mathrm{Fo}) &= \Big\{\xi\Phi_1(\xi,Z,\mathrm{Fo}) - e^{-\xi Z}\Big[(1+\xi Z)\xi\Phi_1(\xi,0,\mathrm{Fo}) \\
&+ \xi Z\Phi_2(\xi,0,\mathrm{Fo})\Big]\Big\}\xi^2 J_0(\xi\varrho), \\
S_{rz}(\xi,\varrho,Z,\mathrm{Fo}) &= \Big\{-\Phi_2(\xi,Z,\mathrm{Fo}) + e^{-\xi Z}\Big[-\xi^2 Z\Phi_1(\xi,0,\mathrm{Fo}) \\
&+ (1-\xi Z)\Phi_2(\xi,0,\mathrm{Fo})\Big]\Big\}\xi^2 J_1(\xi\varrho);
\end{aligned}
\tag{3.16}
$$

$$
T^* = \frac{T}{\Lambda},
$$

$$
\varepsilon_{ij} = \begin{cases} 1 & \text{if } i = j = r \text{ or } \theta, \\ 0 & \text{if } i = j = z \text{ or } i = r, \; j = z, \end{cases}
$$

and where $T$ denotes the temperature defined by (2.19).

## 4. Thermal fracture criteria

The action of the laser beam described by boundary conditions (2.2) – (2.4) results in the thermal stresses determined in Sec. 3. The stresses can produce an initiation and growth of cracks in the body leading to the fracture. For the description of the phenomena, the theory of brittle fracture given by GRIFFITH [4] and MCCLINTOCK and WALSH [5] will be applied.

The normal stresses $\sigma_n$ acting on the boundary of the assumed crack are determined by the following equation [13]:

$$
\sigma_n = \frac{1}{2}\left[(\sigma_1 + \sigma_3) + (\sigma_1 - \sigma_3)\cos(2\theta)\right],
$$

where $\sigma_1$ and $\sigma_3$ denote the maximal and minimal principal stresses (tensile stresses are treated as positive), $\theta_1 = 0.5\tan^{-1}(1/f)$ denotes the angle of crack orientation measured anticlockwise from the direction of $\sigma_3$, and $f$ is the coefficient of surface friction.

If the normal stresses $\sigma_n$ on the crack surface lead to its opening ($\sigma_n < 0$) then the brittle fracture of the body can be determined by using the Griffith and the modified GRIFFITH criteria [4]. According to the first criterion, the fracture process will be started at a given point of the body if

$$\sigma_1 = \sigma_T\,, \tag{4.1}$$

provided that $3\sigma_1 + \sigma_3 > 0$ and where $\sigma_T$ denotes the tensile strength of the material.

In this case the growth of cracks occurs in the plane normal to the direction of principal stresses $\sigma_1$.

If the state of stresses satisfies the condition $3\sigma_1 + \sigma_3 < 0$, the initiation of fracture will be possible under the condition of cracking

$$-(\sigma_1 + \sigma_3)^{-1}(\sigma_1 - \sigma_3)^2 = 8\sigma_T\,. \tag{4.2}$$

In this case the crack growth occurs in the plane inclined by angle $\theta_2$ to the direction of action of the maximal principal stress $\sigma_1$, where angle $\theta_2$ satisfies the equation

$$\cos(2\theta) = 0.5(\sigma_1 + \sigma_3)^{-1}(\sigma_1 - \sigma_3).$$

The fracture caused by compresive stresses $\sigma_n > 0$ will be initiated under the following condition [5]:

$$\sigma_3 - \sigma_1\left[\left(\sqrt{1+f^2}+f\right)/\left(\sqrt{1+f^2}-f\right)\right] = \sigma_c\,, \tag{4.3}$$

where $\sigma_c$ denotes the compressive strength of material. The cracking occurs in the direction of action of the maximal principal stresses.

## 5. Numerical analysis

Figure 1 shows the processes of variation of dimensionless temperature $T^*$ defined by (3.17) at points of surface $Z = 0$ for $\mathrm{Bi} = 0.1$ and $\varrho = 0; 1; 2$. For points lying more closely to the heated region, the process is shorter. For $\mathrm{Fo} = 1$ the temperature at the centre $\varrho = 0$ is equal to 75%, and at the point $\varrho = 2$ it is equal to 50% of the stationary value of temperature given by (2.23). The large gradient of temperature becomes stabilized close to the moment of heating (for $\mathrm{Fo} \approx 0.1$). The values of temperature gradients decrease in the vicinity of the heated region and at the stationary state. Thus, the values of $T^*(\varrho = 0)/T^*(\varrho = 2)$ at times $\mathrm{Fo} = 0.1$; 0.5; 1;2 are equal to 36.2; 17.8; 13.3; 10.9.

