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Research Paper

Large Amplitude Forced Vibrations of Restrained Beams Resting on Elastic Point Supports

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The present paper concerns the study of geometrically non-linear forced vibrations of beams resting on two different types of springs: rotational and translational. Assuming that the motion is harmonic, the displacement is extended as a series of spatial functions determined by solving the linear problem. Hamilton's principle and spectral analysis are used to reduce the problem to a non-linear algebraic system solved using a previously developed approximate method. The effects of the nature of the added springs and their location on the non-linear behaviour of the beam are examined. A multimode approach is used in the forced case to obtain results over a wide range of vibration amplitudes. This leads to examining the non-linear forced dynamic response for different positions of each spring and different levels of excitations. Following a parametric study, the non-linear forced mode shapes and their associated bending moments are presented for different levels of excitations and for different vibration amplitudes to give an estimation of the stress distribution over the beam length.

Key words: geometrical non-linearity; forced vibrations; multimode approach; stress distribution; elastic supports.

1. Introduction

Beams are often found in structures carrying concentric elements such as attached masses, or resting on supports that may be rigid or flexible. In order to design this type of structures, it is necessary to predict the dynamic behaviour of a beam carrying at least one of the above-mentioned concentric elements.

The dynamic behaviour of beams has been the subject of numerous studies and approaches, and has attracted attention of researchers and designers for the last few decades.

A literature survey concerning the vibrations of beams with several concentric elements goes back to DOWELL [1], who investigated the dynamic behaviour of a combined system (mass and spring) using Lagrange multipliers and generalised the results stated by Rayleigh. GÜRGÖZE [2] generalised and systematised the method presented by Thomson in order to provide an approximation of the fundamental frequency and its corresponding mode for a beam with any number of masses and springs. The effect of the added masses and the springs attached and their respective positions on the vibration characteristics of the beam was examined. WANG and QIAO [3] tackled the problem of vibration of Euler-Bernoulli beams with arbitrary types of discontinuities, arbitrary located along the beam span using the Heaviside function to express the modal displacement, solved afterward using the Laplace transform. LIN [4] investigated the free and forced vibration of a multi-span beam carrying a number of various concentrated elements (point masses, rotary inertias, linear springs, rotational springs) by the use of a numerical assembly technique in order to determine the exact natural frequencies and mode shapes in the free vibration case, and the frequency-response function in the forced case. The numerical assembly method was also used by LIN [5] to determine the natural frequencies and mode shapes of a Timoshenko beam carrying different concentric elements including, point masses, rotary inertias, linear and rotational springs, and spring-mass systems. KOHAN *et al.* [6] developed a general algorithm based on the Ritz method and the Timoshenko beam theory to examine the free vibration of stepped and tapered beams resting on multiple elastic supports. The results of the algorithm developed were validated via a comparison with those obtained numerically and experimentally. WU and CHEN [7] used the digital assembly method to determine the amplitude response to the forced vibration of a beam containing arbitrary concentric elements. The analysis was re-established using the finite element method, incorporated with the Newmark method, to perform a comparison, which proved to give a good agreement of the results found via the numerical assembly method with those determined by the finite element method, which validated the accuracy of the proposed method. YESILCE [8] determined the exact natural frequencies, the corresponding mode shapes and the frequency-response curves of axially loaded Timoshenko multi-span beams with a number of various concentrated elements by the numerical assembly technique. WU and CHANG [9] developed a modified continuous-mass transfer matrix to easily obtain the exact natural frequencies and the corresponding mode shapes of an axially loaded Timoshenko beam carrying arbitrary concentrated elements. The results were validated through a comparison with those obtained using the conventional finite element method.

Based on the Euler-Bernoulli beam theory and the finite element method, the vibration of a beam with a number of concentric elements subjected to an axial force was examined by Şakar [10] to determine the effect of the added concentric elements, their positions and the axial force on the natural frequencies of the system.

Our literature survey continues with Farghaly and El-Sayed [11], who, based on Timoshenko's beam theory, analysed the free vibrations of multi-step axially loaded beams containing several attachments, and determined the natural frequencies and mode shapes. The effects of the material properties, the rotational inertia and the shear force were examined for different added centric element combinations. Furthermore, the effect of the supports position, the suspended element positions and their combinations on the system natural frequencies and modes shape was investigated. The free vibrations of thick multi-span beams were tackled by the same authors using the normalised transfer matrix method. To estimate the validity of the method, the results were compared to those obtained experimentally and those obtained by the three-dimensional finite element method. After several and different tests, the comparisons showed an excellent agreement between the proposed method and the MEF (3D) [12]. Lin and Ng used the elementary impedance method to describe the dynamic behaviour of Euler-Bernoulli beams with arbitrary steps resting on elastic supports. The frequencies calculated for many combinations were compared to those in the literature. Experimental studies were carried out to validate the effectiveness of the method, which proved to be practical in predicting the dynamic behaviour in the forced regime. The study eventually led to developing software based on the elementary impedance method and considered by the authors as a practical tool for studying the dynamics of non-uniform structures [13]. Saito *et al.* [14] studied the non-linear dynamic behaviour of a beam carrying a concentrated mass and subjected to an arbitrarily applied excitation e under the influence of gravity. The guiding equation of motion was reduced by applying the Galerkin method and using a single-mode method, and then solved by the harmonic equilibrium method. The effects of the magnitude of the concentrated mass, its position. and the amplitude of vibration on the dynamic behaviour of the beam were also examined. Neglecting the horizontal and rotary inertia forces, a numerical solution for the geometrically non-linear vibrations of multi-span beams resting on elastic supports was proposed by Lewandowski [15]. Based on a variational approach, the dynamic finite element method and iterative procedures, the non-linear eigenvalue problem has been solved, allowing the determination of the amplitude dependence frequencies and associated modes. The author presented several examples showing the effectiveness of the proposed method. It has been clearly noticed that the flexibility of the support modifies the non-linear frequency ratio significantly, but the beams still exhibit a hardening behaviour. The same author analysed the geometrically non-linear vibrations of multi-span beams subjected to harmonic forces using the finite element method. A non-linear system was established, based on Von Kármán's theory, the total Lagrangian formulation and the Ritz method, and then solved by iterative procedures [16].

