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Discussion

The Origins of Newton's Mechanics. Mass, Force, and Gravity[1)]

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Abstract.[2)] In this work, the axiomatic model of dynamics is developed corresponding to the classical model of Newton's dynamics. The key elements of the model are the ability to distinguish isolated systems, and their subsequent division into a selected body (*material particle*) and its surroundings (*attractor*). The material particle (usually assumed to be small relative to surroundings), and the attractor are the axiomatic model's primary concepts. The only fundamental *state parameter* of the model is *acceleration* (kinematic quantity), which is accepted as the starting point of dynamics. The concepts of *mass* (dynamic characteristic of the material particle) and *force* (dynamic characteristic of the surroundings impact on the material particle) are derivative quantities in the model. Two types of (acceleration) measurement procedures deliver precise *operational definitions* of inertial mass and force. Thus, difficulties with the ideas of mass and force present in original formulation of Newton's laws of dynamics are removed. The present model shows that for formulation and interpretations of the laws of dynamics, the general ideas about the particle and the environment affecting its motion, and the concept of *acceleration, are sufficient.* Neither masses nor forces are necessary to formulate the *essence of dynamics.* However, the elegance and power of the concepts of force and mass prompt for their introduction in any isolated system with a separated body as very convenient and useful quantities. Following the developed methodology, an axiomatic model of Newton's *universal gravitation* is formulated, and it is shown that neither inertial mass nor force is actually needed for that purpose. Moreover, in the closed world of gravity, the force concept *cannot be introduced* as a dynamic feature of an attractor only – it must be a feature characterizing a pair of the specific particle and attractor. Reconciliation of the axiomatic model of universal gravity with the axiomatic model of dynamics leads to the equivalence of *gravitational mass* and *inertial mass* concepts.

In Translator opinion, the work contains a very original and elucidating approach towards classical dynamics and due to that deserves worldwide dissemination and knowledge. Nowadays, this can only be achieved, to the extent that the study deserves, by its publication in English language.

Key words: axiomatic model of dynamics; operational definition of inertial mass; operational definition of force; axiomatic model of universal gravity; Newton's dynamics; concept of attractor; concept of material particle; equivalence of inertial mass and gravitational mass.

[1)] *Editorial note:* The present document is an English translation of Appendix C of Jan Rychlewski's book titled *Dimensions and Similarity* (original Polish title *Wymiary i podobieństwo*), Wydawnictwo Naukowe PWN, Warszawa, 1991, pp. 185–208, ISBN 83-01-10557-7.

[2)] *Editorial note:* Abstract by Andrzej Ziółkowski.

The theoretical mastery of the laws of motion took over 1,500 years. Among the contributors to this epic cognitive time were Aristotle and Heron in antiquity, and Buridan in the Middle Ages. The decisive progress was the work of Galileo, as well as Kepler, Huygens, and Wallis. We owe the synthesis of these efforts to Newton [C.1].

At the same time, the theory of gravity was ripening. The beginning came with a Copernican model of the Solar System that was in need of an explanation. Copernicus himself wrote: "In my opinion gravity is nothing more than a natural endeavor, in which the Divine Providence of the Creator has endowed the parts with, in order to join them together in the form of a sphere. This endeavor is proper to the Sun, the Moon and other moving celestial bodies". A century later, Kepler spoke explicitly of the mutual attraction of stone and the Earth as well as the Moon and the Earth, as two manifestations of the same phenomenon. Again, the culmination of these ideas is the work of Newton, who had the independently working Hooke hot on his heels [C.20].

The laws of motion provided by Newton 300 years ago and refined later, mainly by Euler, [C.2], are still today the basis for the presentation and application of mechanics. At the same time, discussions took place and periodically revive, sometimes fiercely, around the shape of Newtonian mechanics as a coherent system (cf. e.g. [C.3–C.15]). Their central points are the basic concepts of *mass* and *force*.

Newton himself, wrongly believing that the concept of mass should precede the laws of motion, describes them in the first definition of his system as follows:

> "quantitas materiae est mensura ejusdem orta ex illius densitate et magnitudine conjunctim".

In modern language, it sounds like this: mass is proportional to density and volume. Therefore, the reader expects an independent definition of density, but does not find it. Today, as a matter of fact, we prefer to define density as a quantity proportional to mass and inversely proportional to volume. Feeling the weakness of his definition, Newton adds immediately that the mass of a body is proportional to its weight. However, later investigations made it clear that it only meant the equivalence of two concepts: *inertial mass* and *gravitational mass*, defined separately.

Mach, feeling the nuances of mechanics deeply, proposed 200 years later to make the following statement the basis for the definition of mass: in an isolated system of two bodies, the ratio of their masses is inversely proportional to the ratio of their accelerations [C.3]. He considered the mass of a body as a measure of its inertia under the influence of all other bodies in the Universe (Mach principle).

Opinions expressed on these matters are still uncompromisingly drastic to this day. We will present the view quoted in [C.6] that "one of the most amazing features of the history of physics is the confusion around the definition of the primary concept of dynamics, namely the concept of mass". The expert on the history of the problem adds: "for introductory physics courses, the concept of mass turns out to be a difficult and complicated matter. No textbook or course contains, as it seems, a logical and scientifically unquestionable presentation of this concept". And further: "no attempts to formalize Newton's mechanics on the basis of a precise and explicit definition of mass have brought great success" [C.6].

The notion of force provoked even greater passions. Let us quote, for example, the words of L. Carnot: "In my mechanics [...] I wanted to circumvent the metaphysical idea of force", H. Hertz: "In many cases, the forces that appear in our mechanics [...] are an empty invention, losing all meaning where it is about reflecting the real facts", and J. d'Alembert: "I have completely removed from mechanics, the forces which constitute a vague concept". Some time ago, there were disputes about the real status of inertia forces in mechanics [C.12–C.15].

Disputes revolve around the interpretation of the meaning of Newton's second law: $\mathbf{F} = m\mathbf{a}$. The following views were and are defended:

(1) This formula is the definition of force when mass was independently defined earlier.

(2) From this formula, the definition of mass is derived as a convenient coefficient of proportionality between the force acting on the body and its acceleration.

Note the futility of the first point of view in which the formula $\mathbf{F} = m\mathbf{a}$ is simply denied the status of a law of nature. H. Poincaré called the second view "a testimony to our powerlessness" [C.5].

Additionally, let us quote A. Whitehead: "We derive our knowledge of forces having a certain theory of mass, and our knowledge of mass on the basis of a certain theory of forces" [C.6]. Finally, let us quote words of H. Poincaré, summarizing his remarks on the proposed reform of mechanics presented by H. Hertz: "we must therefore come to the conclusion that within the framework of the classical system it is impossible to formulate the idea of mass and force in a satisfactory manner" [C.5]. We believe that the great scientist was wrong here. More specifically, we will show that Newton's main ideas can be arranged in such a sequence that the difficulties with mass and force discussed here will not arise.

We will take acceleration as the only fundamental concept for the starting point of dynamics. It will be in the spirit of the geometrization of mechanics, reaching back to Descartes and propagated by E. Mach and H. Hertz [C.3, C.4].

So, we will formulate the laws of dynamics in a form containing only the accelerations of bodies under the action of their surroundings. From the laws formulated in this way, two measurement procedures will directly result: comparing the inertia of bodies and comparing the interaction intensity of surroundings. They define the concept of inertial mass and the concept of force, respectively. By writing down the initial kinematic laws with their use, we give them the form of Newton's laws.

We will show that neither inertial mass nor force is actually needed to formulate and use Newton's theory of universal gravitation. Acceleration and the *gravitational charge*, usually called *gravitational mass*, are sufficient here. The reconciliation of Newton's theory of gravity with his dynamics leads to the equivalence of gravitational mass and inertial mass.

Immodestly, we would like to believe that upon reading the proposed here frameworks of his mechanics foundations, Sir Isaac would simply say that this is what he meant.

Dynamics

1. Space and time. We accept that a proper model of relations describing the mutual position of the objects around us is Euclidean space $\mathcal{E}_{\mathbf{L}}$, with elements called *points*, with dimension dim $\mathcal{E}_{\mathbf{L}} = 3$ and the physical dimension $\mathbf{L}$ = LENGTH (cf. point 5 of Lecture 12[3)]). The basic features of this model – the extreme simplicity of the topological structure, infinity, boundlessness, homogeneity, isotropy – determine the clarity and elegance of classical mechanics.

We accept that a proper model of the sequence of real events is a Euclidean line $\mathcal{T}_{\mathbf{T}}$ with elements called *moments* and of a physical dimension $\mathbf{T}$ = TIME.

Thus, the space-time continuum of classical mechanics is the Cartesian product $\mathcal{E}_{\mathbf{L}} \times \mathcal{T}_{\mathbf{T}}$.[4)]

All of this is a modern reworking and refinement of Newton's original formulations:

> "Absolute space, by its very nature and without connection with anything external, always remains the same and motionless".
>
> "Absolute, mathematical time flows uniformly by itself, by its very nature, with no connection to anything external, and is otherwise called the flow of events".

[3)] *Translator note.* This is reference to Lecture 12, "Miscelanea", point "Dimensional vectors" in book *Dimensions and Similarity* [in Polish].

[4)] W. Noll showed in [C.19] that in reality a slightly more general and less restrictive model of space-time would suffice for the needs of classical mechanics. Euclidean spaces appear to him at separate moments and constitute a kind of loose deck of cards. But we will stick closer to Newton himself here. The complete resignation from the mutual independence of space and time is the starting idea of the theory of relativity.

We call space $\mathcal{E}_{\mathsf{L}}$ *absolute space*, and $\mathcal{T}_{\mathsf{T}}$ Newton's *absolute time*. They are the structures of pure mathematics. We will deliver their experimental interpretations later.

2. Bodies, particles. The geometric models of the surrounding us objects, which we will hereinafter refer to as *real bodies*, are *bodies*. We define bodies as follows: at any moment a body corresponds to a certain part of space $\mathcal{E}_{\mathsf{L}}$, called its *place*. A *particle*, *continuous body*, *rigid body* are the types of bodies defined by appropriate assumptions about the occupied places. By definition, bodies are solid objects, independent of anything, especially independent of time.

