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Research Paper

Analytical Investigation of a Beam on Elastic Foundation with Nonsymmetrical Properties

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The subject of presented analytical and numerical investigation is the stability of an axially compressed beam on an elastic foundation. The shape function of the foundation was assumed. The formula was supplemented with the offset parameter. The critical values of loads were calculated and presented as a function of geometric and mechanical properties of the beam and nonsymmetrical properties of the elastic foundation. The highest values of critical loads can be obtained for the highest values of shape parameter and the lowest values of amplitudes of shape function. The values of critical loads increase with the increase of the value of the offset parameter.

Key words: homogeneous beam; elastic foundation; critical load; nonsymmetrical properties.

1. Introduction

The foundation-structure interaction is an essential obstacle in the design of a variety of constructions, e.g., buildings (reinforced concrete beams resting on elastic foundation), railroads [1] (underlays), airports, highways, sports fields, parking lots, storage capacities, as well as dams and embankments. Furthermore, they have also found applications in geotechnical engineering (improved subgrade or roadbase performance), marine engineering, bio-mechanics, harbor works, buried gas pipeline systems along with constructions of machine foundations. The researchers describe diverse analytical models of beams resting on elastic foundation. A plain mechanical representation of elastic foundation was first presented by Winkler. A number of considerations in the area of foundation-structure interaction have been conducted based on the Winkler hypothesis due to its simplicity and clarity.

Buckling is one of the most disadvantageous types of loss of construction stability. The study of buckling of beams resting on elastic foundation has been presented by several researchers. Mode-jumping instabilities in the post-buckling of a beam on a partial nonlinear foundation was described by Zhang and Murphy [2]. Mode jumping is an instability phenomenon in the post-buckling region, which causes a sudden change in the equilibrium configuration [2]. The analysis acknowledged that certain asymmetric partial foundation configurations could facilitate the smooth transition rather than mode jumping. Mode jumping can be averted by configuring the asymmetric partial foundation. The stability of Euler-Bernoulli beam-columns resting on a two-parameter elastic foundation was investigated by Palacio-Betancur and Aristizabal-Ochoa [3]. The influence of the elastic foundation with alternating properties was discussed. The obtained results were compared with those available in the literature. Buckling of beam-columns on two-parameter elastic foundations was studied by Matsunaga [4]. The outcome of shear deformation, depth change, and rotatory inertia were performed. Buckling stresses of a beam-column with a rectangular cross-section and subjected to axial stresses were obtained. Thermal buckling and post-buckling of a homogeneous Euler-Bernoulli beam resting on a nonlinear elastic foundation were presented by Li and Batra [5]. The influence of a temperature rise on buckling was determined analytically with the use of the linear problem analysis. The post-buckling survey of beams was prepared and analyzed using the shooting method. The analysis revealed that the nonlinear foundation stiffness parameter does not affect the buckling temperature. In addition, it does not affect the post-buckling deformations. Buckling investigation of nonuniform beams on a partial variable elastic foundation was introduced by Zhang *et al.* [6]. The Hencky bar-chain model was used for calculations. The model allowed to obtain results by solving algebraic equations instead of a differential equation. The formulae of the stiffness of the internal spring, end spring, and the stiffness of the elastic foundation were obtained. The impact of the length and stiffness of the foundation on the buckling load was also taken into the consideration. The analysis confirmed that the length parameter of the elastic foundation has a momentous influence on the mode shapes. Buckling analysis of a double-functionally graded Timoshenko beam system on a Winkler-Pasternak elastic foundation was carried out by Deng *et al.* [7]. Two beams were connected by the elastic component. The elastic foundation was presented in the form of two layers: the Winkler layer and a shearing layer. The influence of gradient parameter, foundation parameters, axial load, and connecting stiffness on the buckling load were considered. It can be concluded from the analysis that the buckling load decreases with the increase of the gradient parameter. In addition, for different foundation parameters, the outcome of the stiffness of the shear layer on buckling load was similar to the outcome of the Winkler layer stiffness. The evaluation of buckling of nonuniform and axially functionally graded Timoshenko nanobeams on a Winkler-Pasternak foundation were formulated by Robinson and Adali [8]. It was discovered that the buckling load decreases with the increase of a nonlocal parameter and also depends on the boundary conditions and the elastic foundation. Moreover, the influence of Winkler and Pasternak foundations on the values of buckling loads was disparate for nonlocal beams. Buckling analysis of an axially loaded elastic beam on a linearly elastic nonlocal foundation (Reissner foundation) was conducted by Challamel *et al.* [9]. Analytical and numerical results were obtained. The higher-order boundary conditions did not have a considerable influence on the buckling results. The influence of imperfection modes on the buckling and post-buckling of a beam on a nonlinear foundation were described by Eltaher *et al.* [10]. The beam was resting on a nonlinear elastic foundation and was subjected to an axial load and harmonically distributed force. The values of critical buckling loads were obtained. It was observed in the analysis that the beams with cosine type of geometric imperfections have greater values of buckling loads than beams with sine type of imperfections. Additionally, the values of the elastic foundation parameters (nonlinear) did not influence the buckling loads.