Using two perturbation approaches, Pakdemirli and Nayfeh [17] studied the non-linear response of a simply supported beam carrying a spring-mass system at a primary resonance, considering the effects of stretching and damping of the mid-plane of the beam. The first approach consists of applying the multi-scale method to non-linear partial differential equations and boundary conditions, and the second averages the Lagrangian over the fast time scale and obtains the equations governing the amplitude and phase from the Lagrangian. It has been shown that the frequency-response and force-response curves depend on the mid-plane stretching and the parameters of the spring-mass system. Ghayesh *et al.* [18] have developed a general solution procedure using the multiple timescale method for the vibrations of systems with cubic nonlinearities, subjected to non-linear internal time-dependent boundary conditions.

The effect of the amplitude as well as the position of the mass-spring-damper system on the dynamic behaviour of the beam was studied by Barry *et al.* in [19]. Wielentejczyk and Lewandowski [20] developed a method for analysing geometrically non-linear steady-state vibrations of viscoelastic multi-span beams, considering the Zener rheological model. Furthermore, the von Kármán theory was used to describe the effect of geometrical nonlinearity, and the amplitude equations were derived using the finite element method and the harmonic balance method and subsequently solved by the continuation method. The response curves were illustrated for different cases, and the stability of the steady-state solution was examined. Lotfan and Sadeghi [21] analysed the non-linear vibration of a viscoelastic beam incorporating a mass-spring-damper and described by the Kelvin-Voigt model. The non-linear equations of motion were obtained by Hamilton's principle and solved afterwards by the multiple scales method. Comparison of the results with those in the literature showed the validity of the discrete method for the study of linear and non-linear vibrations of beams with different types of discontinuities. Bukhari and Barry [22] analysed the non-linear vibrations of an Euler-Bernoulli beam carrying a mass-spring system. Hamilton's principle was used to obtain the equations of motion, which were subsequently solved by the multi-scale method.

The purpose of the present paper is to investigate the non-linear free and forced vibrations of a beam resting on rotational and translational springs. Assuming a harmonic motion, the displacement is extended as a series of spatial functions determined by solving the linear problem. The mathematical model is based on the Euler-Bernoulli beams theory and von Kármán geometrical nonlinearity assumptions. The Hamilton principle and spectral analysis reduce the problem to a non-linear algebraic system solved using an approximate method developed previously in [23–26]. The effects of the added springs and their location on the non-linear dynamic behaviour of the beam are examined. Considering the forced case, a multimode approach is used to obtain results over a wide range of vibration amplitudes leading to the investigation of the non-linear forced response for different positions of each spring and different levels of excitations. Following a parametric study, the non-linear forced mode shapes and their associated bending moments are presented for different levels of excitation and different vibration amplitudes.

2. General formulation

Consider a uniform fully clamped beam subjected to transverse vibrations with the following geometrical and material characteristics, i.e., the length, width, thickness, second moment of area of cross-section, Young modulus, area of cross-section and the mass per unit length are, respectively, denoted by: L, b, h, I, E, A, ρ. The beam is subjected to a harmonic concentrated force and supported by two different types of springs, a translational spring of arbitrary position $x_{T.\text{spring}}$ of rigidity $k_{T.\text{spring}}$ and a rotational spring of position $x_{R.\text{spring}}$ and rigidity $k_{R.\text{spring}}$ (see Fig. 1).

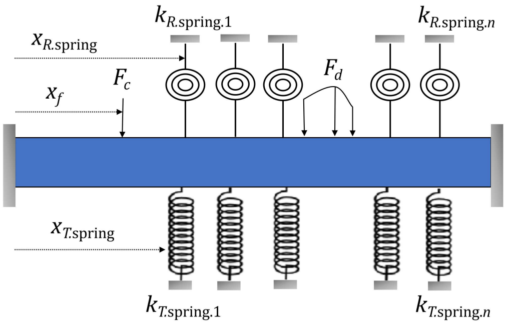

Fig. 1. Physical model of an Euler-Bernoulli beam resting on elastic point supports.

Taking into account the forcing term, the dynamic behaviour of a conservative system can be obtained by applying the Hamilton principle and can be written in the following form [27]:

$$\delta \int_0^{2\pi/\omega} (V - T + W_F)\,\mathrm{d}t = 0, \tag{2.1}$$

where W_F is the work done by the external loads.

The total strain energy V of the beam can be written as the sum of the axial deformation energy V_a due to non-linear stretching forces, the strain energy due to bending V_b, as well as the strain energy introduced by the translation and rotational spring, $V_{T.\text{spring}}$, $V_{R.\text{spring}}$:

$$V_a = \frac{EA}{8L}\left[\int\limits_0^L \left(\frac{\partial W(x,t)}{\partial x}\right)^2 \mathrm{d}x\right]^2, \tag{2.2}$$

$$V_b = \frac{EI}{2}\int\limits_0^L \left(\frac{\partial^2 W(x,t)}{\partial x^2}\right)^2 \mathrm{d}x, \tag{2.3}$$

$$V_{T.\text{spring}} = \frac{1}{2}\sum_{j=1}^{N} K_{T.\text{spring}} W^2(x_{T.\text{spring}}, t), \tag{2.4}$$

$$V_{R.\text{spring}} = \frac{1}{2}\sum_{j=1}^{N} K_{R.\text{spring}} \left(\frac{\partial W(x_{R.\text{spring}}, t)}{\partial x}\right)^2. \tag{2.5}$$

