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CHANGES IN THE EIGENPAIRS DUE TO THE STRUCTURE MODIFICATION THE CONTINUOUS APPROACH

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During the past years, a great deal of time and efforts have been spent in analysing the changes of eigenpairs of the structure due to the reduction of stiffness and especially of cracks as well. These investigations are further used first to identify the crack location and then its magnitude. The whole attention is focussed on cracks and similar types of damage, neglecting the fact that the eigenpairs changes can be also due to other causes, for example variations of the mass. The paper presents the comparison of eigenvector changes due to two types of structure modifications: stiffness and mass variations. The study covers both the discrete approach using finite element method, and also continuous approach. It is evident from both cases that eigenvectors changes different structure modifications exhibit regular patterns and therefore, it is possible to identify not only the location but also to guess the type of the structure modification. Once when the type of the modification is known, it is possible to obtain the magnitude of the modification from the eigenfrequency change.

1. Introduction

The influence of location and magnitude of the structure's change on the response of the structure is an interesting engineering problem. YUEN [1] offers a systematic study where he considers eigenshape's change due to the stiffness reduction. The study is performed on a simple cantilever beam and is limited to the first eigenfrequency change only. In the extensive study PANDEY *et al.* [2] observe the influence of the absolute change on eigenshapes (even higher). A new parameter that indicates the presence of a crack is introduced. BARUH and RATAN [3] present another criterion for the determination of irregularities in the structure combining the dynamic characteristics of an original system with the characteristics of a modified system. SKRINAR [4, 5] has analyzed and expanded all these studies in two directions: first, by introduction of longitudinal displacements and secondly, by considering also the mass variations.

Previous studies using solely the finite element models have proved that locations of the changes can be determined either directly from the eigenvectors (modal shapes) or from the derivatives of their components. The scope of the present investigation is to perform the matching analysis using the continuous approach and thus to verify the results obtained.

2. The analysis process

2.1. *Introduction*

The analysis process in the paper consists in determination of eigenfrequencies and the corresponding eigenvectors (modal shapes) of an undamped continuous model of the mechanical system. The location of modification is in an arbitrarily chosen point of the structure, and the influence of various changes at the same location is examined.

Skrinar [4] has extended the studies given by Yuen [1], Pandey *et al.* [2] and Baruh *et al.* [3] by involving detailed systematisation and generalisation of the type of change. The modification of the structure was not limited to a local reduction of stiffness only, but has included also a local variation of mass as well a simultaneous variation of stiffness and mass. Another essential extension was that all degrees of freedom (longitudinal and transverse displacements, and rotational degrees of freedom) were considered in the finite element analysis, yielding some interesting conclusions.

To complete the previous studies, the same cantilever beam structure will be analysed again using a continuous approach.

The equation of free transverse vibration of a beam is written as follows:

$$m \cdot \frac{\partial^2 y}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) = 0, \tag{2.1}$$

where y – lateral displacement which is a function of the spatial coordinate x and time t, m – the mass distribution of the unit of length, EI – flexural rigidity, the product of the elasticity modulus and the moment of inertia.

The solution of Eq. (2.1) is assumed to be in the form:

$$y(x,t) = v(x) \cdot Y(t), \tag{2.2}$$

where $v(x)$ denotes the eigenshape and $Y(t)$ governs the behaviour of the structure in time.

Introducing Eq. (2.2) into Eq. (2.1), the latter can be separated into two ordinary differential equations. Analysing further and introducing the actual boundary conditions, the solution for the eigenfrequencies is obtained. Knowing eigenfrequencies λ_i $(i = 1, 2, ...)$ it is possible to obtain continuous functions $v_i(x)$ that correspond to the eigenvectors $\mathbf{u}_i$ in the finite element analysis.

The process can be repeated for the modified system to obtain the associated eigenfrequencies λ_i' and eigenshapes $w_i(x)$. From known eigenshapes $v_i(x)$ and $w_i(x)$, the location and type of the modification should be determined.