Beams on elastic foundations were also investigated by Borák and Marcián [11]. Betti's theorem was used for clarifying the problem. The reaction forces, produced by the deflection of the beam, were assumed to be continuously distributed in the supporting medium. The fundamental formulae for the beam on elastic foundation were introduced [11]. Analytical approach for the closed form solution of continuous beams on two-parameter elastic foundations was studied by Aslami and Akimov [12]. The general form of the governing equation was reduced to an arrangement of first-order differential equations with constant coefficients. Different boundary conditions were considered. The bending of beams on a three-parameter elastic foundation was discussed by Avramidis and Morfidis [13]. Parametric evaluation of elastically supported beams of infinite and finite length was carried out, and correlations were made between one, two or three-parameter foundation models and more accurate 2D finite element models. The solutions disclosed the advantages of the Kerr-type foundation model compared to one or two-parameter models [13]. Beams on a three-parameter elastic foundation were also presented by Morfidis [14]. The mathematical analogy of a beam on elastic supports as a beam on an elastic foundation was determined by Sato *et al.* [15]. The authors proposed the hypothesis that a beam on equidistant elastic supports can be considered as a beam on elastic foundation in static and free vibration problems. In order to verify these considerations, the deflections and bending moments of the beams, when the concentrated load was acting at the center of these beams, as well as natural frequencies and modal shapes were compared with each other. No significant distinctions were observed. The mesh-free method, called the radial point interpolation method, was carried out by Binesh [16] to analyze a beam on a two-parameter elastic foundation. The beam and the foundation were designed separately. Karamanlidis and Prakash [17] as well as Morfidis and Avramidis [18] performed finite element solutions for beams resting on an elastic foundation. The analysis of propagation of a flexural wave in the periodic beam on elastic foundations was conducted by Yu *et al.* [19]. The number of waves and their characteristic were discussed. Parametric instability of an inextensional beam on an elastic foundation was described by Wang *et al.* [20]. The beam was also assumed to be subjected to the uniformly distributed harmonic excitation.

Various foundation models were analyzed. Taking into account the parametric instability, the parameters of diverse foundations have an influence on modes of the nonlinear response of the beam. The problem of beams resting on the elastic foundation was studied by several researchers. They assumed that the physical model of the foundation is presented as a great number of small springs. In this case, the results of calculations are dependent on the characteristics of springs. Models of structures, available in the literature, describe the properties of elastic foundations as constant parameters. The authors introduced a generalization of common models of elastic foundation by adopting its variable properties.

This work deals with the analysis and the calculations of the values of critical loads of homogeneous beam with variable properties of the foundation. An analytical model is studied. The beam with a symmetrical shape function was presented by the authors in [21]. The presented research refers to the nonsymmetrical shape function that designates the shape of the elastic foundation. The formula is supplemented with the offset parameter relative to the right end of the beam. The value of the offset parameter for the symmetrical function is equal to $p = 0$. In addition, the numerical analysis is prepared and performed. Sample analytical and numerical calculations are carried out, showing a good agreement between the results obtained for both models. The scheme of considered beam, geometry, and load is shown in Fig. 1. The beam is subjected to a compressive axial force F_0. The work describes an original approach to the problem of beams on elastic foundations. The original shape function, which represents the characteristics of the elastic foundation, is presented. In addition, the original function of deflection is proposed.

2. Analytical model of the beam

The model is derived with linear mechanical properties of the homogeneous beam. The scheme of the beam is shown in Fig. 1.