The kinetic energy T can be expressed by:

$$T = \frac{1}{2}\rho A\int\limits_0^L \left(\frac{\partial W(x,t)}{\partial t}\right)^2 \mathrm{d}x, \tag{2.6}$$

where $W(x,t)$ is the transverse displacement of the beam, $k_{T.\text{spring}}$ and $k_{R.\text{spring}}$ are respectively the rigidity of the translational and rotational springs. Assuming harmonic motion and expanding the displacement in the form of a series of functions, the transverse displacement can be expressed as in [28]:

$$W(x,t) = a_i w_i \sin(\omega t). \tag{2.7}$$

Substituting W in the expressions of energies V_a, V_b, T can be written in the form:

$$V_a = \frac{1}{2} a_i a_j a_k a_l \mathbf{b}_{ijkl} \sin^4(\omega t), \tag{2.8}$$

$$V_b = \frac{1}{2} a_i a_j \mathbf{k}_{ij} \sin^2(\omega t), \tag{2.9}$$

$$T = \frac{1}{2}\omega^2 a_i a_j \mathbf{m}_{ij} \cos^2(\omega t), \tag{2.10}$$

where $\mathbf{k}_{ij}$ denotes the classical rigidity tensor:

$$\mathbf{k}_{ij} = \int_0^L \left(\frac{\partial^2 w_i}{\partial x^2}\right)\left(\frac{\partial^2 w_j}{\partial x^2}\right) \mathrm{d}x + \sum_i k_{T.\mathrm{spring}} w_i(x_{T.S}) w_j(x_{T.S}) + \sum_i k_{R.\mathrm{spring}} \left(\frac{\partial w_i(x_{R.S})}{\partial x}\right)\left(\frac{\partial w_j(x_{R.S})}{\partial x}\right), \tag{2.11}$$

$\mathbf{b}_{ijkl}$ is the nonlinearity tensor:

$$\mathbf{b}_{ijkl} = \frac{ES}{4L} \int_0^L \left(\frac{\partial w_i}{\partial x}\right)\left(\frac{\partial w_j}{\partial x}\right) \mathrm{d}x \int_0^L \left(\frac{\partial w_k}{\partial x}\right)\left(\frac{\partial w_l}{\partial x}\right) \mathrm{d}x, \tag{2.12}$$

and $\mathbf{m}_{ij}$ is the mass tensor:

$$\mathbf{m}_{ij} = \rho S \int_0^L w_i(x)\, w_j(x)\, \mathrm{d}x. \tag{2.13}$$

Now, an Euler-Bernoulli beam subjected to a force $F(x,t)$ over the range S is considered. The force excites the modes of the beam via a set of generalised forces F_i depending on the expression of F, the excitation point for concentrated excitation, the excitation length for distributed forces, and the mode considered. The generalised forces $F_i(t)$ are given by:

$$F_i(t) = \int_S F(x,t) w_i(x)\, \mathrm{d}x. \tag{2.14}$$

The geometrically non-linear transverse vibrations are examined for a beam subjected to two types of excitations. The first concerns a centred point force F_c applied to the point x_f, and the second- a distributed harmonic force F_d. The excitations are defined by:

$$F_d(x,t) = f_d \sin(\omega t), \tag{2.15}$$

$$F_c(x,t) = f_c \sin(\omega t) \delta(x - x_f), \tag{2.16}$$

where δ is the Dirac function, $F_{d,\,i}(t)$ and $F_{c,\,i}(t)$ are the corresponding generalized forces which can be expressed as:

$$F_{d,\,i}(t) = F_d(t) \sin(\omega t) \int_0^L w_i(x)\, \mathrm{d}x = f_{d,\,i} \sin(\omega t), \tag{2.17}$$

$$F_{c,\,i}(t) = F_c \sin(\omega t) w_i(x_f) = f_{c,\,i} \sin(\omega t). \tag{2.18}$$

After calculations, the following non-linear system is obtained:

$$[\mathbf{K}]\{\mathbf{A}\} + \frac{3}{2}\left[\mathbf{B}\left(\{\mathbf{A}\}\right)\right]\{\mathbf{A}\} - \omega^2\left[\mathbf{M}\right]\{\mathbf{A}\} = \{\mathbf{F}\}, \tag{2.19}$$

where $[\mathbf{K}]$ and $[\mathbf{M}]$ are respectively the linear rigidity and mass matrices, $[\mathbf{B}(\{\mathbf{A}\})]$ defines the non-linear geometrical stiffness matrix, $\{\mathbf{F}\}$ is the column matrix of generalised forces, and $\{\mathbf{A}\}$ is the column vector of the basic function contribution coefficients. To obtain non-dimensional parameters, one formulates:

$$\begin{gathered}\frac{w_i(x)}{w_i^*} = h\left(x^*\right), \qquad \frac{\mathbf{m}_{ij}}{\mathbf{m}_{ij}^*} = \rho A h^2 L,\\ \frac{\mathbf{k}_{ij}}{\mathbf{k}_{ij}^*} = \frac{EIh^2}{L^3}, \qquad \frac{\mathbf{b}_{ijkl}}{\mathbf{b}_{ijkl}^*} = \frac{EIh^2}{L^3}, \qquad \frac{\omega^2}{\left(\omega^*\right)^2} = \frac{EI}{\rho A L^4},\\ \frac{k_{T.\mathrm{spring}}}{k_{T.\mathrm{spring}}^*} = \frac{EI}{L^3}, \qquad \frac{k_{R.\mathrm{spring}}}{k_{R.\mathrm{spring}}^*} = \frac{EI}{L}.\end{gathered} \tag{2.20}$$