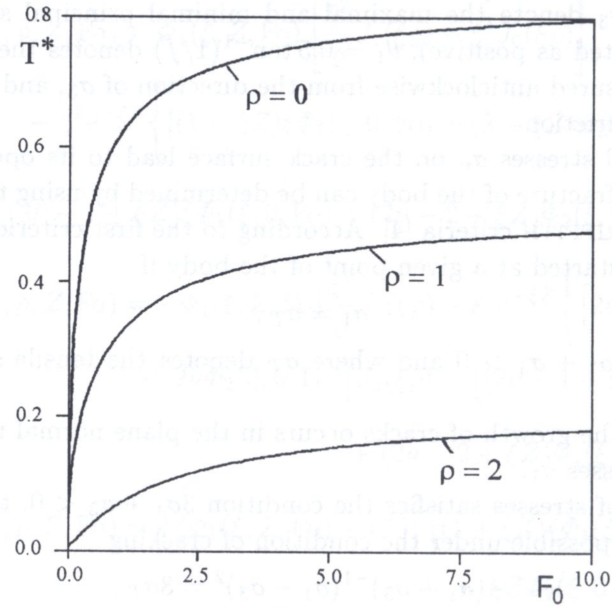


FIG. 1.

It can be seen that the value Fo $\approx 10$ corresponds to the stationary state given by (2.23) at the moment of heating for Bi $= 0.1$. For the considered materials $k \approx 10^{-5}\,\mathrm{m}^2/\mathrm{s}$, so for $a = 0.1\,\mu\mathrm{m}$ and $a = 1\,\mu\mathrm{m}$ the characteristic times of the process are 0.1 ms or 10 ms.

The distributions of temperature $T^*$ along lines $\varrho = 0$, $\varrho = 1$ and $\varrho = 2$ for Bi $= 0.1$, Fo $= 1$ are shown on Fig. 2. The temperature decreases fast with the growth of depth $Z$, and for $Z \geq 1$ it is almost equal to zero.

It should to be emphasized that the accuracy of temperature determination depends on the conductivity $h$ of surface temperature. In the paper [10] determining the surface temperature, the relation between $h$ and $K$ is given, and the evaluation of $h \approx 0.02K/a$ is presented under the assumption that a convective heat exchange reduces the maximal temperature of the body by no more than 10%.

The variation of maximal value of $\sigma_1^* = \sigma_1/C$ and minimal value of $\sigma_3^* = \sigma_3/C$ of dimensionless principal stresses with respect to depth along lines $\varrho = 0$; 1; 2 for Fo $= 1$, Bi $= 0.1$, $f = 0.9$ is shown in Fig. 3 and Fig. 4, respectively. The principal stresses $\sigma_1^*$ are positive for $Z > 0$ and they reach the maximal value near the boundary surface of the half-space for $Z \approx 2.4$. The stresses $\sigma_3^*$ are negative for $0 \leq Z \leq 1.8$ and they almost vanish for $Z > 1.8$.

By substituting values of $\sigma_1^*$ and $\sigma_3^*$ into the criterial equations (4.1) – (4.3), three zones of the state of stresses produced by laser heating can be established in region $0 \leq \varrho \leq 3$, $0 \leq Z \leq 5$ for Bi $= 0.1$, Fo $= 0.1$, see Fig. 5.

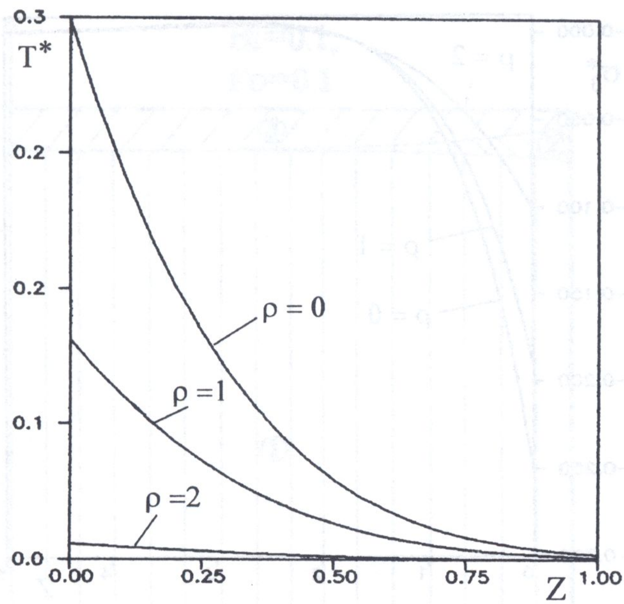


FIG. 2.