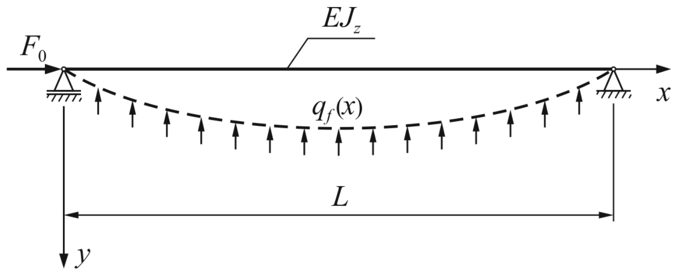

FIG. 1. The scheme of the homogeneous beam.

The differential equation of the beam is as follows:

$$EJ_z \frac{\mathrm{d}^4 v}{\mathrm{d}x^4} + F_0 \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = -q_f(x), \tag{2.1}$$

where $q_f(x)$ is the intensity of load – reaction of the elastic foundation [N/mm] and $q_f = c(x) \cdot v(x)$, where $c(x)$ is the property – foundation constant (a mathematical function describing the shape of the elastic foundation) [N/mm^2], and $v(x)$ is the deflection of the beam [mm].

Therefore, the differential Eq. (2.1) can be written in the following form:

$$EJ_z \frac{\mathrm{d}^4 v}{\mathrm{d}x^4} + F_0 \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + c(x) \cdot v(x) = 0. \tag{2.2}$$

The shape function, which represents variable properties of the elastic foundation, is assumed as:

$$c(x) = c_0 - c_1 \sin^k(\pi\xi), \tag{2.3}$$

where $\xi = \frac{x}{L}$ (L – length of the beam, ξ – dimensionless length of the beam), $0 \leq \xi \leq 1$, and k is a natural number.

Initially, it was assumed that the shape of the function (2.3) is symmetrical for both ends of the beam [21]. Figure 2 presents the function for variable values of parameter k. The figure demonstrates that parameter k is the essential factor that affects the shape of the function (2.3).

The shape function is a variable in this model. The function (2.3) has been supplemented with the parameter p – the offset parameter – relative to one end of the beam (Fig. 3). The offset of the function was assumed relative to the right end of the beam (nonsymmetrical properties of the elastic foundation). The new shape function is in the following form:

$$c(x) = c_0 - c_1 \sin^k [\pi(\xi - p)]. \tag{2.4}$$

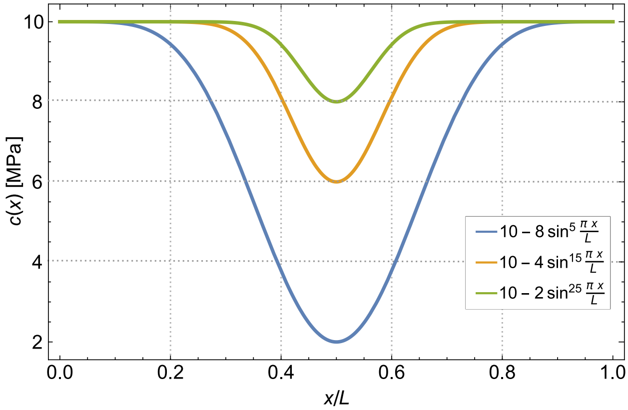

FIG. 2. The shape of the function (2.3) for variable values of parameter k (symmetrical properties of the foundation).

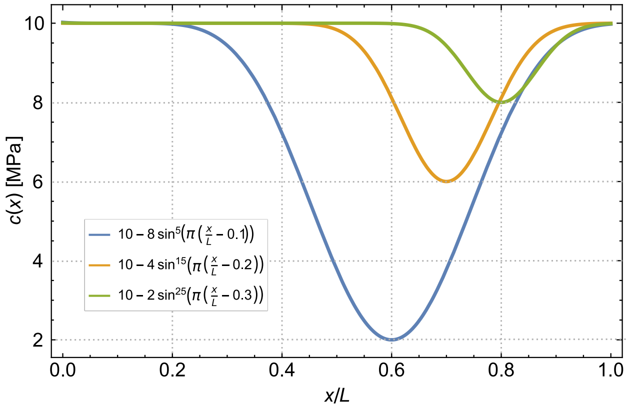

FIG. 3. The shape of the function (2.3) for variable values of parameters k and p (nonsymmetrical properties of the foundation).

The function of deflection of the homogeneous beam is assumed in the following form:

$$v(x) = v_a \cdot \sin(m\pi\xi) \cdot \sin^n(\pi\xi), \tag{2.5}$$

where v_a is the amplitude of the deflection, and m and n are the natural numbers.