The dimensionless generalised forces f_i^{*d} and f_i^{*c} can be expressed as:

$$f_i^{*d} = F^d \frac{L^4}{EIh} \int\limits_0^1 w_i^*(x^*)\,\mathrm{d}x, \tag{2.21}$$

$$f_i^{*c} = F^c \frac{L^3}{EIh} w_i^*(x_f). \tag{2.22}$$

After substituting these notations in Eq. (2.19), one obtains the following non-linear algebraic equation:

$$\left[\mathbf{K}^*\right]\{\mathbf{A}\} + \frac{3}{2}\left[\mathbf{B}^*\left(\{\mathbf{A}\}\right)\right]\{\mathbf{A}\} - \left(\omega^*\right)^2\left[\mathbf{M}^*\right]\{\mathbf{A}\} = \{\mathbf{F}^*\}, \tag{2.23}$$

which may be written in the following tensor form:

$$a_i\mathbf{k}_{ir}^* - \left(\omega^*\right)^2 a_i\mathbf{m}_{ir}^* + \frac{3}{2}a_i a_j a_k \mathbf{b}_{ijkr}^* = \mathbf{F}_r^*, \qquad r = 1, ..., n, \tag{2.24}$$

ω^{*2} is given as in [29] by:

$$\left(\omega^*\right)^2 = \frac{\{\mathbf{A}\}^{\mathrm{T}}[\mathbf{K}]\{\mathbf{A}\} + k\{\mathbf{A}\}^{\mathrm{T}}[\mathbf{B}](\{\mathbf{A}\})\{\mathbf{A}\}}{\{\mathbf{A}\}^{T}[\mathbf{M}]\{\mathbf{A}\}} \tag{2.25}$$

with $k = 3/2$.

It has previously been shown that the contribution of one mode remains predominant relatively to the others for the range of amplitudes considered. To indicate that the contributions of the other modes remain small, they are denoted by ε_i, while the predominant mode is denoted by a_i.

According to [30], by separating in the non-linear expression $a_i a_j a_k \mathbf{b}^*_{ijkr}$ terms proportional to a_1^3, terms proportional to $a_1^2\varepsilon_i$, and by neglecting the terms proportional to $a_1\varepsilon_i\varepsilon_j$ and terms proportional to $\varepsilon_i\varepsilon_j\varepsilon_k$ one may write:

$$a_i a_j a_k \mathbf{b}_{ijkr} = a_1^3\mathbf{b}_{111r} + a_1^2\varepsilon_i\mathbf{b}_{11ir}, \qquad r = 1, ..., n. \tag{2.26}$$

Equation (2.23) can then be expressed in the vicinity of the r^{th}mode as follows:

$$\left([\mathbf{K}_r^*]_R - (\omega^*)^2\,[\mathbf{M}_r^*]_R\right)\{\mathbf{A}_r\}_R + \frac{3}{2}\,[\alpha_r^*]\,\{\mathbf{A}_r\}_R = \left\{\mathbf{F}_r - \frac{3}{2}a_r^3 b_{irrr}\right\} \tag{2.27}$$

with $[\alpha_r^*]_R = \left[(a_r^*)^2\ b^*_{ijrr}\right]_R$.

Equation (2.27) is an approximate linear system easy to solve in order to get the contribution coefficients to the non-linear beam forced response.

3. Results and discussion

3.1. Case of a clamped-clamped beam resting on translational support

To verify the validity of the present approach for the analysis of the nonlinear vibrations of a beam resting on a translational spring, a comparison was made with the results obtained using the variational approach, the dynamic finite element and iterative procedure. It is clearly shown in Fig. 2 that the results of the present method have a very good agreement with those obtained in [15] since the average of relative difference does not exceed 2.77% for the

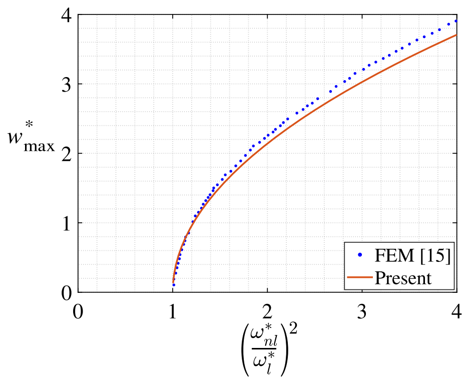

Fig. 2. Comparison of the frequency ratios of a beam resting on translational spring near the second mode.

same maximum non-dimensional amplitude $w^*_{\max} = 2$ and a spring attached to the beam centre with rigidity $k^*_{T.\text{spring}} = 240$.

The preliminary step to study the forced case is to determine the predominant mode corresponding to the excitation. In this sense, the generalised forces have been calculated and are presented in Table 1 for a beam subjected to a uniformly distributed force.

Table 1. Percentage of generalised forces exciting the first five symmetric modes of a clamped-clamped beam resting on translational spring.

	Modes	1	3	5	7	9
$F_d = 100$	$\int w_i^*(x)\,\mathrm{d}x$	83.60	35.11	23.27	16.92	13.41
	$\int w_i^*(x)\,\mathrm{d}x \Big/ \sum_{i=1}^{n} \left\lvert \int w_i^*(x)\,\mathrm{d}x \right\rvert$	48.51%	20.37%	13.51%	9.82%	7.79%
$F_d = 500$	$\int w_i^*(x)\,\mathrm{d}x$	418.01	175.56	116.38	84.61	67.09
	$\int w_i^*(x)\,\mathrm{d}x \Big/ \sum_{i=1}^{n} \left\lvert \int w_i^*(x)\,\mathrm{d}x \right\rvert$	48.51%	20.37%	13.51%	9.82%	7.79%
$F_d = 1000$	$\int w_i^*(x)\,\mathrm{d}x$	836.02	351.12	232.76	169.23	134.18
	$\int w_i^*(x)\,\mathrm{d}x \Big/ \sum_{i=1}^{n} \left\lvert \int w_i^*(x)\,\mathrm{d}x \right\rvert$	48.51%	20.37%	13.51%	9.82%	7.79%

Table 1 shows that the first mode remains predominant for the excitation levels considered relative to the others. Consequently, the study will focus on resolving the system near the first mode as described in the formulation.