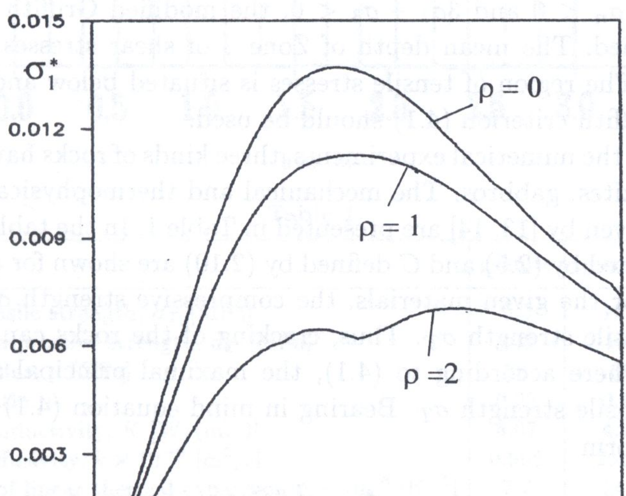


FIG. 3.

In the region $0 \leq Z \leq 0.4$ marked by 3 and situated directly below the heated region, the stress state is suitable for applying the McClintock - Walsh condition given by (4.3) and to determine the compresional fracture. The zone marked

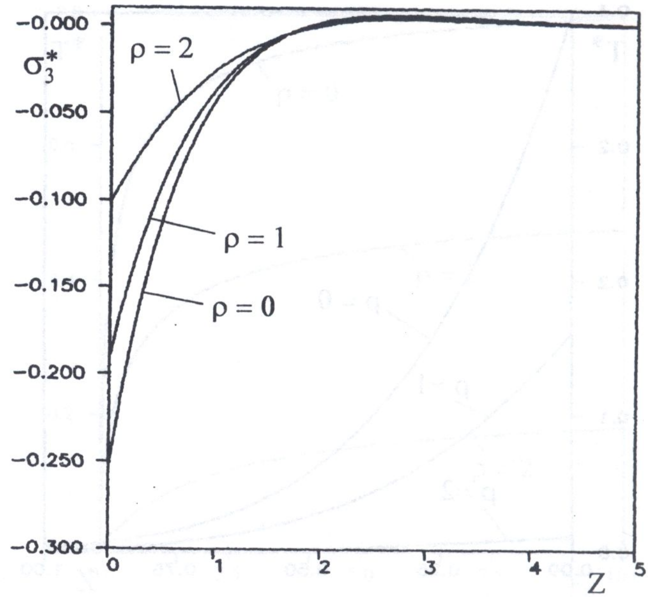


FIG. 4.

by 2, in which $\sigma_n < 0$ and $3\sigma_1 + \sigma_3 < 0$, the modified Griffith criterion (4.2) should be applied. The mean depth of Zone 2 of shear stresses is not greater than $Z = 0.5$. The region of tensile stresses is situated below and is marked by 1. Here the Griffith criterion (4.1) should be used.

To carry out the numerical experiments, three kinds of rocks have been chosen: granites, quartzites, gabbros. The mechanical and thermophysical properties of the materials given by [13, 14] are presented in Table 1. In the table, the values of constans $\Lambda$ defined by (2.5) and $C$ defined by (2.10) are shown for $q = 10^8\,\mathrm{W/m^2}$, $a = 0.1\,\mathrm{mm}$. For the given materials, the compressive strength $\sigma_c$ considerably exceeds the tensile strength $\sigma_T$. Thus, cracking of the rocks can occur in Zone 1, see Fig. 5, where according to (4.1), the maximal principal stresses $\sigma_1$ are equal to the tensile strength $\sigma_T$. Bearing in mind equation (4.1) written in the dimensionless form

$$\sigma_1^* = \sigma^T / C \,, \tag{5.1}$$

the values of $\sigma_T$ and material constants of the considered rocks are presented in Table 1.

The set of points of the plane $\varrho Z$, where the Griffith criterion is satisfied can be illustrated by a continuous curve of equal stresses (isolines). The isolines of the value of 0.002 for quartzites, of the value of 0.005 for granites and of the value of 0.011 for gabbro, are shown in Fig. 6. Since the crack growth in Zone 1 can be generated in the direction normal to the direction of action of the maximal