A particle we call a body whose places are only points. A particle is a geometrical model of a real body with a small enough size when compared to the distances considered. To emphasize this, we sometimes call a particle a *body-particle*. In the past, the name *material point* was also used.

Next, we will gradually equip the particles with kinematic and dynamic properties.

3. Kinematics. *The motion of a body* is continuous – in an easily identifiable sense – as the sequence of places it occupies over time. *The motion of the particle* will then be a continuous curve in $\mathcal{E}_{\mathsf{L}}$, along with its parameterization by moments. By identifying moments with scalars $t \in \mathsf{T}$, we record the motion of a particle $\mathcal{B}$ in absolute space $\mathcal{E}_{\mathsf{L}}$ in the form:

$$p = \chi(\mathcal{B}, t), \qquad t \in \mathsf{T}, \;\; p \in \mathcal{E}_{\mathsf{L}}. \tag{C.1}$$

Velocity and *acceleration* of the particle $\mathcal{B}$ at instant t relative to absolute space $\mathcal{E}_{\mathsf{L}}$ are defined as:

$$\mathbf{v}(\mathcal{B}, t) \equiv \dot{\chi}(\mathcal{B}, t) = \lim_{\Delta_t \to 0} \frac{1}{\Delta t} \overrightarrow{\chi(\mathcal{B}, t)\chi(\mathcal{B}, t + \Delta t)}, \tag{C.2}$$

$$\mathbf{a}(\mathcal{B}, t) \equiv \ddot{\chi}(\mathcal{B}, t) = \dot{\mathbf{v}}(\mathcal{B}, t). \tag{C.3}$$

As it follows, these are dimensional vectors, $\mathbf{v} \in \ni_{\mathsf{LT}^{-1}}$, $\mathbf{a} \in \ni_{\mathsf{LT}^{-2}}$ (see Lecture 12).

By establishing the origin $o \in \mathcal{E}_{\mathsf{L}}$, we write the motion in the form:

$$\chi(\mathcal{B}, t) = o + \mathbf{r}(\mathcal{B}, t), \tag{C.4}$$

where

$$\dot{\mathbf{v}} = \dot{\mathbf{r}}, \qquad \mathbf{a} = \dot{\mathbf{v}}. \tag{C.5}$$

N o t e. Hereinafter, we refer only to motions with sufficiently low speeds compared to the speed of light.

4. The experimental status of space-time: the kinematic aspect. In fact, we observe not the points from $\mathcal{E}_{\mathsf{L}}$ and the moments from $\mathcal{T}_{\mathsf{T}}$ but real bodies, their mutual positions and the change of these positions. Space $\mathcal{E}_{\mathsf{L}}$ and time $\mathcal{T}_{\mathsf{T}}$, being objects of pure Mathematics, must therefore receive an experimental interpretation.

The experimental status of time $\mathcal{T}_{\mathsf{T}}$ establishes the concept of an observer of time. The *time observer* we call an ordered pair of real events. By identifying them with moments with ω_0, ω_1 from $\mathcal{T}_{\mathsf{T}}$, we obtain every other moment $\omega \in \mathcal{T}_{\mathsf{T}}$ as $\omega = \omega_0 + t$, where $t = x\overrightarrow{\omega_0\omega_1}$, $x \in R$, is a scalar from the dimension T.

We call a *space observer* any four ordered real bodies considered as bodies – particles $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ – that meet two conditions: experimental observations allow to assume a priori that (1) the distances between them remain constant, and (2) there is no plane containing them. The *place of space observer* $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ will be the frame of reference of space $\mathcal{E}_{\mathsf{L}}$, i.e., the four ordered vertices p_0, p_1, p_2, p_3 of the non-degenerate tetrahedron[5),6)] in $\mathcal{E}_{\mathsf{L}}$. Every space observer is rigid by definition, that is, any two of its places are congruent. The motion of the space observer is the motion of four of its particles satisfying the above conditions.

The experimental status of space $\mathcal{E}_{\mathsf{L}}$ is assigned to it by recognizing *a priori* that the observed positions of the four described real bodies are its points. In other words, we assume *a priori* that the chosen space observer is stationary. Every other point in space $\mathcal{E}_{\mathsf{L}}$ is then an object $p = p_0 + \mathbf{r}$, where $\mathbf{r} = x_1\overrightarrow{p_0p_1} + \ldots$, $x_1 \in R, \ldots$, is the dimensional vector from $\ni_{\mathsf{L}}$.

A pair of observers: of time and of space, we simply call an *observer* or a *reference frame*. The observer encodes all information about particle motions in the form of sets of quaternion numbers $(x_0,\ x_1,\ x_2,\ x_3)$.

5. Material particles and attractors: motivations. Let us consider real fields and real bodies that move in a certain part of space, interacting with each other and with nothing else. We understand the non-interaction with the rest of the Universe as the inability of instruments to detect such an interaction with a predetermined accuracy. We will call this situation a *real isolated system*, or simply a *system*.

The distinguishing of isolated systems was Newton's first (implicit) preliminary step.

Newton's second preliminary step (also implicit) was to split this overall situation into two parts:

[5)] That is a simplex of point Euclidean space.

[6)] *Translator note.* Actually, *observer* (*reference frame*) can be defined with only *three*, non-collinear points, e.g., $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2 \in \mathcal{E}_{\mathsf{L}}$. First two vectors can be created, e.g., $\overrightarrow{\mathcal{B}_0\mathcal{B}_1}$, $\overrightarrow{\mathcal{B}_0\mathcal{B}_2}$. Then, they can be orthonormalized and the third orthonormal vector is obtained as cross product of the first two. $\mathcal{B}_0$ is an anchor point.

(C.6) (separated body; its surroundings),

and posing the problem of describing the influence on the motion of a separated body of its surroundings, i.e., the remaining bodies and fields of the system. We will assume immediately that the separated body is sufficiently small.

Let us call a pair (C.6), together with all the information about the interaction of bodies and fields, a *system with a separated body.*

At first, it seems that the path now leads to the creation of separate models of the members of the pair (body, environment) and then linking these models. It is not so.

As a Newtonian model of a system with a separated body, observed at any given moment, we will assume a pair

(C.7) (material particle $\mathcal{B}$, attractor $\mathcal{U}$).

We will provide explanations and motivations in this point, and clarify this decomposition in the next point.

We will gradually describe the manner of division of the information about the entirety of the situation into a pair of members ($\mathcal{B}$, $\mathcal{U}$), starting with the following declaration of intentions.

The *material particle* $\mathcal{B}$ is to be a dynamic model of the body separated by division (C.6). This model is to be based solely on information about this body, as an unchanging object, independent of the location and time.

The *attractor* $\mathcal{U}$ is to be a dynamic model of influencing the motion of a separated body by the remaining bodies and fields of the system, at the moment under consideration. This model is to be based on information about the entire system, except information about the separated body used in the model of "material particle $\mathcal{B}$". The attractor $\mathcal{U}$ is to contain, in particular, all information about the quantities describing internal geometry and mutual position of all bodies and fields of the system, including the separated body, as well as information about the rates of change of these quantities, at the moment under consideration. In a specific situation, it may happen that no information about the separated body will be needed when creating the model of "attractor $\mathcal{U}$".

We assume by definition that a material particle $\mathcal{B}$, in particular, is a particle $\mathcal{B}$, and therefore its every place is a point. This highly restrictive kinematic postulate strips off in one fell swoop the original – the real body – of its geometry: shape and size. In many situations, however, the interaction of the body with the environment depends, sometimes decisively, on its geometry. In such a case, the information about the body geometry must be included in the description of attractor $\mathcal{U}$.

Putting information about a real isolated system in a pair ($\mathcal{B}$, $\mathcal{U}$) is always a matter of the interaction between theory and experiment and is obtained by trial and error. We'll give some leading examples.

Crown Example: Solar System planetary motion. The isolated system here consists of the Sun, Planets and their Satellites, $N = 1 + 9 + 60 = 70$. For each planet considered to be a material particle $\mathcal{B}$, the attractor $\mathcal{U}$ does not contain any information about that planet and is different at every point in its orbit.

Example: the motion of a child on a slide. The isolated system certainly includes a slide, the Earth, perhaps the supporting mother's hand, but certainly not Mars. The information constituting the attractor $\mathcal{U}$ includes the shape of the slide, gravitational acceleration, the shape of the contact surface, the smoothness of the trousers and slide, perhaps even the speed of the child relative to the slide. The color of the pants, the gender of the child, and the infinite amount of other information are not included in $\mathcal{B}$ nor in $\mathcal{U}$.

Example: a stationary body on a spring scale. The bodies acting on this suspended body are the spring and the Earth. The characteristics of the spring's compliance and its specific elongation under the influence of a suspended body belong to the information describing the attractor $\mathcal{U}$, as well as the gravitational acceleration. The attractor $\mathcal{U}$ does not contain any information about the body here.

Example: a real charged body moving in an electromagnetic field. The electric charge q of the body is not included in the description of its model $\mathcal{B}$. The set of information describing the attractor $\mathcal{U}$ consists of: the electric charge q, its velocity $\mathbf{v}$ in relation to the magnetic field, electric field intensity $\mathbf{E}$, and magnetic field intensity $\mathbf{B}$, at the place p occupied by the particle $\mathcal{B}$ at a given moment t.[7)]

School Example: the motion of a trolley under the action of a spring on a perfectly smooth horizontal plane. The system consists of two bodies. The attractor is a stretched spring.

Example: a satellite in an orbit around the Earth. The factors affecting the motion of the satellite should always include the Earth, most often the Moon, often the Earth's atmosphere, then the speed of the satellite in relation to the gas along with its aerodynamic data, sometimes the incident beam of sunlight along with the shape of the satellite and the optical properties of its surface, and sporadically the Earth's electromagnetic field along with the charge and relative speed of the satellite. The attractor $\mathcal{U}$ is different at each point of the orbit due to the varying height, heterogeneity of the distribution of matter

[7)] *Translator note*: The text of this example should be rather corrected to (as follows to deliver factually accurate information):
Example: a real charged body moving in an electromagnetic field. The electric charge q of the body and its velocity $\mathbf{v}$ in relation to the magnetic field is included in the description of its model $\mathcal{B}$. The set of information describing the attractor $\mathcal{U}$ consists of: electric field intensity $\mathbf{E}$ and magnetic field intensity $\mathbf{B}$, in the place p occupied by the particle $\mathcal{B}$ at a given moment t.

inside the Earth, inhomogeneity of the atmosphere, light conditions and the electromagnetic field.