The non-linear forced bending moments are illustrated in Fig. 3 for different excitation levels of a force uniformly distributed over the beam length $F_d = 0$,

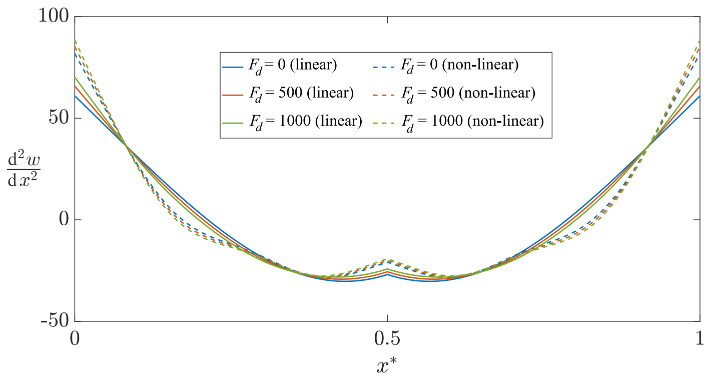

Fig. 3. Curvatures corresponding to the non-linear deflection response function for various excitation levels.

$F_d = 500$, $F_d = 1000$, the same maximum non-dimensional amplitude $w^*_{\max} = 2$, and a translational spring with rigidity $k^*_{T.\text{spring}} = 100$. It can be noticed that the distributed force increases the stress near the clamps but decreases at the location of the attached spring. Table 2 shows the effect of force variation on the correction percentages introduced by this non-linear theory relative to the linear theory. It should be noted that by increasing the force, the correction percentages tend to decrease progressively either in the middle of the beam or in the vicinity of the clamps. The non-linear forced bending moments are respectively plotted in Fig. 4 for different values of the maximum non-dimensional amplitude $w^*_{\max} = 0.8$, $w^*_{\max} = 1$, $w^*_{\max} = 1.5$, $w^*_{\max} = 2$, the same spring rigidity $k^*_{T.\text{spring}} = 100$, and an excitation level of a uniformly distributed force $F_d = 100$. It is clearly shown that the amplitude of vibration increases the stress significantly, both at the spring location and near the clamps. It is clearly noticeable in Table 3 that the increase in vibration amplitude considerably increases the percentage correction provided by the non-linear theory and remains more pronounced near the clamps. Figure 5 shows the non-linear response curves obtained by the multimode approach for three levels of excitation of a force uniformly distributed over the beam length. It is clear that the forced response curves show the hardening type behaviour due to cubic non-linearity. The jump phenomenon can also be observed since, for

Table 2. Effect of force variation on the percentage correction introduced by the non-linear theory compared to the linear theory.

	percentage correction $= \dfrac{\text{absolute change (linear and non-linear model)}}{\text{reference value (linear model)}}$		
Force	0	500	1000
Clamps	25.44	22.83	20.63
Middle of the beam	28.83	26.89	26.29

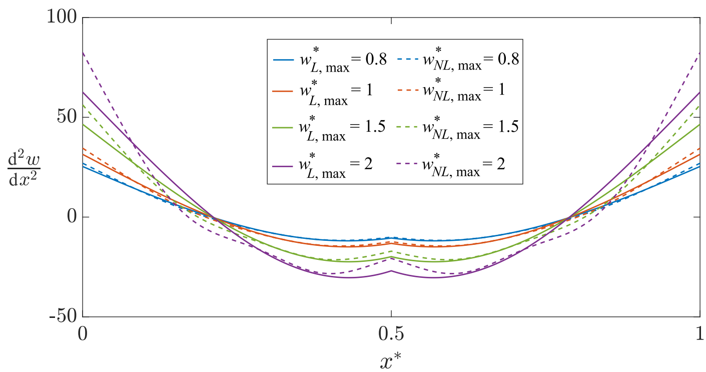

FIG. 4. Curvatures corresponding to the non-linear deflection response function for various amplitudes.

Table 3. Effect of amplitudes variation on the percentage correction introduced by the non-linear theory compared to the linear theory.

	percentage correction $= \dfrac{\text{absolute change (linear and non-linear model)}}{\text{reference value (linear model)}}$			
Amplitudes	0.8	1	1.5	2
Clamps	6.04	8.94	17.3	24.07
Middle of the beam	4.68	7.33	15.96	29.4

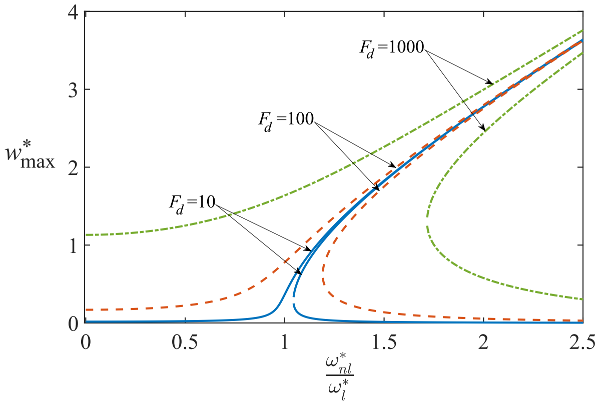

Fig. 5. The response curves based on the multimode approach of a clamped-clamped beam resting on translational spring for various excitation levels.

a given frequency range, three solutions exist for a single frequency. In Fig. 6, the frequency response curves are plotted for different positions of the translational spring of stiffness $k^*_{T.\text{spring}} = 100$ and a uniform beam subjected to a distributed force $F_d = 500$. It is clear that the closer the spring gets to the clamps, the more significantly the hardening effect increases.