2.2. *The process of analysis*

In the finite element analysis the discrete eigenvectors $\mathbf{u}$ and $\mathbf{v}$ are first normalised with respect to the corresponding mass matrices $\mathbf{u}_i^T \cdot \mathbf{M} \cdot \mathbf{u}_i = 1$ and $\mathbf{v}_i^T \cdot \mathbf{M} \cdot \mathbf{v}_i = 1$, and further to the corresponding circular eigenfrequencies. Each vector $\mathbf{u}$ and $\mathbf{v}$ is then divided into three vectors – the components of the longitudinal displacements $\mathbf{X}_i^*$, transverse displacements $\mathbf{Y}_i^*$ and rotations φ_i^*.

$$\mathbf{X}_i^* = \frac{\mathbf{X}_i^u}{\omega_i^u} - \frac{\mathbf{X}_i^v}{\omega_i^v}, \qquad \mathbf{Y}_i^* = \frac{\mathbf{Y}_i^u}{\omega_i^u} - \frac{\mathbf{Y}_i^v}{\omega_i^v}, \qquad \varphi_i^* = \frac{\varphi_i^u}{\omega_i^u} - \frac{\varphi_i^v}{\omega_i^v},$$

where i stands for the frequency, indices u and v stand for the original and modified structure, respectively.

In the continuous approach, the eigenshapes $v_i(x)$ and $w_i(x)$ are therefore first normalised over the mass. Normalised eigenshapes $\overline{v}_i(x)$ and $\overline{w}_i(x)$ are computed from:

$$\int_{x=0}^{L} \overline{v}_i(x) \cdot m(x) \cdot \overline{v}_i(x) \cdot dx = 1 \qquad \text{and} \qquad \int_{x=0}^{L} \overline{w}_i(x) \cdot m(x) \cdot \overline{w}_i(x) \cdot dx = 1.$$

Normalised eigenshapes are further normalised over the corresponding circular eigenfrequency

$$v_i^*(x) = \frac{\overline{v}_i(x)}{\omega_i} \qquad \text{and} \qquad w_i^*(x) = \frac{\overline{w}_i(x)}{\omega_i'}.$$

Normalised eigenshapes $w_i^*(x)$ of the modified structure are afterwards compared with normalised eigenshapes $v_i^*(x)$ of the original structure:

$$\xi_i(x) = v_i^*(x) - w_i^*(x). \tag{2.3}$$

In the finite element analysis, the vector of rotational components φ_i^* is obtained by dividing the eigenvectors into three vectors ($\mathbf{X}_i^*, \mathbf{Y}_i^*$ and φ_i^*) according to the degrees of freedom. In the continuous analysis, a vector equivalent to the vector of rotations components φ_i^* is obtained as the first derivative of vector $\xi_i(x)$ from Eq. (2.3).

3. Numerical studies for different structure modifications

3.1. Numerical example

As a demonstration example, a simple cantilever beam with a rectangular cross-section, fixed at one end, is chosen (Fig. 1). In previous studies [4, 5] the same cantilever was discretized with 21 nodes and 20 elements, and the stiffness and mass matrices were obtained from the finite element theory, using 2-noded beam elements with Hermite polynomials as interpolation functions.

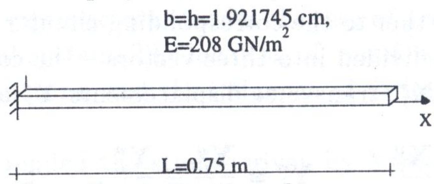

FIG. 1. Cantilever beam.

In further examination the following two modifications of the structure are considered:

1) local reduction of stiffness or cracked part of the structure,

2) added mass of the structure – the increase of the mass of the structure without affecting the stiffness of the structure.

3.1.1. Damaged part of the structure. The simplest way to model the local damage in a finite model is to reduce the modulus of elasticity of the damaged finite element ($E_{\text{damaged}} < E_{\text{intact}}$). This approach, widely used in the literature requires only a minimal change in the mathematical model of the structure with the finite elements and does not require any implementation of a special element. In the continuous approach such method cannot be easily used.

For more realistic description of the crack, a special finite element for plane beams of a rectangular cross-section with a uniform transverse crack was introduced [6] in previous analysis [4, 5]. The implementation of such an element requires precise information about the crack location and crack depth. The computational model of this element consists of two elastic beams (representing uncracked part of the beam), connected by a rotational spring, representing the crack. This model embodies the computational model that has been widely used in the preceding literature, where also a large variety of rotational stiffness spring constants is given.