Therefore, Eq. (2.2) can be rewritten in the following form:

$$\Phi(\xi) = \pi^4 EJ_z \frac{v_a}{L^4} \cdot f_4(\xi) + F_0 \cdot \pi^2 \frac{v_a}{L^2} \cdot f_2(\xi) + \Big\{c_0 - c_1 \sin^k[\pi(\xi - p)]\Big\} \cdot v_a \sin(m\pi\xi) \sin^n(\pi\xi) = 0,$$

which gives

$$(2.6)\quad \Phi(\xi) = \frac{\pi^4 E J_z}{L^4} \cdot f_4(\xi) + \frac{F_0 \cdot \pi^2}{L^2} \cdot f_2(\xi) + \left\{ c_0 - c_1 \sin^k[\pi(\xi - p)] \right\} \cdot \sin(m\pi\xi) \sin^n(\pi\xi) = 0,$$

where f_2 and f_4 are the derivates of the Eq. (2.5) of the second and fourth-order, respectively.

The critical value of load (2.14) will be calculated with the use of the Galerkin method. The main assumed condition of this method is as follows:

$$(2.7)\quad \int_0^1 \Phi(\xi) \cdot \sin(m\pi\xi) \sin^n(\pi\xi)\, \mathrm{d}\xi = 0.$$

The general solution is defined in the following form:

$$(2.8)\quad \left(\frac{\pi}{L}\right)^2 E J_z \cdot J_4 - F_0 \cdot J_2 + \left(\frac{L}{\pi}\right)^2 \cdot J_0 = 0,$$

from which

$$(2.9)\quad F_0 = \frac{1}{J_2} \left[J_4 \cdot F_{\mathrm{Euler}} + \left(\frac{L}{\pi}\right)^2 \cdot J_0 \right],$$

where

$$(2.10)\quad F_{\mathrm{Euler}} = \frac{\pi^2 E J_z}{L^2},$$

$$(2.11)\quad J_4 = \left(\frac{1}{\pi}\right)^4 \int_0^1 \frac{\mathrm{d}^4 \widetilde{v}}{\mathrm{d}\xi^4} \cdot \sin(m\pi\xi) \cdot \sin^n(\pi\xi)\, \mathrm{d}\xi,$$

$$(2.12)\quad J_2 = -\left(\frac{1}{\pi}\right)^2 \int_0^1 \frac{\mathrm{d}^2 \widetilde{v}}{\mathrm{d}\xi^2} \cdot \sin(m\pi\xi) \cdot \sin^n(\pi\xi)\, \mathrm{d}\xi,$$

$$(2.13)\quad J_0 = \int_0^1 \left\{ c_0 - c_1 \sin^k[\pi(\xi - p)] \right\} \cdot [\sin(m\pi\xi) \sin^n(\pi\xi)]^2\, \mathrm{d}\xi,$$

where $\widetilde{v}$ is the dimensionless value of deflection.

The critical load $F_{0,CR}$ is a function of geometric parameters and mechanical properties of the homogeneous beam as well as nonsymmetrical properties of the elastic foundation. The function is in the following form:

$$F_{0,CR} = \min_{m,n} \left\{ \frac{J_4 \cdot F_{\text{Euler}} + \left(\frac{L}{\pi}^2\right) \cdot J_0}{J_2} \right\}. \tag{2.14}$$

The homogeneous beam is resting on the elastic foundation. The presented foundation is flat, but it has a variable intensity of the reaction. The intensity is similar to the intensities of common foundations, e.g., soil foundations. The soil is flat, but it also has a variable structure. It can be denser locally.

3. Results

Sample analytical values of critical loads and stresses in a function of variable parameters m and n have been performed for the following data: $c_0 = 10$ MPa, $E = 200\,000$ MPa, $L = 1200$ mm, $J_z = 240$ mm^4, and $A = 180$ mm^2. The results for diverse proportions of c_1/c_0 (amplitudes of shape function) as well as k (shape parameter) and p (offset parameter) are presented in Tables 1–3.