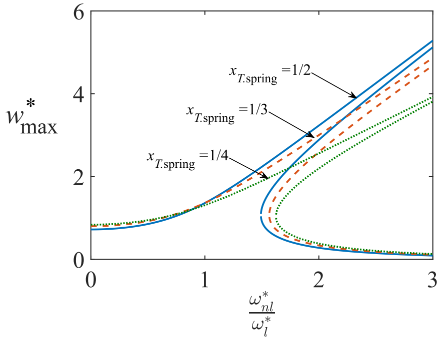

Fig. 6. The response curves based on the multimode approach of a clamped-clamped beam resting on translational spring for various spring positions.

3.2. *Case of a clamped-clamped beam resting on rotational support*

As described before, the forced analysis requires the determination of the predominant mode, so Table 4 shows the calculation of the generalised forces for each excitation level.

Table 4. Percentage of generalised forces exciting the first five symmetric modes of a clamped-clamped beam resting on rotational support.

	Modes	1	3	5	7	9
$F_d = 100$	$\int w_i^*(x)\,\mathrm{d}x$	83.08	36.37	23.14	16.97	13.40
	$\int w_i^*(x)\,\mathrm{d}x \Big/ \sum_{i=1}^{n} \left\lvert \int w_i^*(x)\,\mathrm{d}x \right\rvert$	48.03%	21.03%	13.38%	9.81%	7.75%
$F_d = 500$	$\int w_i^*(x)\,\mathrm{d}x$	415.43	181.88	115.74	84.88	67.00
	$\int w_i^*(x)\,\mathrm{d}x \Big/ \sum_{i=1}^{n} \left\lvert \int w_i^*(x)\,\mathrm{d}x \right\rvert$	48.03%	21.03%	13.38%	9.81%	7.75%
$F_d = 1000$	$\int w_i^*(x)\,\mathrm{d}x$	830.86	363.76	231.49	169.76	134.01
	$\int w_i^*(x)\,\mathrm{d}x \Big/ \sum_{i=1}^{n} \left\lvert \int w_i^*(x)\,\mathrm{d}x \right\rvert$	48.03%	21.03%	13.38%	9.81%	7.75%

From Table 4, it can be concluded that for a beam resting on a rotational spring and subjected to a distributed force, the first mode remains predominant with respect to the modes considered. The same parametric study of the translational spring is conducted for a beam subjected to a uniformly distributed force and resting on a rotational spring. The non-linear forced curvatures are presented in Fig. 7 for different excitation levels of a force uniformly distributed over the

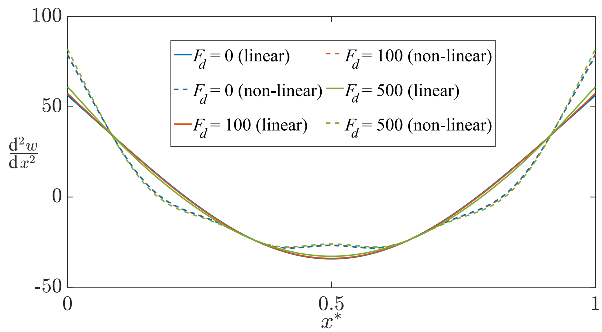

FIG. 7. Curvatures corresponding to the non-linear deflection response function for various excitation levels.

beam length $F_d = 0$, $F_d = 100$, $F_d = 500$, the same maximum non-dimensional amplitude $w^*_{\max} = 2$, and the same rotational spring rigidity $k^*_{R.\text{spring}} = 100$. The figure shows a progressive increase in the stress distribution at the beam due to the increase in the levels of the applied force. The non-linear forced curvatures are illustrated in Fig. 8 for different maximum non-dimensional amplitudes $w^*_{\max} = 0.8$, $w^*_{\max} = 1$, $w^*_{\max} = 1.5$, $w^*_{\max} = 2$, the same rotational spring rigidity $k^*_{R.\text{spring}} = 100$, and uniformly distributed force $F_d = 100$. It seems clear that the increase in vibration amplitude considerably increases the stress distribution but is more pronounced at the clamps than at the spring location. Table 5 shows that for a higher level of the distributed force, the correction percentage of the non-linear theory compared to the linear theory becomes smaller, while the increase in amplitude increases this percentage considerably as shown in Table 6.

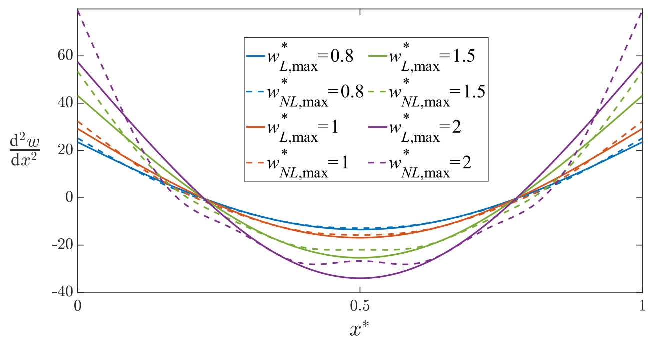

FIG. 8. Curvatures corresponding to the non-linear deflection response function for various amplitudes.

Table 5. Effect of force variation on the percentage correction introduced by the non-linear theory compared to the linear theory.

	percentage correction $= \frac{\text{absolute change (linear and non-linear model)}}{\text{reference value (linear model)}}$		
Force	0	100	500
Clamps	28.27	27.50	25.02
Middle of the beam	27.58	27.28	26.21

Table 6. Effect of amplitude variation on the percentage correction introduced by the non-linear theory compared to the linear theory.

	percentage correction $= \frac{\text{absolute change (linear and non-linear model)}}{\text{reference value (linear model)}}$			
Amplitudes	0.8	1	1.5	2
Clamps	6.62	9.73	19.14	27.5
Middle of the beam	4.59	7.24	15.42	27.28

For example, for a beam supported by a spring with the same characteristics, the percentage increases from 6.62% for a maximum non-dimensional of 0.8 to 27.5% for $w^*_{\max} = 2$. The non-linear response curves obtained by the multimode approach are shown in Fig. 9 for three levels of excitation with force uniformly distributed over the length of the beam. The hardening effect appears clearly on the forced response curves and increases considerably with the increase in the level of the applied force. The frequency response curves are plotted in Fig. 10 for different positions of the rotational spring of stiffness $k^*_{R.\text{spring}} = 100$ for a uniform beam subjected to a uniformly distributed force $F_d = 500$. It should be noted that the closer the rotational spring gets to the clamps, the more considerably the hardening effect decreases.