Figure 2 compares vectors $\mathbf{Y}_1^*$ and φ_1^* with the corresponding vectors $\xi(x)$ and $\xi'(x)$ from the continuous approach. In the FEM model, vector φ_1^* clearly indicates the influence of the modification of the structure by discontinuity of vector φ_1^* at the damaged element. The modification of the structure is evident

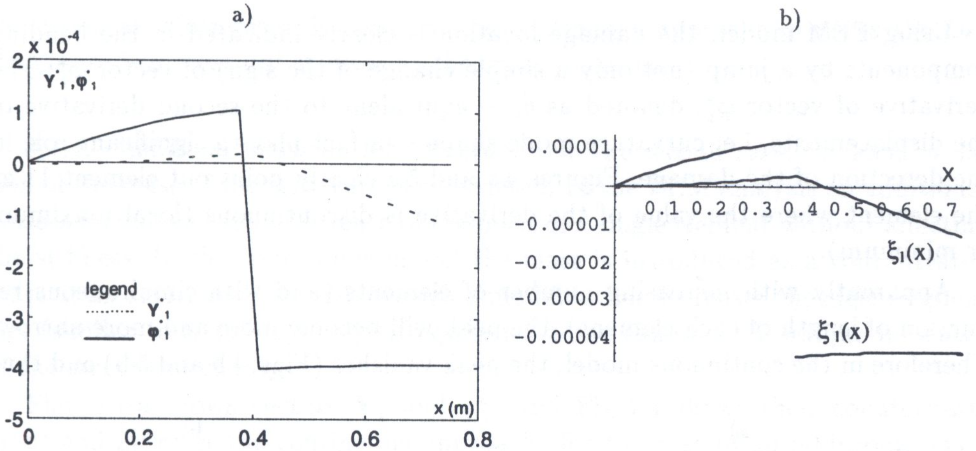

FIG. 2. a. Vectors $\mathbf{Y}_1^*$ and φ_1^* for the damage at element 11 (FEM model). b. Vectors $\xi_1(x)$ and $\xi_1'(x)$ for the damage at element 11 (continuous model).

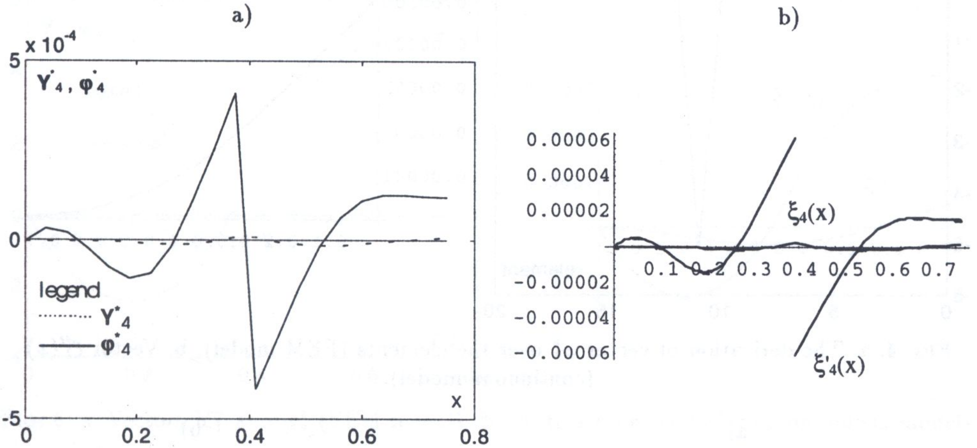

FIG. 3. a. Vectors $\mathbf{Y}_4^*$ and φ_4^* (FEM model). b. Vectors $\xi_4(x)$ and $\xi_4'(x)$ (continuous model).

also in higher frequencies (Fig. 3 a). Comparing Fig. 2 a with Fig. 2 b it is evident that similar behaviour can be detected also in the continuous approach, even in higher eigenmodes (Fig. 3).

To locate the damage, PANDEY *et al.* [2] introduce the approximations of second derivatives of transverse displacements, called curvature mode shapes. The maximum absolute differences of the curvature mode shapes change in fact indicate the location of the damage in the structure, but there is no physical interpretation of absolute values. In their study they consider only transverse displacements (neglecting longitudinal displacements and rotations as well). Further, eigenvectors are not normalised over the corresponding eigenfrequency before the comparison of modal shapes.

Using FEM model, the damage location is clearly indicated in the bending components by a jump (not only a simple change of the sign) of vector φ_i^*. The derivative of vector φ_i^*, denoted as θ_i, – equivalent to the second derivative of the displacements, i.e. curvature mode shape – in fact plays a significant role in the detection of the damage. Figures 4 a and 5 a clearly point out element 11 as the element where the value of the derivative is discontinuous (local maximum or minimum).