H a r m f u l n o t e: The reader of this work unfortunately inevitably knows, in more or less successful terms, the concepts of mass and force. Perhaps the reader is even an expert with an unshakable view on the matter. We hope this will not weaken his or her logical sensitivity and criticism. To facilitate the reception, we will repeal our cards: we condemn the pair (particle $\mathcal{B}$, attractor $\mathcal{U}$) to death from the beginning, and the afterlife in the Mechanics' paradise will only receive the soul – the pair (mass $m(\mathcal{B})$, and force $\mathbf{F}(\mathcal{U})$).

H a r m l e s s n o t e: The purpose of Natural Science, as we humbly understand it 300 years after Newton, is not to search for absolute truths, but to construct models that describe reality in a manner that is completely internally consistent, as clear and elegant as possible, and sufficiently accurate.

6. Material particles and attractors: clarifications. We make material particles and attractors the primary concepts of Newtonian mechanics as a mathematical system. Their axiomatics will be based on the laws of dynamics given in the next section.

In the examples, we have noticed that it is necessary to make the description of the pairs $(\mathcal{B},\mathcal{U})$ more precise, because real isolated systems are observed at any moment t, from a certain point p. It would be best to choose for this point the place occupied by the selected moving body-particle $\mathcal{B}$, at the moment in question. By writing down the information constituting the attractor $\mathcal{U}$ in relation to (p,t), we obtain its *location* in space-time $\mathcal{U}(p,t)$. As we write on $(\mathcal{B},\mathcal{U})$, we always mean for each pair (p,t).

$$(\mathcal{B},\mathcal{U}(p,t)) \qquad \text{for} \qquad p \equiv \chi(\mathcal{B},t). \tag{C.8}$$

We will start equipping a pair $(\mathcal{B},\mathcal{U})$ with properties using the postulate of material particles and attractors' independence: for every two pairs $(\mathcal{B},\mathcal{U})$ and $(\mathcal{B}_*,\mathcal{U}_*)$, which are models of certain real isolated systems with a selected body, there exist real isolated systems with a selected body, the models of which are $(\mathcal{B}_*,\mathcal{U})$ and $(\mathcal{B},\mathcal{U}_*)$. Without this postulate, it is impossible to introduce mass or force into science.

Let us consider a real isolated system consisting of one real body $\mathcal{B}$.[8)] We will denote the attractor corresponding to this system with $\mathcal{U}_0$ and call it the *empty attractor*.

We will introduce the operation of *merging of attractors*:

$$(\mathcal{U}_1,\mathcal{U}_2) \to \mathcal{U}_1 \circ \mathcal{U}_2 \equiv \mathcal{U}_2 \circ \mathcal{U}_1. \tag{C.9}$$

8) The Voyager-2 spacecraft, which left the solar system in 1989 after passing successively near Jupiter, Saturn, Uranus and Neptune, will be such a system with enormous accuracy in, say, 100,000 years.

By $\mathcal{U}_1 \circ \mathcal{U}_2$ we understand an attractor being a model of a real, isolated model, which is created by assembling systems, whose models are attractors $\mathcal{U}_1$, $\mathcal{U}_2$, with a separated body identical in both systems. In the example with a satellite, the attractor is a merger of five attractors $\mathcal{U} = \mathcal{U}_1 \circ ... \circ \mathcal{U}_5$.

To express ourselves more precisely, we assume that for each pair of attractors $\mathcal{U}_1$, $\mathcal{U}_2$ there exists an attractor $\mathcal{U}_1 \circ \mathcal{U}_2$, and the operation (C.9) has the properties described further by the laws of dynamics. For an empty attractor $\mathcal{U}_0$ by definition:

(C.10) $$\mathcal{U}_0 \circ \mathcal{U}_0 = \mathcal{U}_0 \circ \mathcal{U} = \mathcal{U}, \qquad \text{for every attractor } \mathcal{U}.$$

We do not exclude the possibility of two material particles $\mathcal{B}_1$, $\mathcal{B}_2$ occupying the same place at some time. We will introduce the operation of *merging material particles:*

(C.11) $$(\mathcal{B}_1, \mathcal{B}_2) \to \mathcal{B}_1 \circ \mathcal{B}_2 \equiv \mathcal{B}_2 \circ \mathcal{B}_1.$$

It is a model of a situation where two small real bodies, joined together or not, occupy close places at any moment, in relation to the distances considered.

More precisely, we assume that for any two particles $\mathcal{B}_1$, $\mathcal{B}_2$ a particle $\mathcal{B}_1 \circ \mathcal{B}_2$ exists such that for each of its motions χ a pair of motions χ_1, χ_2 exist of particles $\mathcal{B}_1$, $\mathcal{B}_2$ satisfying the condition

(C.12) $$\chi(\mathcal{B}_1 \circ \mathcal{B}_2, t) = \chi_1(\mathcal{B}_1, t) = \chi_2(\mathcal{B}_2, t),$$

at any time t. The properties of the operation (C.11) will be further described by the laws of dynamics.

7. The laws of dynamics. Let us choose any pair (p, t). By $\mathcal{B}$ we further understand any particle occupying a point p at time t, $p \equiv \chi(\mathcal{B}, t)$, by $\mathbf{a}$ its acceleration at that moment, $\mathbf{a} = \ddot{\chi}(\mathcal{B}, t)$. By $\mathcal{U}$ we mean an attractor $\mathcal{U}(p, t)$ describing the influence of particle $\mathcal{B}$ environment on its motion.

We will write the acceleration vector as:

(C.13) $$\mathbf{a} = A \otimes \mathbf{n} \equiv A\mathbf{n}, \qquad \mathbf{nn} = 1.$$

The acceleration modulus is a dimensional scalar $A \in \mathsf{LT}^{-2}$, and its direction is a dimensionless vector $\mathbf{n} \in \ni_3$. By B we denote the set of all material particles, and by U the set of all attractors.

The extract of observations, *experiences and reflections on the motion of bodies*, *from antiquity to Newton*, will be summarized into the following axioms about particles and attractors.

LAW OF INSTANTANEOUS ACTION. *The attractor $\mathcal{U}$ and the particle $\mathcal{B}$ itself determine unequivocally, at any moment, the acceleration of this particle*[9],

$$(\mathcal{B},\mathcal{U}) \to \mathbf{a}(\mathcal{B},\mathcal{U}). \tag{C.14}$$

The assignment **a** fulfills the postulate of the independence of particles and attractors:

$$\text{Dom}\,\mathbf{a} = B \times U, \tag{C.15}$$

and the following A, B, C, D *laws.*

A. ZERO LAW OF DYNAMICS. *For every attractor $\mathcal{U}$ there is an alternative: it is either dynamic, i.e.,*

$$\mathbf{a}(\mathcal{B},\mathcal{U}) \neq \mathbf{0} \qquad \text{for every } \mathcal{B}, \tag{C.16}$$

or it is static, i.e.,

$$\mathbf{a}(\mathcal{B},\mathcal{U}) = \mathbf{0} \qquad \text{for every } \mathcal{B}. \tag{C.17}$$

B. FIRST LAW OF DYNAMICS. *The empty attractor $\mathcal{U}_0$ is a static attractor, hence*

$$\mathbf{a}(\mathcal{B},\mathcal{U}_0) = \mathbf{0} \qquad \text{for every } \mathcal{B}. \tag{C.18}$$

C. SECOND LAW OF DYNAMICS. *There is a class of basic dynamic attractors, defined as follows:*

C.1 (the condition of the independence of the direction of acceleration from the particle). *For every basic attractor $\mathcal{U}$ and every particle $\mathcal{B}$*

$$\mathbf{a} = A(\mathcal{B},\mathcal{U})\mathbf{n}(\mathcal{U}). \tag{C.19}$$

C.2 (condition for distinguishing particles). *For every basic attractor $\mathcal{U}$ there are particles $\mathcal{B}$, $\mathcal{B}_*$, such that*

$$A(\mathcal{B},\mathcal{U}) \neq A(\mathcal{B}_*,\mathcal{U}). \tag{C.20}$$

C.3 (condition for separation of particles and attractors). *For each pair of basic attractors $\mathcal{U}, \mathcal{U}_*$ and each pair of particles $\mathcal{B}$, $\mathcal{B}_*$*

$$A(\mathcal{B},\mathcal{U})A(\mathcal{B}_*,\mathcal{U}_*) = A(\mathcal{B},\mathcal{U}_*)A(\mathcal{B}_*,\mathcal{U}). \tag{C.21}$$

[9] Here and further we denote, as it is commonly accepted in Physics, the function and its value with the same letter. It is really about the mapping $\alpha : B \times U \to \ni_{\mathsf{LT}^{-2}}$, $\mathbf{a} = \alpha(\mathcal{B},\mathcal{U})$.