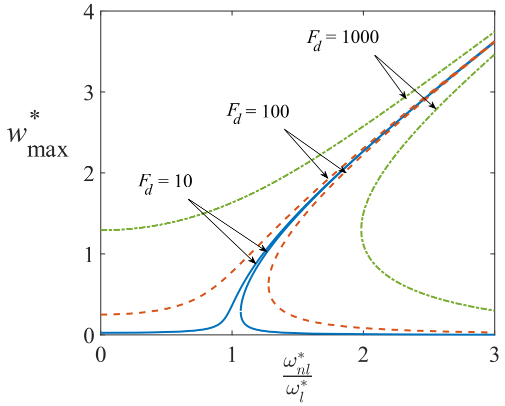

Fig. 9. The response curves based on the multimode approach of a clamped-clamped beam resting on rotational spring for various excitation levels.

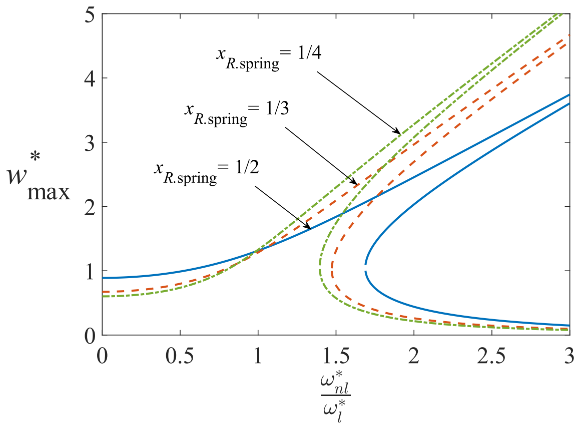

Fig. 10. The response curves based on the multimode approach of a clamped-clamped beam resting on rotational spring for various spring positions.

4. Conclusion

The free and forced non-linear vibrations were investigated for a clamped-clamped beam with an internal point support described by translational or rotational rigidity. Hamilton's principle and spectral analysis were used to reduce the problem to a non-linear algebraic system solved by the approximate method, called the second formulation. Firstly, the results obtained by the presented method were compared with those obtained by the finite element method, and showed a very good agreement since the relative difference did not exceed 2.77% for the maximum non-dimensional amplitude $w^*_{\text{max}} = 2$. To investigate the forced case, a multimode approach was developed to examine the non-linear behaviour of a beam subjected to a uniformly distributed force, resting on translational and rotational springs. In the case of a translational spring, it was observed that as the spring got closer to the clamps, the stronger the hardening effect became, while it decreased considerably in the case of a rotational spring.

Furthermore, the effects of the added springs and their location on the forced non-linear behaviour were studied. The non-linear bending moments were illustrated for different levels of excitations and for different vibration amplitudes, showing the stress distribution over the beam. The force and vibration amplitude effect on the percentage correction introduced by the non-linear theory compared to the linear theory was calculated for different case studies. For a beam supported by a translational spring, the percentage of correction reached up to 29% for the maximum non-dimensional amplitude $w^*_{\text{max}} = 2$, and 27% for the beam supported by a rotational spring. This result, among others previously presented, shows that the results obtained by the linear theory are likely to lead to significant calculation errors in the case of large amplitude vibrations where deformations should not be neglected.

References

1. Dowell E.H., On some general properties of combined dynamical systems, *Journal of Applied Mechanics*, **46**(1): 206–209, 1979, doi: 10.1115/1.3424499.
2. Gürgöze M., A note on the vibrations of restrained beams and rods with point masses, *Journal of Sound and Vibration*, **96**(4): 461–468, 1984, doi: 10.1016/0022-460X(84)90633-3.
3. Wang J., Qiao P., Vibration of beams with arbitrary discontinuities and boundary conditions, *Journal of Sound and Vibration*, **308**(1–2): 12–27, 2007, doi: 10.1016/j.jsv.2007.06.071.
4. Lin H.-Y., Dynamic analysis of a multi-span uniform beam carrying a number of various concentrated elements, *Journal of Sound and Vibration*, **309**(1–2): 262–275, 2008, doi: 10.1016/J.JSV.2007.07.015.
5. Lin H.-Y., On the natural frequencies and mode shapes of a multispan Timoshenko beam carrying a number of various concentrated elements, *Journal of Sound and Vibration*, **319**(1–2): 593–605, 2009, doi: 10.1016/j.jsv.2008.05.022.

6. Kohan P.H., Nallim L.G., Gea S.B., Dynamic characterization of beam type structures: Analytical, numerical and experimental applications, *Applied Acoustics*, **72**(12): 975–981, 2011, doi: 10.1016/j.apacoust.2011.06.007.

7. Wu J.S., Chen J.H., An efficient approach for determining forced vibration response amplitudes of a MDOF system with various attachments, *Shock and Vibration*, **19**(1): 57–79, 2012, doi: 10.3233/SAV-2012-0616.

8. Yesilce Y., Free and forced vibrations of an axially-loaded Timoshenko multi-span beam carrying a number of various concentrated elements, *Shock and Vibration*, **19**(4): 735–752, 2012, doi: 10.3233/SAV-2012-0665.

9. Wu J.-S., Chang B.-H., Free vibration of axial-loaded multi-step Timoshenko beam carrying arbitrary concentrated elements using continuous-mass transfer matrix method, *European Journal of Mechanics – A/Solids*, **38**: 20–37, 2013, doi: 10.1016/j.euromechsol. 2012.08.003.