Apparently with increasing number of elements (and with simultaneous reduction of length of each element), the peak will become more and more narrow. Therefore in the continuous model, the peak vanishes (Figs. 4 b and 5 b) and thus

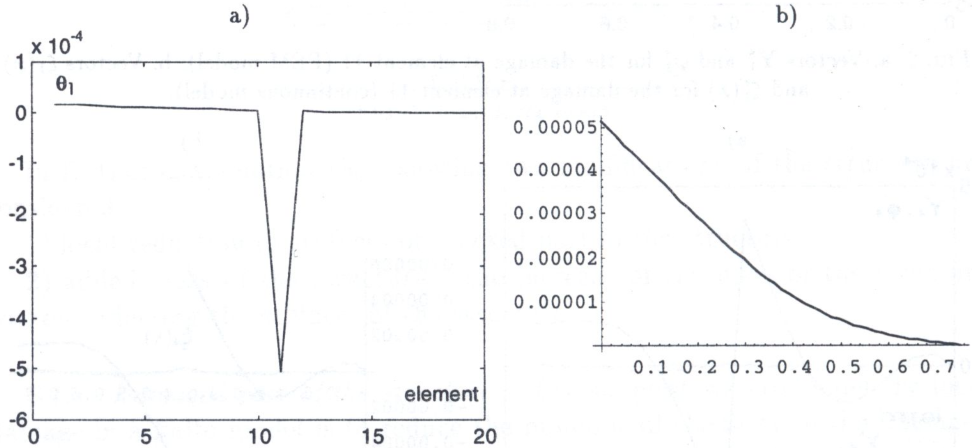

FIG. 4. a. The derivation of vector φ_1^* over the elements (FEM model). b. Vector $\xi_1''(x)$ (continuous model).

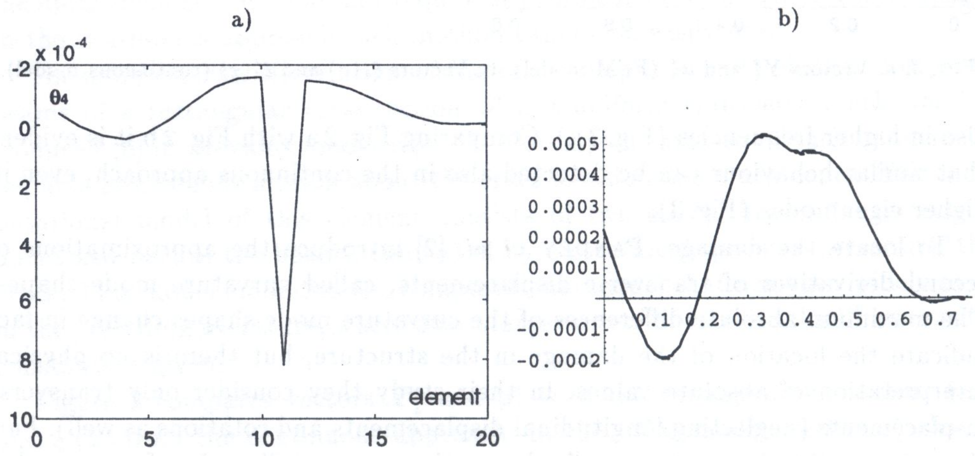

FIG. 5. a. The derivation of vector φ_4^* over the elements (FEM model). b. Vector $\xi_4''(x)$ (continuous model).

the essential information is omitted: the continuous approach neglects the fact that there is increase of stresses at the crack location.

3.1.2. Additional mass on the structure. In the second case, the local variation of the mass of the structure is considered. In the FEM analysis an additional mass is introduced as an increase of the mass of a single element without affecting the stiffness. In the continuous model the mass is introduced as a concentrated mass at an arbitrary point. Since such a modification decreases the eigenfrequencies (similar to a damage), eigenfrequencies alone cannot be a unique indicator of the structural change.