C.4 (condition of universality). *For every non-zero vector* $\mathbf{c} \in \ni_{\mathsf{LT}^{-2}}$ *and every particle* $\mathcal{B}$ *there is a basic attractor* $\mathcal{U}$, *such that*

$$\mathbf{a}(\mathcal{B},\mathcal{U}) = \mathbf{c}. \tag{C.22}$$

D. Rule of merging particles. *For any particles* $\mathcal{B}_1$, $\mathcal{B}_2$ *and any dynamic attractor* $\mathcal{U}$

$$\frac{1}{A(\mathcal{B}_1 \circ \mathcal{B}_2,\mathcal{U})} = \frac{1}{A(\mathcal{B}_1,\mathcal{U})} + \frac{1}{A(\mathcal{B}_2,\mathcal{U})}. \tag{C.23}$$

E. Rule of merging attractors. *For any attractors* $\mathcal{U}_1$, $\mathcal{U}_2$ *and any particle* $\mathcal{B}$

$$\mathbf{a}(\mathcal{B},\mathcal{U}_1 \circ \mathcal{U}_2) = \mathbf{a}(\mathcal{B},\mathcal{U}_1) + \mathbf{a}(\mathcal{B},\mathcal{U}_2). \tag{C.24}$$

The set of these laws we will call further the *laws of dynamics*. Writing them with respect to (p, t) (see (C.8)), we add *the principle of homogeneity of time*: the laws of dynamics remain valid after any shift of time

$$t \to t + \mathrm{const}, \tag{C.25}$$

and *the principle of homogeneity and isotropy of space and particles*: the laws of dynamics remain valid after any spatial shift

$$p \to p + \mathrm{const}, \tag{C.26}$$

and any constant rotation around the point p

$$p + \mathbf{r} \to p + \mathbf{Q}\mathbf{r}, \qquad \mathbf{Q} = \mathrm{const}, \tag{C.27}$$

wherein

$$\mathbf{Q} * \mathcal{B} \equiv \mathcal{B}, \qquad \mathbf{a}(\mathcal{B}, \mathbf{Q} * \mathcal{U}) = \mathbf{Q} * \mathbf{a}(\mathcal{B},\mathcal{U}). \tag{C.28}$$

Here $\mathcal{U} \to \mathbf{Q} * \mathcal{U}$ is an operation of the rotation group in the set of attractors that satisfies the conditions:

$$\mathbf{1} * \mathcal{U} = \mathcal{U}, \qquad (\mathbf{Q}_1\mathbf{Q}_2) * \mathcal{U} = \mathbf{Q}_1 * (\mathbf{Q}_2 * \mathcal{U}),$$

where $\mathbf{Q}$ is any orthogonal tensor, $\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{1}$.

The interpretation of this action is as follows. If $\mathcal{U}$ is an attractor corresponding to a certain isolated system with a separated body, then $\mathbf{Q} * \mathcal{U}$ is an attractor corresponding to this system after a rigid rotation, together with the separated body.

Therefore, we require that the acceleration, and hence its direction, are isotropic functions of the attractor, and the acceleration modulus is its invariant, i.e.,

$$A(\mathcal{B}, \mathbf{Q} * \mathcal{U}) = A(\mathcal{B}, \mathcal{U}), \qquad \mathbf{n}(\mathcal{B}, \mathbf{Q} * \mathcal{U}) = \mathbf{Q}\mathbf{n}(\mathcal{B}, \mathcal{U}). \tag{C.29}$$

We could conclude the presentation of Newton's dynamics as a formal structure with the following description:

> *The classical dynamics of material particles* is a collection of sets $\mathcal{E}_{\mathsf{L}}$, $\mathcal{T}_{\mathsf{T}}$, B, U with elements called points, moments, material particles and attractors, equipped with a structure by the axiomatics of space $\mathcal{E}_{\mathsf{L}}$ and time $\mathcal{T}_{\mathsf{T}}$, motions (C.1), operations (C.9), (C.11), six laws of dynamics and principles of homogeneity of time and the homogeneity and isotropy of space and particles, (C.9), (C.11).

We note, moreover, that *descriptions* of this kind can never be fully complete. We do not mention here, e.g., the laws of analysis, and even less so, axioms of set theory and logic.

We emphasize that for the formulation and interpretations of the laws of dynamics the general ideas about the particle, the environment affecting its motion and the concept of *acceleration* are sufficient. Neither masses nor forces, the more so weights and dynamometers are needed to formulate the essence of the mechanics.

Are the laws of dynamics valid? This question does not make sense. The meaning, and it is the essential meaning, has its undertone: can this model be interpreted in such a way that the predicted by it behavior of real bodies will be in a satisfactory agreement with the results of observations and measurements?

To answer this question, one needs to be able to:

(1) identify and distinguish material particles $\mathcal{B}$ and attractors $\mathcal{U}$ for real systems considered to be isolated,

(2) measure the acceleration.

If we know the above, the procedure for experimentally checking the formulated laws of dynamics is conceptually simple: take different particles $\mathcal{B}$ and various attractors $\mathcal{U}$, measure accelerations $\mathbf{a}$ and substitute them for formulas (C.14)–(C.24). There is no need, at the same time, for any special knowledge on how to describe the attractors and particles. For example $\mathcal{U}$ is here a symbol of a situation, in which a separate real body has been determined, and $\mathcal{U}_1 = \mathcal{U}_2$ or $\mathcal{U}_1 \neq \mathcal{U}_2$ simply means that we consider these situations to be identical or non-identical.

When the pair $(\mathcal{B},\mathcal{U})$ obeys the laws of dynamics, we will say for the imagery of language, that the attractor $\mathcal{U}$ *acts* on the material particle $\mathcal{B}$.

The law of instantaneous action (C.14) specifies the word "acts": the attractor determines, for a fixed particle, the second – and not any other one! – derivative of its motion. The result of this action – the acceleration value – depends, in general, not only on the attractor, but also on the particle itself. Let us write down the law (C.14) explicitly:

$$\ddot{\chi}\,(\mathcal{B},t) = A\,(\mathcal{B},\mathcal{U}(p,t))\;\mathbf{n}\,(\mathcal{U}(p,t))\,, \qquad p \equiv \chi(\mathcal{B},t). \tag{C.30}$$

The zero law of dynamics has no explicit counterpart in Newton's laws.

The first law of motion (C.18) corresponds to Newton's first law: the isolated body remains at rest or uniform rectilinear motion.

The second law of dynamics is at the heart of the classical mechanics.

The meaning of the conditions (C.1) and (C.2) follows from their names.

The condition of separation (C.21) seems intricate and somewhat alien. The impression disappears at once after writing it a little further in two equivalent forms (C.34), (C.41). We will show without difficulty that for the basic attractors, the separation condition (C.21) implies the existence of mass, the existence of force, and binding them Newton's second law.

The universality condition means that each particle can be given any acceleration using a suitably selected basic attractor. It follows from here, in particular, that the number of basic attractors is infinite.

Experience shows that, for example, the attractors in the examples "charged particle" and "frictionless trolley" are the basic attractors.

The equivalents of the laws (C.23), (C.24) in Newton's model will be the additivity of mass, taken as obvious in the light of his description of mass, and the additivity of force, respectively.

8. The experimental status of space-time: the dynamic aspect. When formulating the laws of dynamics, we referred to observations and experiments. Now we will explain how to measure accelerations experimentally.

In practice, we only observe changes in the positions of bodies in relation to other bodies. Meanwhile, the motion of a particle (C.1) was defined as a change of its place in absolute, stationary space $\mathcal{E}_{\mathsf{L}}$.

Let us take a moving space observer. Observing the motion of a particle as a sequence of positions relative to this observer, we will obtain (in a manner known from the kinematics of rigid bodies) its acceleration in relation to the observer.

An observer is called an observer or an *inertial* system if this acceleration satisfies the laws of dynamics for any motion of each particle. In other words, an inertial observer is an observer who can be regarded in dynamics *a priori* as stationary.

Clearly, any observer who moves in a uniform, rectilinear and non-rotating motion with respect to a certain inertial observer, himself will be an inertial observer.

Considering in succession as stationary two inertial observers, moving in relation to each other, corresponds to the transformation of space-time:

$$p \to p + \mathbf{v}t + \text{const}, \qquad t \to t + \text{const}, \qquad \mathbf{v} = \text{const}, \tag{C.31}$$

called the *Galileo transformation.*

Thus, the experimental validation of the laws of dynamics always consists in recognizing a priori a certain set of real bodies as immobile. Measurements of acceleration relative to this system are done by measuring distances and time intervals. There are also special instruments for measuring acceleration, called Newton-meters[10)].

Experience shows that the Sun and three distant stars that are not in one plane with it provide an excellent example of an inertial system. Possible deviations of the behavior of bodies from the behavior predicted by the laws of dynamics (C.14)–(C.24), in which **a** is understood as acceleration in relation to this system, are undetectable by any known devices.[11)] With this in mind, I.I. Worowicz pragmatically writes in [C.14]: "The choice of a system [...] related to the Sun and distant stars, it seems, solves the problem of absolute space for the whole, conceivable period of human existence".

Moreover, in most of the situations described by the laws of dynamics, the perfect approximation of the inertial system is the Earth itself. The reason lies in the fact that the centrifugal acceleration (at the equator) caused by the Earth's spinning is only $\sim 3.4\ \text{cm} \cdot \text{s}^{-2}$, and the centrifugal acceleration of the Earth itself as it moves around the Sun even less, as it is $\sim 0.6\ \text{cm} \cdot \text{s}^{-2}$.

9. Necessity of factorization – of identifying particles and of identifying attractors. Material bodies-particles $\mathcal{B}$, we at once deprived of geometry. However, they are still too close to the originals – real bodies, to constitute real objects of the theory. All the more, it concerns the attractors, with their ostentatious tangibility (springs, planets, surface friction, ...). Of course, it is obvious that one has to stop differentiating between particles and attractors that behave alike.

The identifications are obvious: two material particles $\mathcal{B}_1$ and $\mathcal{B}_2$ we recognize as *dynamically indistinguishable* and write $\mathcal{B}_1 \sim \mathcal{B}_2$, when

10) Their construction and nuances of measurements are described, for example, in [C.16].

11) With one famous exception. This exception is the change of the perihelion of Mercury's orbit by 578″ over a century, instead of the value of 535″ predicted by the laws of dynamics. The reason, however, is not a disturbance in the inertia of the Sun-stars system, but the inaccuracy of the Euclidean model of space itself in the vicinity of the gigantic mass of the Sun. This effect is a spectacular experimental fact in favor of Einstein's general theory of gravity.

$$\mathbf{a}(\mathcal{B}_1,\mathcal{U}) = \mathbf{a}(\mathcal{B}_2,\mathcal{U}), \tag{C.32}$$

for each dynamic attractor $\mathcal{U}$; two attractors $\mathcal{U}_1$ and $\mathcal{U}_2$ we consider a *dynamically indistinguishable* and write $\mathcal{U}_1 \sim \mathcal{U}_2$, when

$$\mathbf{a}(\mathcal{B},\mathcal{U}_1) = \mathbf{a}(\mathcal{B},\mathcal{U}_2), \tag{C.33}$$

for every material particle $\mathcal{B}$.

These identifications have a clear experimental meaning. In this way, a huge part of the information about a real isolated system with a separated body is removed, which is irrelevant in dynamics.