10. Şakar G., The effect of axial force on the free vibration of an Euler-Bernoulli beam carrying a number of various concentrated elements, *Shock and Vibration*, **20**(3): 357–367, 2013, doi: 10.3233/SAV-120750.

11. Farghaly S.H., El-Sayed T.A., Exact free vibration of multi-step Timoshenko beam system with several attachments, *Mechanical Systems and Signal Processing*, **72–73**: 525–546, 2016, doi: 10.1016/j.ymssp.2015.11.025.

12. El-Sayed T.A., Farghaly S.H., A normalized transfer matrix method for the free vibration of stepped beams : comparison with experimental and FE (3D) methods, *Shock and Vibration*, **2017**: Article ID 8186976, 2017, doi: 10.1155/2017/8186976.

13. Lin R.M., Ng T.Y., Exact vibration modes of multiple-stepped beams with arbitrary steps and supports using elemental impedance method, *Engineering Structures*, **152**: 24–34, 2017, doi: 10.1016/j.engstruct.2017.07.095.

14. Saito H., Sato K., Yutani T., Non-linear forced vibrations of a beam carrying concentrated mass under gravity, *Journal of Sound and Vibration*, **46**(4): 515–525, 1976, doi: 10.1016/0022-460X(76)90677-5.

15. Lewandowski R., Nonlinear free vibrations of multispan beams on elastic supports, *Computers & Structures*, **32**(2): 305–312, 1989, doi: 10.1016/0045-7949(89)90042-4.

16. Lewandowski R., Non-linear, steady-state analysis of multispan beams by the finite element method, *Computers & Structures*, **39**(l–2): 83–93, 1991, doi: 10.1016/0045-7949(91)90075-W.

17. Pakdemirli M., Nayfeh A.H., Nonlinear vibrations of a beam- spring-mass system, *Journal of Vibration and Acoustics*, **116**(4): 433–439, 1994, doi: 10.1115/1.2930446.

18. Ghayesh M.H., Kazemirad S., Darabi M.A., A general solution procedure for vibrations of systems with cubic nonlinearities and nonlinear/time-dependent internal boundary conditions, *Journal of Sound and Vibration*, **330**(22): 5382–5400, 2011, doi: 10.1016/j.jsv.2011.06.001.

19. Barry O.R., Oguamanam D.C.D., Zu J.W., Nonlinear vibration of an axially loaded beam carrying multiple mass-spring-damper systems, *Nonlinear Dynamics*, **77**(4): 1597–1608, 2014, doi: 10.1007/s11071-014-1402-5.

20. Wielentejczyk P., Lewandowski R., Geometrically nonlinear, steady state vibration of viscoelastic beams, *International Journal of Non-Linear Mechanics*, **89**: 177–186, 2016, doi: 10.1016/j.ijnonlinmec.2016.12.012.

21. Lotfan S., Sadeghi M.H., Large amplitude free vibration of a viscoelastic beam carrying a lumped mass–spring–damper, *Nonlinear Dynamics*, **90**(2): 1053–1075, 2017, doi: 10.1007/s11071-017-3710-z.
22. Bukhari M.A., Barry O.R., Nonlinear vibrations of a beam- spring-large mass system, *Proceedings of the ASME 2017 International Mechanical Engineering Congress and Exposition. Volume 4B: Dynamics, Vibration, and Control*, Tampa, Florida, USA, November 3–9, 2017, Art. ID V04BT05A061, 2017, doi: 10.1115/IMECE2017-70444.
23. Fakhreddine H., Adri A., Rifai S., Benamar R., A multimode approach to geometrically non-linear forced vibrations of Euler-Bernoulli multispan beams, *Journal of Vibration Engineering & Technologies*, **8**(2): 319–326, 2020, doi: 10.1007/s42417-019-00139-8.
24. Fakhreddine H., Adri A., Chajdi M., Rifai S., Benamar R., A multimode approach to geometrically non-linear forced vibration of beams carrying point masses, *Diagnostyka*, **21**(4): 23–33, 2020, doi: 10.29354/diag/128603.
25. Chajdi M., Ahmed A., El Bikri K., Benamar R., Analysis of the associated stress distributions to the nonlinear forced vibrations of functionally graded multi-cracked beams, *Diagnostyka*, **22**(1): 101–112, 2021, doi: 10.29354/diag/133702.
26. El Hantati I., Adri A., Fakhreddine H., Rifai S., Benamar R., A multimode approach to geometrically nonlinear free and forced vibrations of multistepped beams, *Shock and Vibration*, **2021**: 6697344, 2021, doi: 10.1155/2021/6697344.
27. Géradin M., Rixen D.J., *Mechanical Vibrations: Theory and Application to Structural Dynamics*, 3rd ed., John Wiley & Sons, 2015.
28. Rahmouni A., Beidouri Z., Benamar R., A discrete model for geometrically nonlinear transverse free constrained vibrations of beams carrying a concentrated mass at various locations, *Proceedings of the 9th International Conference on Structural Dynamics, EURODYN 2014, Porto, Portugal, 30 June – 2 July 2014*, A. Cunha, E. Caetano, P. Ribeiro, G. Müller [Eds], pp. 2093–2099, 2014.
29. El Bikri K., Benamar R., Bennouna M.M., Geometrically non-linear free vibrations of clamped-clamped beams with an edge crack, *Computers & Structures*, **84**(7): 485–502, 2006, doi: 10.1016/j.compstruc.2005.09.030.
30. El Kadiri M., Benamar R, White R.G., Improvement of the semi-analytical method, for determining the geometrically non-linear response of thin straight structures. Part I: Application to clamped-clamped and simply supported-clamped beams, *Journal of Sound and Vibration*, **249**(2): 263–305, 2002, doi: 10.1006/jsvi.2001.3808.

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