Figure 6 a shows vectors $\mathbf{Y}_1^*$ and φ_1^* (and Fig. 7 a shows their counterparts $\xi_1(x)$ and $\xi_1'(x)$ in the continuous approach) for the case of an additional mass

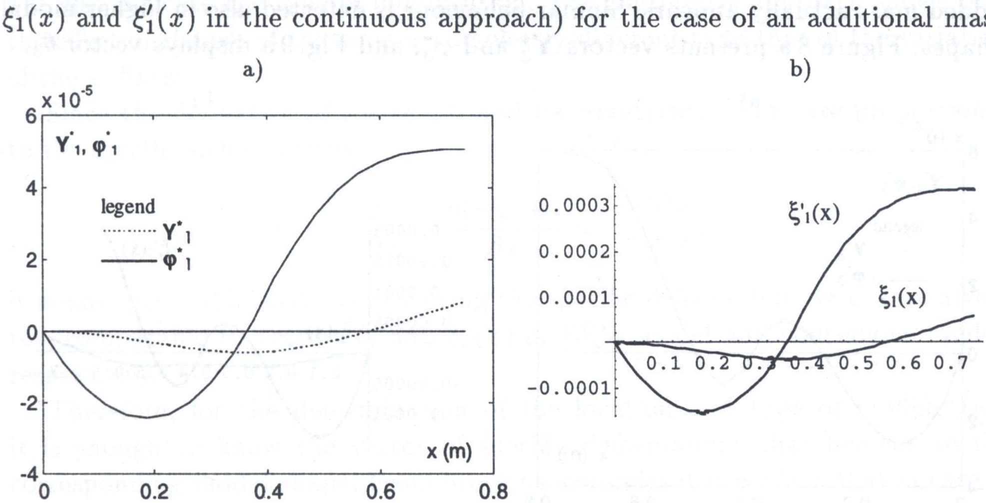

FIG. 6. a. Vectors $\mathbf{Y}_1^*$ and φ_*^1 (FEM model). b. Vectors $\xi_1(x)$ and $\xi_1'(x)$ (continuous model).

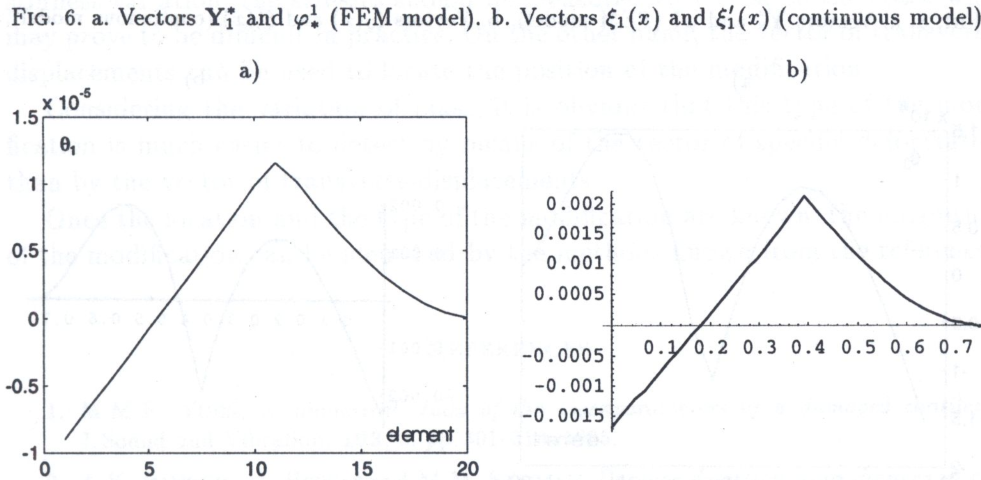

FIG. 7. a. Derivative of vector φ_1^*. b. Vector $\xi_1''(x)$.

on element 11. Vectors do not indicate the location of the modification so clearly as in the case of the damaged structure. However, vectors φ_1^* and $\xi_1'(x)$ also in this case offer enough information about the location of the added mass. While the local reduction of the stiffness is reflected as a discontinuity of the vector φ^* and $\xi_1(x)$ at the point of the damage, an added mass is manifested by a reflection point at the added mass position. In the case of vector θ_i the reflection point is again represented as an extremum at the point of modification (Fig. 7 a). The essential difference is in the way in which this extremum is reached. While in the case of the stiffness reduction the extremum is reached only at the position of the modification, in the case of an added mass the values are smoothly increasing (or decreasing) to reach finally the extremum value at the element where the added mass actually appears. Similar behaviour is detected also in higher modal shapes. Figure 8 a presents vectors $\mathbf{Y}_3^*$ and φ_3^*, and Fig. 9 a displays vector θ_3.

a) b)