Let us note at once that all static attractors are dynamically indistinguishable from the empty attractor $\mathcal{U}_0$.

It is also clear that each attractor dynamically indistinguishable from the basic attractor is itself the basic attractor.

The genuine objects of dynamics are the classes of dynamically indistinguishable particles $\underline{\mathcal{B}}$ and dynamically indistinguishable attractors $\underline{\mathcal{U}}$.

Mass and force

The particularly compact and clear form can be given to the dynamics of particles with basic attractors. This is achieved by introducing mass and force.

Throughout this chapter, by "attractor" we mean the basic attractor, and by "particle" a material particle.

10. Inertial mass. The separation condition (C.21) can be expressed in the following equivalent form: for any pair of particles $\mathcal{B}$, $\mathcal{B}_*$ and any pair of basic attractors $\mathcal{U}$, $\mathcal{U}_*$:

$$\frac{A(\mathcal{B},\mathcal{U})}{A(\mathcal{B}_*,\mathcal{U})} = \frac{A(\mathcal{B},\mathcal{U}_*)}{A(\mathcal{B}_*,\mathcal{U}_*)}, \tag{C.34}$$

that is, the ratio of the acceleration moduli of two particles, exposed independently to the action of an arbitrarily chosen basic attractor, does not depend on this attractor, but only on the particles themselves.

This suggests the idea of introducing an experimental procedure of *particle measurement*, $(\mathcal{B},\mathcal{B}_*) \to \mathcal{B} : \mathcal{B}_* \in R$, defined by a formula:

$$\mathcal{B} : \mathcal{B}_* \equiv \left[\frac{A(\mathcal{B},\mathcal{U})}{A(\mathcal{B}_*,\mathcal{U})}\right]^{-1}, \tag{C.35}$$

for a certain (and then each) basic attractor $\mathcal{U}$. It is clear that the MEASUREMENT POSTULATE

$$(\mathcal{B} : \mathcal{B}_*)\,(\mathcal{B}_* : \mathcal{B}_{**}) = \mathcal{B} : \mathcal{B}_{**}, \tag{C.36}$$

is fulfilled (cf. (1.6), i.e. $(\mathcal{A} : \mathcal{U})(\mathcal{U} : \mathcal{W}) = \mathcal{A} : \mathcal{W}$ for all objects $\mathcal{A}$ and all standards $\mathcal{U}$, $\mathcal{W}$).

So, we introduce, in accordance with the general procedure for all concepts of physics (see Lecture 1), the dimension

$$\mathbf{M} = \text{INERTIAL MASS}, \tag{C.37}$$

with elements called inertial masses. Each particle $\mathcal{B}$ is assigned in $\mathbf{M}$ exactly one dimensional scalar $m(\mathcal{B})$ called the *inertial mass* of the particle $\mathcal{B}$, and the assignment $\mathcal{B} \to m(\mathcal{B})$ is by definition such that for any freely chosen standard particle $\mathcal{B}_*$

$$m(\mathcal{B}) = (\mathcal{B} : \mathcal{B}_*)m(\mathcal{B}_*). \tag{C.38}$$

In this way, through a clear and general measurement procedure (C.35) carried out by measuring the acceleration moduli the inertial mass is introduced to Mechanics, and thus to whole of Physics.

Particles $\mathcal{B}_1$ and $\mathcal{B}_2$ are dynamically indistinguishable, $\mathcal{B}_1 \sim \mathcal{B}_2$, if and only if their mass is the same, $m(B_1) = m(B_2)$. It results immediately from (C.32), (C.35), (C.38). Thus, a scalar $m(\mathcal{B})$ is a mathematical representation of a class $\underline{\mathcal{B}}$ composed of all particles which cannot be distinguished from a particle $\mathcal{B}$ in any mechanical experiment, i.e., by observing accelerations under the action of test attractors.

As soon as the mass of a particle is determined, the material particle itself loses all other individual properties in Mechanics. Its motion under the influence of any chosen attractor is determined only by its mass. This does not mean, of course, that all individual features are lost in Mechanics by the very real body separated in a real isolated system. Some of its features may contain an attractor.

Note. The dimension $\mathbf{M}$ used here, similarly like the previously used Euclidean space $\mathcal{E}_{\mathbf{L}}$ are constructions of pure Mathematics, determined axiomatically.[12] They are built in the edifice of Physics as ready-made bricks. In Lectures 1, 2, and 6, we drew attention to the enormous extravagance of this procedure.

Note. Taking the reverse measuring procedure

$$(\mathcal{B} : \mathcal{B}_*) \equiv \frac{A(\mathcal{B},\mathcal{U})}{A(\mathcal{B}_*,\mathcal{U})}, \tag{C.39}$$

we would get a feature that could be described by the word *volatility*, which is a measure of the particle's mobility. Historically, it has become common to use mass rather than volatility. The reason is the rule (C.23).

[12] On the freedom of axiomatics it has already been said somewhere that although a mathematician has the right – and usually a desire – to sew any clothes, even for ants with seven legs, only some clothes are worn.

11. Additivity of mass. The observation of the disjointed motions of the particles $\mathcal{B}_1$ and $\mathcal{B}_2$ and the motions of the merged particle $\mathcal{B}_1 \circ \mathcal{B}_2$ led to the formulation of the axiom (C.23). It is not difficult to conclude from (C.38) and (C.35) that it is equivalent to the following *law of additivity of the masses*:

$$m(\mathcal{B}_1 \circ \mathcal{B}_2) = m(\mathcal{B}_1) + m(\mathcal{B}_2), \tag{C.40}$$

for any particles $\mathcal{B}_1$, $\mathcal{B}_2$.

This suggests the idea of interpreting mass as a *measure of the amount of a substance*. This is what Newton himself understood as the mass of bodies. This point of view could not be considered to be correct throughout Physics, due to the mass defect phenomenon in the interactions of elementary particles.

Due to the formulas (C.35), (C.38), (C.40) mass $m(\mathcal{B})$ is called a *scalar measure of inertia* of particle $\mathcal{B}$. According to (C.35), the acceleration of a particle moving under the influence of an attractor is the smaller, the greater the mass of this particle. This is, of course, an interpretation of the word "inertia" and not the word "mass".

12. Force. Quite similarly, we will free ourselves from the irrelevant individual characteristics of the basic attractors.

The separation condition (C.21) can be rewritten in another equivalent form: for any pair of basic attractors $\mathcal{U}$, $\mathcal{U}_*$ and any pair of particles $\mathcal{B}$, $\mathcal{B}_*$

$$\frac{A(\mathcal{B},\mathcal{U})}{A(\mathcal{B},\mathcal{U}_*)} = \frac{A(\mathcal{B}_*,\mathcal{U})}{A(\mathcal{B}_*,\mathcal{U}_*)}, \tag{C.41}$$

i.e. the ratio of the acceleration moduli which give a particle two different basic attractors, does not depend on the particle but only on the attractors themselves.

This suggests the introduction of an experimental procedure for *measuring basic attractors*, $(\mathcal{U}_1,\mathcal{U}_2) \to \mathcal{U}_1 : \mathcal{U}_2 \in R$, defined by the formula:

$$\mathcal{U} : \mathcal{U}_* \equiv \frac{A(\mathcal{B},\mathcal{U})}{A(\mathcal{B},\mathcal{U}_*)}, \tag{C.42}$$

for a certain (and then every) particle $\mathcal{B}$. Clearly, the MEASUREMENT POSTULATE (cf. (1.6))

$$(\mathcal{U} : \mathcal{U}_*)(\mathcal{U}_* : \mathcal{U}_{**}) = \mathcal{U} : \mathcal{U}_{**} \tag{C.43}$$

is fulfilled.

Hence, we will introduce the dimension

$$\mathbf{F} = \text{FORCE}, \tag{C.44}$$

with elements called force moduli. We will assign to each attractor $\mathcal{U}$ in F exactly one dimensional scalar $F(\mathcal{U})$ called modulus of its force, and the assignment $\mathcal{U} \to F(\mathcal{U})$ is by definition such that for the freely chosen standard attractor $\mathcal{U}_*$

$$F(\mathcal{U}) = (\mathcal{U} : \mathcal{U}_*) F(\mathcal{U}_*). \tag{C.45}$$

Force of basic attractor $\mathcal{U}$ we will call a vector:

$$\mathbf{F}(\mathcal{U}) \equiv F(\mathcal{U}) \mathbf{n}(\mathcal{U}), \tag{C.46}$$

where $\mathbf{n}(\mathcal{U})$ is a unit vector from formula (C.19). It is a unit vector from space Э (see Lecture 12).

In this way, through a clear and general measuring procedure (C.42), carried out by measuring the acceleration moduli the force is introduced into Mechanics.

Basic attractors $\mathcal{U}_1$, $\mathcal{U}_2$ are dynamically indistinguishable, $\mathcal{U}_1 \sim \mathcal{U}_2$, if and only if the corresponding forces are equal, $\mathbf{F}(\mathcal{U}_1) = \mathbf{F}(\mathcal{U}_2)$. It results immediately from (C.33) and (C.45). In other words, a vector $\mathbf{F}(\mathcal{U})$ is a mathematical image of a class $\underline{\mathcal{U}}$ composed of all basic attractors, which cannot be distinguished from an attractor $\mathcal{U}$ in any mechanical experiment, i.e., by observing the test motions of particles.

As soon as the force of the basic attractor is determined, the attractor itself loses all other individual features in Mechanics. Its influence on the motion of any chosen particle is determined only by its force.

13. Additivity of force. Observations of motions under the action of attractors $\mathcal{U}_1$ and $\mathcal{U}_2$, separately and together, lead to the formulation of the axiom (C.24).

It is not difficult to show that the set of basic attractors is closed with respect to the merging operation: for any basic attractors $\mathcal{U}_1$, $\mathcal{U}_2$ the attractor $\mathcal{U}_1 \circ \mathcal{U}_2$ is the basic attractor.

The law (C.24) takes the form of the *law of force additivity*:

$$\mathbf{F}(\mathcal{U}_1 \circ \mathcal{U}_2) = \mathbf{F}(\mathcal{U}_1) + \mathbf{F}(\mathcal{U}_2), \tag{C.47}$$

for any basic attractors $\mathcal{U}_1$, $\mathcal{U}_2$.