Table 1. Critical values of loads for the homogeneous beam on the elastic foundation with various properties ($k = 5$).

c_1/c_0	0.2	0.4	0.6	0.8	p
$F_{0,CR}$ [kN]	42.721	40.118	36.579	32.853	0.1
σ_{CR} [MPa]	237.3	222.9	203.2	182.5	
m	8	8	7	7	
n	1	2	3	3	
$F_{0,CR}$ [kN]	43.267	41.407	39.547	37.249	0.2
σ_{CR} [MPa]	240.4	230	219.7	206.9	
m	8	8	8	7	
n	1	1	1	1	
$F_{0,CR}$ [kN]	43.954	42.781	41.608	40.436	0.3
σ_{CR} [MPa]	244.2	237.7	231.2	224.6	
m	8	8	8	8	
n	1	1	1	1	
$F_{0,CR}$ [kN]	44.591	44.055	43.519	42.984	0.4
σ_{CR} [MPa]	247.7	244.8	241.8	238.8	
m	8	8	8	8	
n	1	1	1	1	

Table 2. Critical values of loads for the homogeneous beam on the elastic foundation with various properties ($k = 15$).

c_1/c_0	0.2	0.4	0.6	0.8	p
$F_{0,CR}$ [kN]	43.567	41.916	40.092	37.731	0.1
σ_{CR} [MPa]	242	232.9	222.7	209.6	
m	8	8	8	7	
n	1	2	2	3	
$F_{0,CR}$ [kN]	43.969	42.811	41.653	40.495	0.2
σ_{CR} [MPa]	244.3	237.8	231.4	225	
m	8	8	8	8	
n	1	1	1	1	
$F_{0,CR}$ [kN]	44.465	43.804	43.143	42.482	0.3
σ_{CR} [MPa]	247	243.4	239.7	236	
m	8	8	8	8	
n	1	1	1	1	
$F_{0,CR}$ [kN]	44.876	44.626	44.376	44.126	0.4
σ_{CR} [MPa]	249.3	247.9	246.5	245.1	
m	8	8	8	8	
n	1	1	1	1	

Table 3. Critical values of loads for the homogeneous beam on the elastic foundation with various properties ($k = 30$).

c_1/c_0	0.2	0.4	0.6	0.8	p
$F_{0,CR}$ [kN]	43.986	42.845	41.488	40.130	0.1
σ_{CR} [MPa]	244.4	238	230.5	222.9	
m	8	8	8	8	
n	1	1	2	2	
$F_{0,CR}$ [kN]	44.274	43.421	42.568	41.715	0.2
σ_{CR} [MPa]	246	241.2	236.5	231.8	
m	8	8	8	8	
n	1	1	1	1	
$F_{0,CR}$ [kN]	44.668	44.210	43.752	43.294	0.3
σ_{CR} [MPa]	248.2	245.6	243.1	240.5	
m	8	8	8	8	
n	1	1	1	1	
$F_{0,CR}$ [kN]	44.975	44.824	44.673	44.521	0.4
σ_{CR} [MPa]	249.9	249	248.2	247.3	
m	8	8	8	8	
n	1	1	1	1	

It can be inferred that critical values of loads are dependent on the values of parameter k and c_1/c_0 ratio. The highest values of critical loads can be obtained for the highest values of k – the shape parameter and the lowest values of c_1/c_0 ratio – amplitudes of the shape function.

The offset parameter also affects the values of critical loads. The critical loads increase with the increase of the values of parameter p, i.e., a higher value of load should be applied to the beam in order to obtain similar values of the parameters as in the case of a symmetrical structure. The highest value (in the studied area) is equal to $F_{0,CR} = 45.030$ kN and was acquired for $k = 50$ and $c_1/c_0 = 0.2$ ($p = 0.4$). The results for different proportions of c_1/c_0 as well as parameters k and p are also summarized in Figs. 4–6.

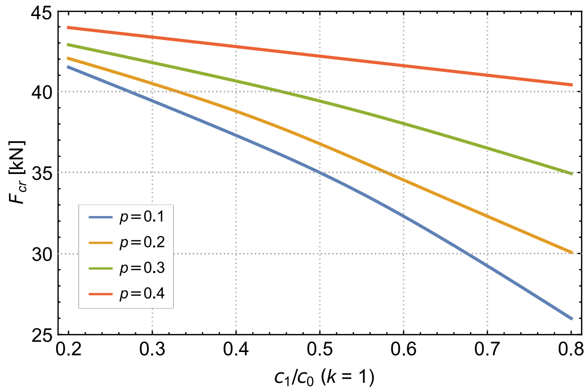

FIG. 4. The influence of parameter p on the values of critical loads ($k = 1$).