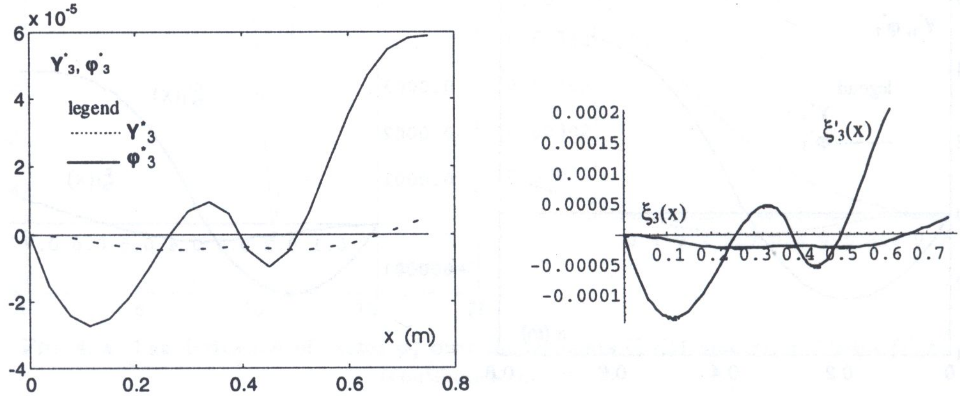

FIG. 8. a. Vectors $\mathbf{Y}_3^*$ and φ_3^* (FEM model). b. Vectors $\xi_3(x)$ and $\xi_3'(x)$ (continuous model).

a) b)

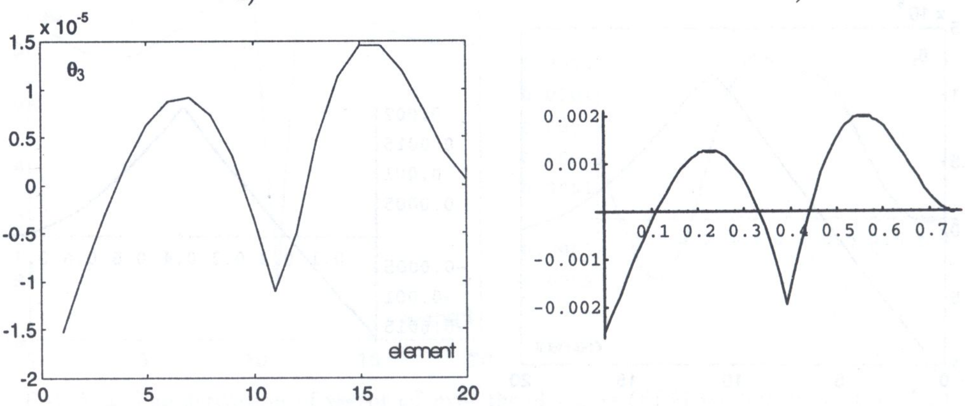

FIG. 9. a. The derivative of vector φ_3^*. b. Vector $\xi_3''(x)$.

Using the continuous model we see that the corresponding eigenshapes (Figs. 6 b, 7 b, 8 b and 9 b) behave in the same way as the eigenshapes belonging to the FEM model. Therefore we can assume that also all conclusions drawn before are valid for this case.

4. Discussion of the results and conclusions

From the presented study it is once more evident that each modification is clearly reflected in the modal shapes. Either eigenvectors or their derivatives give enough information not only about the location of the structure modification, but also indicate the type of modification. The continuous approach shows that the effect of change of the mass is completely different than that of the variation of the stiffness.

Since the derivative of vector φ_i^*, and its counterpart $\xi_i'(x)$ are proportional to the specific deformations:

$$\varepsilon = \frac{\partial u}{\partial x} = \frac{\partial(-y \cdot \varphi)}{\partial x} = -y\frac{\partial^2 v}{\partial x^2},$$

it seems reasonable that by measuring the specific deformation we obtain a vector, comparable to vectors θ_i and $\xi_i''(x)$ in FEM model and continuous model, respectively.

Therefore, for the determination of the location and type of modification it is enough to know the vector of specific deformations that belongs to the corresponding modal shape. From previous examples it is evident that in case of stiffness variation, the stresses should be measured very close to the crack what may prove to be difficult in practice. On the other hand, the vector of transversal displacements can be used to locate the position of the modification.

Considering the variation of mass, it is obvious that this type of the modification is much easier to detect by means of the vector of specific deformation than by the vector of transverse displacements.

Once the location and the type of the modification are known, the magnitude of the modification can be identified by the methods known from the references.

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