Bearing in mind the formulas (C.42), (C.45), (C.47), we say that force $\mathbf{F}(\mathcal{U})$ is a *vector measure of the intensity and operation direction of the basic attractor* $\mathcal{U}$.

14. Newton's second law. The mass of the material particle and the force of the basic attractor are introduced in such a manner that a dimensional scalar

$$R \equiv \frac{m(\mathcal{B}) A(\mathcal{B}, \mathcal{U})}{F(\mathcal{U})} \in \mathsf{F}^{-1} \mathsf{M}^1 \mathsf{T}^{-2} \mathsf{L}^2 \tag{C.48}$$

depends neither on the particle $\mathcal{B}$ nor on the attractor $\mathcal{U}$. Indeed, substituting for (C.48) the quantities (C.38), (C.45) and using the definition of measurement

(C.35), (C.42), we obtain the exchange of the pair ($\mathcal{B}$, $\mathcal{U}$) for any other pair ($\mathcal{B}_*$, $\mathcal{U}_*$). The quantity R is therefore a universal constant in dynamics.

The separation condition (C.21) now has the form:

$$A\left(\mathcal{B},\mathcal{U}\right) = R\frac{F\left(\mathcal{U}\right)}{m\left(\mathcal{B}\right)}, \qquad R = \text{const.} \tag{C.49}$$

This wonderful formula uncovers the purposefulness of introducing the concepts of force as an attractor characteristic and mass as a particle characteristic. The role of the attractor as the cause of the acceleration of the particle and the role of the particle itself have been s p l i t and r e v e a l e d: the acceleration modulus is directly proportional to the attractor force modulus and inversely proportional to the mass of the particle.

However, universal constant R irritates. In Newton's times, relations between units of physical quantities were not used. For example, instead of saying, as we do today, that the area of a rectangle is *equal* to the product of the base length and height, it was said that it was *proportional* to them (see formula (6.31)). In the further development of mechanics, since the time of Euler, dimensions and units have been coupled so that

$$R = 1, \quad \text{from where} \quad \mathsf{F} = \mathsf{LMT}^{-2}. \tag{C.50}$$

The conditions (C.19), (C.49) now take the form of *Newton's second law*:

acceleration $\mathbf{a} = A\mathbf{n}$, $\mathbf{nn} = 1$ of particle $\mathcal{B}$ under the action of basic attractor $\mathcal{U}$ is determined as follows:

$$A = A(\mathcal{B},\mathcal{U}) = \frac{F(\mathcal{U})}{m(\mathcal{B})}, \tag{C.51}$$

$$\mathbf{n} = \mathbf{n}(\mathcal{U}). \tag{C.52}$$

This is the pinnacle of classical mechanics.

This law is usually written in a mathematically equivalent but misleading form:

$$\mathbf{F}(\mathcal{U}) = m(\mathcal{B})\mathbf{a}(\mathcal{B},\mathcal{U}), \tag{C.53}$$

or even more abbreviated and confusingly:

$$\mathbf{F} = m\mathbf{a}. \tag{C.54}$$

We will introduce the quantity

$$\mathbf{M} = m\mathbf{v}, \tag{C.55}$$

called by us *momentum*, and by Newton "motus"[13]. Since we have assumed that the particle is a constant object and, by definition, independent of time, the law $\mathbf{F} = m\mathbf{a}$ can be written as:

$$\mathbf{F} = \dot{\mathbf{M}}. \tag{C.56}$$

This corresponds to Newton's formulation:

LEX II

Mutationem motus proportionalem esse vi motrice impressiae et fieri secundum lineam rectam qua vis ilia imprimitur.[14]

Comparing this wording with (C.51), (C.52), we judge carefully and with due respect that Sir Isaac did not care too much about clarity.

Note. A body of variable mass (e.g., a rocket on an active section of a track) is actually a shrinking over time part of a body (a rocket at launch time). Continuous variation of this kind can only be accurately described in continuum mechanics.

15. Unveiling the backstage. Let $\mathcal{X}$ and $\mathcal{Y}$ be non-empty sets formed in any way from anything.

THEOREM. *A real positive function f on $\mathcal{X}\times\mathcal{Y}$ satisfies a functional equation:*

$$f(x_1, y_1)f(x_2, y_2) = f(x_1, y_2)f(x_2, y_1), \tag{C.57}$$

for all x_1, y_1, x_2, y_2 if and only if there exist such positive real functions, μ on X and φ on Y, that for all x, y

$$f(x, y) = \mu(x)\varphi(y). \tag{C.58}$$

Proof. Sufficiency is obvious.

Necessity. Let us arbitrarily fix x_* and y_* and take two positive numbers k, l such that

$$kl = f(x_*, y_*), \tag{C.59}$$

where f is a function satisfying (C.57). Let us introduce the functions μ, φ with formulas:

$$\begin{aligned} \mu(x) &\equiv \frac{f(x, y_*)}{f(x_*, y_*)}k, \\ \varphi(x) &\equiv \frac{f(x_*, y)}{f(x_*, y_*)}l. \end{aligned} \tag{C.60}$$

[13] One of Newton's immediate predecessors, J. Wallis, used the term "momentum", which remained in English for unknown reasons.

[14] *Translator note*: in English: Alteration of momentum is proportional to the exerted force and takes place along straight line on which the force is exerted.

Now

$$\mu(x)\varphi(x) = \frac{f(x,y_*)f(x_*,y)}{(f(x_*,y_*))^2}kl = f(x,y). \quad \blacksquare \tag{C.61}$$

Operations $x_1 : x_2$ and $y_1 : y_2$ are defined by the formulas:

$$\begin{aligned} x_1 : x_2 &\equiv \frac{f\,(x_1,y)}{f\,(x_2,y)}, \\ y_1 : y_2 &\equiv \frac{f(x,y_1)}{f(x,y_2)}, \end{aligned} \tag{C.62}$$

and can be called, if one prefers it, *measuring* in $\mathcal{X}$ and $\mathcal{Y}$.

We can see that this is the general template that we have used to introduce mass and force.

Thus, as we have announced, the *separation condition* (C.21), directly verifiable in experiment, is a necessary and sufficient condition for the introduction of the *mass* and the *force* modulus, and written with their use together with (C.19) becomes the *Newton's second law.*

Gravity

The basic attractors considered in the previous chapter, adequately describing a large number of isolated systems, in particular technical devices, do not exhaust all types of attractors.

The dynamic *gravitational* attractors are the most important among dynamic non-basic attractors, and they are defined as follows:

$$A(\mathcal{B}_1,\mathcal{U}) \equiv A(\mathcal{B}_2,\mathcal{U}), \tag{C.63}$$

for any material particles $\mathcal{B}_1$, $\mathcal{B}_2$. The acceleration of the body under the action of the gravitational system does not depend on the accelerated particle itself:

$$A = A(\mathcal{B},\mathcal{U}) = A(\mathcal{U}). \tag{C.64}$$

For gravitational attractors, the separation condition (C.21) occurs identically. The road on which we previously walked is therefore closed.

16. Gravitational isolated systems. Let us consider an isolated system composed of N interacting real bodies, small enough in relation to their distances. The problem of describing all possible motions of this system is called the *N bodies problem.* The crowning already cited example is the Solar system without small bodies (asteroids and comets); here $N = 70$. Taking one of the bodies, e.g., our Moon, we get a system with a distinguished body. In fact, almost all bodies can be left out here except the Earth and the Sun. Then we have a pair $(\mathcal{B}, \mathcal{U})$, where particle $\mathcal{B} = \{\text{Moon}, t\}$, attractor $\mathcal{U} = \{$Earth, Sun, their radius vectors at a given instant, from a point $p = \chi$ (Moon, t)$\}$.

The simplest system, for $N = 1$, is an isolated body, subject to the first law of dynamics (C.18).

Let us take the *problem of two-bodies* $\mathcal{B}$ and $\mathcal{G}$. Considering $\mathcal{B}$ as the selected body, and $\mathcal{G}$ as the interacting body, we have an attractor

$$\mathcal{U}_{\mathcal{G}} = \{\mathcal{G}, \mathbf{r}\}, \tag{C.65}$$

where $\mathbf{r}(t)$ is the radius vector from $\chi(\mathcal{B}, t)$ to $\chi(\mathcal{G}, t)$.

Each attractor with $n-1$ bodies influencing the motion of the selected body is of the form:

$$\mathcal{U} = \mathcal{U}_{\mathcal{G}_1} \circ \ldots \circ \mathcal{U}_{\mathcal{G}_{N-1}}, \tag{C.66}$$

therefore it is enough to describe only the attractor in the two-body problem.

Note that by changing the roles of $\mathcal{B}$ and $\mathcal{G}$, we obtain an attractor

$$\mathcal{U}_{\mathcal{B}} = \{\mathcal{B}, -\mathbf{r}\}. \tag{C.67}$$

The attractors (C.65) and all their mergers are gravitational attractors in the sense of the definition (C.63).[15)]

The equality of acceleration of all bodies falling in the Earth's field of gravity was one of Galileo's most surprising and controversial discoveries. Today, we probably teach it in kindergartens.

17. The law of universal gravitation. Based on Kepler's laws derived from lasting 21 years astronomical observations by Tycho de Brahe, Newton formulated the *law of universal gravitation*, which we will formulate as follows:

> *Acceleration* $\mathbf{a} = A\mathbf{n}$, $\mathbf{nn} = 1$, *of a particle-body* $\mathcal{B}$ *under the action of an attractor* $\mathcal{U}_{\mathcal{G}}$ *is described by the formulas:*
>
> $$A = C(\mathcal{G})\frac{i}{r^2}, \qquad \mathbf{n} = \mathbf{r}/|\mathbf{r}|, \tag{C.68}$$
>
> *where* $\mathbf{r} \equiv \overrightarrow{\chi(\mathcal{B})\chi(\mathcal{G})}$.

We see that acceleration really does not depend on the particle being accelerated.

---

15) Are there other gravitational attractors with real equivalents?