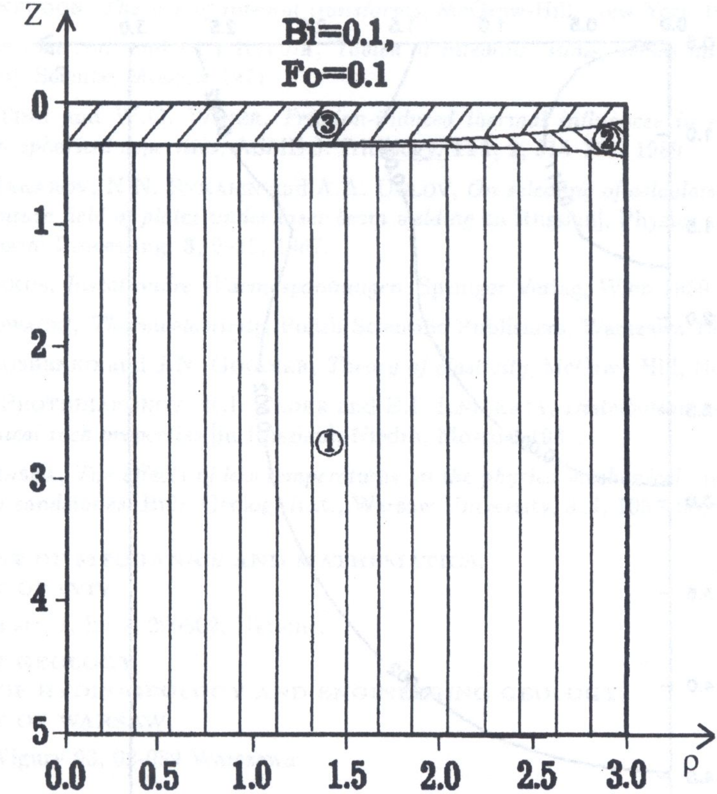


FIG. 5.

**Table 1.**

| Parameters | granite | quartzite | gabbro |
|---|---|---|---|
| Uniaxial tensile strength, $\sigma_T$ [MPa] | 9.0 | 13.5 | 16.0 |
| Uniaxial compressive strength, $\sigma_C$ [MPa] | 205 | 190 | 162 |
| Shear modulus, $\mu$ [GPa] | 28 | 36 | 34 |
| Poisson's ratio, $\nu$ | 0.23 | 0.16 | 0.24 |
| Thermal conductivity, $K$ [W/(mK)] | 4.07 | 4.21 | 3.67 |
| Thermal diffusivity, $k \times 10^{-6}$ [m$^2$/s] | 0.505 | 2.467 | 0.458 |
| Coefficient of linear thermal expansion $\alpha \times 10^{-6}$ [K$^{-1}$] | 7.7 | 24.2 | 4.7 |
| $\Lambda \times 10^4$ [K] | 0.246 | 0.237 | 0.272 |
| $C$ [GPa] | 1.69 | 5.70 | 1.42 |
| $(\sigma_T/C) \times 10^{-3}$ | 5.319 | 2.367 | 11.280 |

principal stresses, the isolines shown are normal to the directon of $\sigma_1$ at every point and they represent the fracture trajectories. Moreover, because values of the principal stress $\sigma_1$ increase with the growth of $Z$, the isolines determine a limiting depth, at which the thermal fracture is possible.

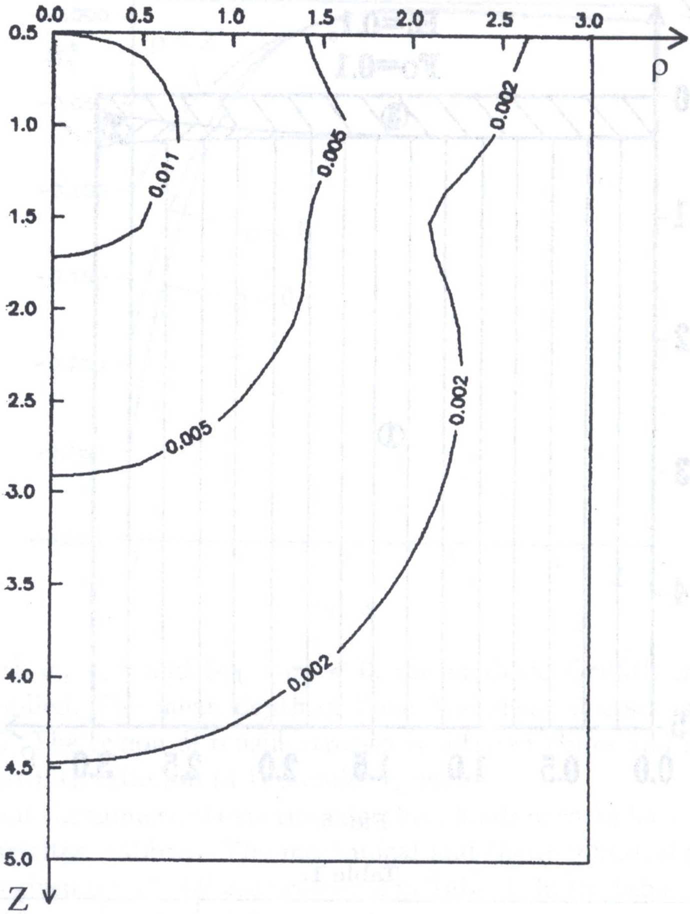


FIG. 6.

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