Instead of (C.68) one could take a little less, only considering that the direction of acceleration is a function of the radius vector from the site of particle $\mathcal{B}$ to the site of particle $\mathcal{G}$:

$$\mathbf{n} = f(\mathbf{r}). \tag{C.69}$$

According to the isotropy principle of space (C.29), we have then

$$f(\mathbf{Q}\mathbf{r}) = \mathbf{Q}f(\mathbf{r}). \tag{C.70}$$

This functional equation has only two versor solutions:

$$f(\mathbf{r}) = +\mathbf{r}/|\mathbf{r}|, \qquad f(\mathbf{r}) = -\mathbf{r}/|\mathbf{r}|. \tag{C.71}$$

In the Universe known to us, the "+" sign is realized, which corresponds to the gravitational attraction.

The body-particle $\mathcal{G}$ in (C.68) we call the *center of attraction*, and the field around $\chi(\mathcal{G}, t)$ described by the formulas (C.68) its *gravitational field.*

18. Gravitational charge. Let us take two attractors

$$\mathcal{U} = \{\mathcal{G}, \mathbf{r}\}, \qquad \mathcal{U}_* = \{\mathcal{G}_*, \mathbf{r}\}, \tag{C.72}$$

differing only in the center of attraction used.

We will introduce the following procedure for *measuring particles as centers of attraction*:

$$\mathcal{G} : \mathcal{G}_* \equiv \frac{A(\mathcal{B}, \mathcal{U}_{\mathcal{G}})}{A(\mathcal{B}, \mathcal{U}_{\mathcal{G}_*})} = \frac{C(\mathcal{G})}{C(\mathcal{G}_*)}. \tag{C.73}$$

The measurement postulate (1.6), cf. note below (C.36), is obviously fulfilled. The physical feature of particles measured in this way (an empirical, physical concept in the terminology of Lecture 1) will be called a *gravitational charge* or, according to the well-established tradition, the meaning of which will appear in a moment, a *gravitational mass.*

The dimension corresponding to the measuring procedure (C.73) we will call

$$\mathbf{M}_{\mathrm{gr}} = \text{GRAVITATIONAL CHARGE}. \tag{C.74}$$

Each body-particle $\mathcal{G}$ is assigned a dimensional scalar $\mathfrak{m}(\mathcal{G}) \in \mathbf{M}_{\mathrm{gr}}$ called the *gravitational charge* or the *gravitational mass* of this body-particle. Upon choosing a standard body-particle $\mathcal{G}_*$, we have

$$\mathfrak{m}(\mathcal{G}) = (\mathcal{G} : \mathcal{G}_*)\mathfrak{m}(\mathcal{G}_*). \tag{C.75}$$

This scalar is the *measure of the body-particle* $\mathcal{G}$ *as the center of attraction.* By introducing a universal constant

$$D \equiv \frac{C(\mathcal{G})}{\mathfrak{m}(\mathcal{G})} = \frac{C(\mathcal{G}_*)}{\mathfrak{m}(\mathcal{G}_*)} = ..., \tag{C.76}$$

we will write the main formula of the law of universal gravitation in the form

$$A = D\frac{\mathfrak{m}\,(\mathcal{G})}{r^2}. \tag{C.77}$$

Taking the attractor

$$\mathcal{U}_1 \circ \mathcal{U}_2 = \{\mathcal{G}_1 \circ \mathcal{G}_2, \mathbf{r}\}, \tag{C.78}$$

according to the law of the merging attractors (C.24) we obtain the law of additivity of the gravitational charge:

$$\mathfrak{m}(\mathcal{G}_1 \circ \mathcal{G}_2) = \mathfrak{m}(\mathcal{G}_1) + \mathfrak{m}(\mathcal{G}_2). \tag{C.79}$$

Note. The ratio of accelerations of particles $\mathcal{B}$, $\mathcal{G}$ in a two-body system is

$$\frac{A(\mathcal{B},\mathcal{U}_\mathcal{G})}{A(\mathcal{G},\mathcal{U}_\mathcal{B})} = \frac{C(\mathcal{G})}{C(\mathcal{B})} = \frac{\mathfrak{m}(\mathcal{G})}{\mathfrak{m}(\mathcal{B})}, \tag{C.80}$$

hence, Mach's proposal, which we wrote about in the introduction, concerned the measurement of gravitational charges.

19. The specific character of the theory of gravity. The obtained formulas close the theory of gravity as a physical theory. However, there remain difficult problems of pure Mathematics regarding the solution of the relevant differential equations. Only for the two-body problem, the solution is obtained in a closed form. For the mechanics of the sky and astronautics, powerful computer methods for solving the problem of N bodies, for larger N, have been developed.

Gravitation as a separated part of dynamics describing the gravitational attractors of the type (C.65) *is thus dealt with entirely without the concept of mass and force.* Moreover, in the closed world of gravity, these concepts *cannot be introduced.*

The thing is, however, that we are dealing with real isolated systems in which gravitational attractors act next to and together with non-gravitational attractors. Crowning examples are pendulums and satellites in low orbits. In order to consider such situations, we must somehow assign a force to gravitational attractors.

20. Force of attraction. We assume the following definition of the force of attraction: *the force of attraction of a particle* $\mathcal{B}$ by a particle $\mathcal{G}$ is a vector $\mathbf{f}(\mathcal{B},\mathcal{U}_\mathcal{G})$ with a modulus

$$f(\mathcal{B},\mathcal{U}_\mathcal{G}) \equiv D\frac{\mathfrak{m}(\mathcal{G})}{r^2}m(\mathcal{B}), \tag{C.81}$$

and direction

$$\mathbf{n}(\mathcal{B},\mathcal{G}) \equiv \mathbf{r}/r, \tag{C.82}$$

where $r \equiv |\mathbf{r}|$, $\mathbf{r} \equiv \overrightarrow{\chi(\mathcal{B})\chi(\mathcal{G})}$.

We emphasize the fundamental difference between the force $\mathbf{F}(\mathcal{U})$ of the basic attractor and the force of attraction $\mathbf{f}(\mathcal{B},\mathcal{U}_\mathcal{G})$. The first is a vector measure of the basic attractor. The second one depends not only on the attractor, but also on the particle that this attractor acts on. According to the adopted definition:

$$\mathbf{f}(\mathcal{B},\mathcal{U}_\mathcal{G}) \equiv m(\mathcal{B})\mathbf{a}(\mathcal{B},\mathcal{U}_\mathcal{G}), \tag{C.83}$$

thus it reconciles the law of universal gravitation with Newton's second law.

21. Newton's third law. Under the law of universal gravitation, we now have

$$\mathbf{f}(\mathcal{B},\mathcal{U}_\mathcal{G}) \equiv -\mathbf{f}(\mathcal{G},\mathcal{U}_\mathcal{B}). \tag{C.84}$$

This equality is Newton's third law, stated by him as follows:

Lex III

Actioni contrariam semper et equalem esse reactionem; sive corporom duorom actiones in se mutuo semper esse equales et in partes contrarias dirigi.[16)]

22. Mass as a gravitational charge. Substituting into the formula (C.84) the reconciling definition of the force of attraction (C.81), we obtain the law of proportionality of inertial mass and gravitational mass:

$$\frac{\mathfrak{m}(\mathcal{G})}{\mathfrak{m}(\mathcal{B})} = \frac{m(\mathcal{G})}{m(\mathcal{B})}. \tag{C.85}$$

for any body-particles $\mathcal{G}$, $\mathcal{B}$.

This proportionality is checked again and again with increasing accuracy. Newton himself, in experiments with pendulums made of wood, gold, silver, lead, glass, salt, water, sand and even wheat, achieved an accuracy of 10^{-3}. Bessel corrected it to 10^{-4}. Many researchers improved the result. Eötvös was particularly distinguished here. In Braginski's recent experiments, a record accuracy of $0.9 \cdot 10^{-12}$ has been achieved. The equivalence of inertial mass $m(\mathcal{B})$ and gravitational charge $\mathfrak{m}(\mathcal{B})$ in the sense of (C.84) was the starting point for Einstein in the construction of general relativity theory.

16) *Translator note*: in English: To action there is always opposed an equal reaction; or mutual actions of two bodies are always equal and directed opposite.

According to (C.84), a scalar

$$C \equiv \frac{\mathfrak{m}\,(\mathcal{B})}{m\,(\mathcal{B})} \in \mathsf{M}_{\mathrm{gr}}\mathsf{M}^{-1} \tag{C.86}$$

does not depend on the particle $\mathcal{B}$, so it is a universal constant. On adopting $C = 1$, we write the law (C.84) in the form

$$\mathfrak{m}(\mathcal{B}) = m(\mathcal{B}), \qquad \mathsf{M}_{\mathrm{gr}} = \mathsf{M} \tag{C.87}$$

for each particle $\mathcal{B}$.

23. The law of universal gravitation for the second time. Considering the reconciling definition of the force of attraction (C.81) and the convention (C.87), we obtain *the law of universal gravitation* in the form we are used to:

An isolated pair of particles $(\mathcal{B}, \mathcal{G})$ *moves according to Newton's second law* (C.83), *where the force of action of* $\mathcal{G}$ *on* $\mathcal{B}$ *has a modulus and a direction:*

$$\mathbf{f}(\mathcal{B},\mathcal{U}_{\mathcal{G}}) = G\frac{m(\mathcal{B})m(\mathcal{G})}{r^2}, \qquad \mathbf{n}(\mathcal{B},\mathcal{G}) = \mathbf{r}/r, \tag{C.88}$$

where $r \equiv |\mathbf{r}|$, $\mathbf{r} \equiv \overrightarrow{\chi(\mathcal{B})\chi(\mathcal{G})}$, *and* $\mathcal{G}$ *is the universal constant* (as defined by Eq. (12.122): $\mathcal{G} = 6.672 \cdot 10^{-11}\ \mathrm{m}^3/(\mathrm{kg}\cdot\mathrm{s}^2)$).

24. Any gravitational fields. We generalize these formulas immediately to the case when a particle $\mathcal{B}$ moves in the gravitational field of matter occupying at the considered instant any domain V. We denote by $\rho(\mathbf{x})$ the *density of matter* at the point $q = \chi(\mathcal{B}, t) + \mathbf{x}$. We will denote this attractor with $\mathcal{U}_{\mathrm{gr}}$.

The force of attraction of a particle $\mathcal{B}$ by the distributed matter is

$$\mathbf{f}(\mathcal{B},\mathcal{U}_{\mathrm{gr}}) = Gm(\mathcal{B})\int \frac{\rho(\mathbf{x})}{|\mathbf{x}|^3}\mathbf{x}\,\mathrm{d}V. \tag{C.89}$$

For the center of attraction $\mathcal{G}$ at the point $q = p + \mathbf{r}$, $\mathbf{r} = \mathrm{const}$, we have

$$\rho(\mathbf{x}) = m(G)\delta(\mathbf{x} - \mathbf{r}), \tag{C.90}$$

where δ is the Dirac distribution, and we recover the previous formula. The superimposition of the centers of attraction corresponds to the sum of such expressions.

Force (general case)

25. Need for unification. The existence of basic attractors allowed for the introduction of mass as the only dynamic feature of material particles. It is different with the attractors.

Force we introduced for basic attractors as their only dynamic feature. Its determination was entirely based on the features (C.19), (C.20), (C.21) distinguishing basic attractors.

Although dynamics can be developed without the concept of force, its elegance and power presented above prompt the introduction of force for any isolated system with a separated body. We will do it similarly to the theory of gravity, assigning the force not to the attractor itself, but to the pair (particle, attractor).

26. Unification. So we introduce a mapping

$$(\mathcal{B},\mathcal{U}) \to \mathbf{f}(\mathcal{B},\mathcal{U}), \tag{C.91}$$

defined as follows: for each particle $\mathcal{B}$

(1) if the attractor $\mathcal{U}$ is static then

$$\mathbf{f}(\mathcal{B},\mathcal{U}) \equiv \mathbf{0}, \tag{C.92}$$

(2) if the attractor $\mathcal{U}$ is dynamic then

$$\mathbf{f}(\mathcal{B},\mathcal{U}) = \mathbf{F}(\mathcal{P}), \tag{C.93}$$

where the attractor $\mathcal{P}$ is the basic attractor defined by the formula:

$$\mathbf{a}(\mathcal{B},\mathcal{U}) = \mathbf{a}(\mathcal{B},\mathcal{P}), \qquad \mathcal{B} = \text{const}. \tag{C.94}$$

Such an attractor exists for each value $\mathbf{a}(\mathcal{B},\mathcal{U})$ due to the universality condition (C.22) of the set of basic attractors. Of course, $\mathbf{F}(\mathcal{P}_1) = \mathbf{F}(\mathcal{P}_2)$ for any attractors $\mathcal{P}_1$, $\mathcal{P}_2$ satisfying (C.94).

A vector $\mathbf{f}(\mathcal{B},\mathcal{U})$ is called *the force of action of attractor $\mathcal{U}$ on particle $\mathcal{B}$.*

According to the definition (C.93), for basic attractors

$$\mathbf{f}(\mathcal{B},\mathcal{U}) = \mathbf{F}(\mathcal{U}), \tag{C.95}$$

for gravity fields

$$\mathbf{f}(\mathcal{B},\mathcal{U}_{\mathrm{gr}}) = Gm(\mathcal{B}) \int \frac{\rho(\mathbf{x})\mathbf{x}}{|\mathbf{x}|^3}\, \mathrm{d}V, \tag{C.96}$$

and for their mergers

$$\mathbf{f}(\mathcal{B},\mathcal{U} \circ \mathcal{U}_{\mathrm{gr}}) = \mathbf{F}(\mathcal{U}) + Gm(\mathcal{B}) \int \frac{\rho(\mathbf{x})\mathbf{x}}{|\mathbf{x}|^3}\, \mathrm{d}V. \tag{C.97}$$

We are not sure if more than the last formula presented above is needed in real situations. In all our examples, we deal with the attractors $\mathcal{U} \circ \mathcal{U}_{\mathrm{gr}}$, where

$\mathcal{U}$ is the composition of the basic attractors, and $\mathcal{U}_{\text{gr}}$ the composition of the gravitational fields.

Force $\mathbf{f}(\mathcal{B},\mathcal{U})$ meets the condition of additivity:

$$\mathbf{f}(\mathcal{B},\mathcal{U}_1 \circ \mathcal{U}_2) = \mathbf{f}(\mathcal{B},\mathcal{U}_1) + \mathbf{f}(\mathcal{B},\mathcal{U}_2). \tag{C.98}$$

We emphasize that force $\mathbf{f}(\mathcal{B},\mathcal{U})$ is not generally a measure of an attractor $\mathcal{U}$, but of its interaction with a specific particle $\mathcal{B}$. Only on the subset – admittedly the most important for applications – composed of *static* attractors and *basic dynamic* attractors the vector $\mathbf{f}(\mathcal{B},\mathcal{U}) = \mathbf{F}(\mathcal{U})$ becomes a measure of the attractor itself.

At the same time, as it is not difficult to show, a pair $(\mathcal{B}_1, \mathcal{U}_1)$ is dynamically indistinguishable from a pair $(\mathcal{B}_2, \mathcal{U}_2)$, i.e.,

$$B_1 \sim B_2, \qquad U_1 \sim U_2, \tag{C.99}$$

(see definition (C.32), (C.33)) if and only if

$$m(\mathcal{B}_1) = m(\mathcal{B}_2), \qquad \mathbf{f}(\mathcal{B}_1,\mathcal{U}_1) = \mathbf{f}(\mathcal{B}_2,\mathcal{U}_2), \tag{C.100}$$

27. Newton's laws. Newton's first law is written now in the form of

$$\mathbf{f}(\mathcal{B},\mathcal{U}) = \mathbf{0} \quad \Leftrightarrow \quad A(\mathcal{B},\mathcal{U}) = 0. \tag{C.101}$$

The original wording is:

Lex I

> *Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus illud a viribus impressis cogitur statum suum mutare.*[17)]

Applying the law of force additivity, we obtain the *equilibrium condition of attractors*:

$$\mathbf{f}(\mathcal{B},\mathcal{U}_1) + \ldots + \mathbf{f}(\mathcal{B},\mathcal{U}_2) + \ldots = \mathbf{0} \quad \Leftrightarrow \quad A(\mathcal{B},\mathcal{U}_1 \circ \mathcal{U}_2 \circ \ldots) = 0, \tag{C.102}$$

for a given particle $\mathcal{B}$.

We introduced the force $\mathbf{f}(\mathcal{B},\mathcal{U})$ in such a way that for each particle $\mathcal{B}$ and for each attractor $\mathcal{U}$ holds *Newton's second law* in the form:

$$\mathbf{f}(\mathcal{B},\mathcal{U}) = m(\mathcal{B})\mathbf{a}(\mathcal{B},\mathcal{U}). \tag{C.103}$$

It results directly from (C.91), (C.92), (C.53). This unifies the dynamics by covering all types of attractors.

This formula is by no means a definition of force $\mathbf{f}(\mathcal{B},\mathcal{U})$. The force is independently defined by the formulas (C.95), (C.96), (C.97).

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17) *Translator note*: in English: Every body perseveres in its state of being at rest or moving uniformly in a straight line, except insofar as it is compelled to change its state by exerted force.

Notes

28. Experimental mechanics and analytical mechanics of material particles. We can now clearly see that the Mechanics of material particles is divided into two interdependent but fundamentally different parts.

The first part is the experimental and theoretical determination of particle measures and basic attractor measures

$$m(\mathcal{B}), \qquad \mathbf{F}(\mathcal{U}),$$

based on the information about real isolated systems. This is the hardest part of Mechanics. It is by its nature not fully formalizable, as it concerns real objects.

Determining the mass of a particle $m(\mathcal{B})$, equal to its gravitational charge $\mathfrak{m}(\mathcal{B})$, is a simple procedure. Spring or analytical balances and the law of mass additivity are sufficient.

It is much more intricate to condense into one vector $\mathbf{F}(\mathcal{U})$ all, usually very complex, information about the circumstances of the motion of the real body. This requires an apt selection and description of an isolated system, i.e., bodies and fields that actually affect the motion of a separated body, as well as the selection and description of its features essential for interaction with this environment. This can go well beyond Mechanics. As a result, we establish the assignment $\mathcal{U} \to \mathbf{F}(\mathcal{U})$. There is usually an intermediate stage here, in which we describe $\mathcal{U}$ using scalars, vectors, tensors and other objects of pure Mathematics.

In the example "a charged body in an electromagnetic field" (see point 5), the force of the field is the Lorentz force:

$$\mathbf{F} = q\mathbf{E} + \frac{q}{c}\mathbf{v} \times \mathbf{B}.$$

For a body moving uniformly in a viscous gas (see point 7 of Lecture 11), the modulus of the gas resistance force is

$$F = \rho v^2 S \varphi(\mathrm{Re}, M, L).$$

The second part of the Mechanics of material particles make the consequences of Newton's laws. Formally, it belongs to pure Mathematics and is identified by mathematicians with the entire Mechanics of Particles. A fresh and aesthetic approach to Mechanics as part of Mathematics is the monograph [C.18].

29. Mechanics of rigid bodies and mechanics of continuous media. When we consider congruent parts of space $\mathcal{E}_{\mathsf{L}}$ as locations of the body $\mathcal{B}$ – we obtain the Mechanics of *rigid bodies*. The model of the body will become a moving rigid reference frame. Its acceleration will consist of a term describing the translational motion and a term corresponding to rotation. The laws of dynamics will have to be written for both terms. As a result, for the basic attractors $\mathcal{U}$, instead of mass, we will obtain a pair (mass $m(\mathcal{B})$, inertia tensor $\mathbf{I}(\mathcal{B})$), and instead of a force, a pair (force $\mathbf{F}(\mathcal{U})$, moment $\mathbf{M}(\mathcal{U})$).

When for the location of the body $\mathcal{B}$ we recognize tied together by continuous bijections parts of space $\mathcal{E}_{\mathsf{L}}$ – we obtain the Mechanics of Continuous Media. The place of $m(\mathcal{B})$, $\mathbf{I}(\mathcal{B})$, $\mathbf{F}(\mathcal{U})$, $\mathbf{M}(\mathcal{U})$ will take their densities.

It is possible to follow the opposite path, obtaining the Mechanics of particles and the Mechanics of rigid bodies by imposing conditions on the locations of bodies in the axiomatically built Mechanics of Continuous Media.

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