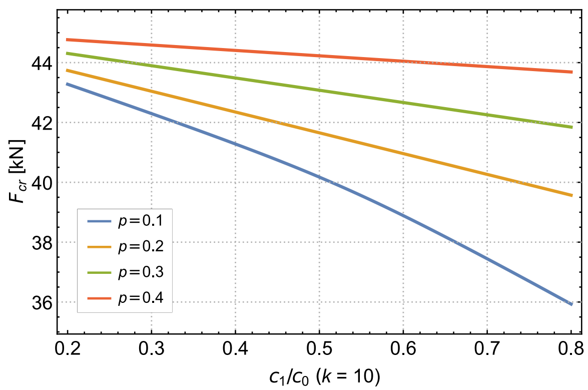

FIG. 5. The influence of parameter p on the values of critical loads ($k = 10$).

The charts confirm the preceding assumptions. The highest values of critical loads can be obtained for the highest values of parameter k (the shape para-

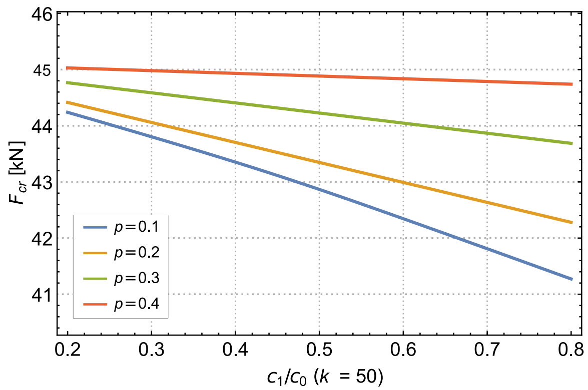

FIG. 6. The influence of parameter p on the values of critical loads ($k = 50$).

meter) and the lowest values of c_1/c_0 ratio (amplitudes of the shape function). The offset parameter also affects the values of critical loads. The critical loads increase with the increase of the values of parameter p.

4. Numerical model of the beam

Sample analytical values of critical loads in a function of alternating parameters of the elastic foundation are depicted for the following data: $b = 45$ mm, $t = 4$ mm, $E = 200\,000$ MPa, $L = 1200$ mm, $A = 180$ mm^2, $J_z = 240$ mm^4. The results for disparate proportions of c_1/c_0 (amplitudes of the shape function) and k (the shape parameter) are computed. The numerical analysis is divided into symmetrical and nonsymmetrical ones.

Linear buckling analysis of the homogeneous beam is performed. Varied parameters of the elastic foundation and its symmetry/asymmetry are taken into the consideration. The values of critical loads and buckling modes are obtained. The finite element analysis is carried out with the use of SolidWorks software. The compressive force is applied to the beam's axial plane (Fig. 1). The elastic foundation is substituted by 24 (symmetrical analysis) and 48 (nonsymmetrical analysis) elastic supports with the values of stiffness calculated based on formulae (2.3) and (2.4), respectively. The entire geometrical model includes 301 668 nodes and 108 881 finite elements. The size of an individual element is set to 2 mm with a tolerance equal to 0.1 mm. SOLID elements are applied to model the beam. The elements are defined in the form of the tetrahedron by 10 nodes (a parabolic tetrahedral element with 4 corner nodes, 6 mid-side nodes, and 6 edges) having three degrees of freedom at each node: translations in the nodal x, y, and z orthogonal directions. The bonded contact between the elements is adopted. Compatible mesh is used for the buckling analysis.

The boundary conditions follow from the supports:

- pinned support – the displacements in two perpendicular directions have been blocked,
- roller support – the displacements have been blocked in a perpendicular direction to the plane, in which the support can be moved.

4.1. Symmetrical analysis

The first analysis concerned the symmetrical properties of the elastic foundation (symmetry relative to the plane passing through the center of the beam's length). Initially, the values of the differences between the calculations obtained with the use of the analytical and numerical methods did not differ significantly in the analyzed range. For the amplitude of $c(x)$ function equal to 0.1, the difference was equal to 1.93%, while for the maximum analyzed value $c_1/c_0 = 0.8$, it was equal to 1.07% ($k = 1$). With the increase of the values of parameter k, higher discrepancies in the values of critical loads are observed. This distinction appeared for the highest analyzed values of the amplitude c_1/c_0 and resulted from simplifications in the FE model. In the analytical solution, the model of the elastic foundation is presented in the form of Eqs. (2.3) and (2.4). These equations are the mathematical description of the studied phenomenon. The shape of the elastic foundation is influenced by the c_1/c_0 amplitude and the shape parameter k. The higher the value of these two parameters, the peak on the graph of the function (2.3) or (2.4) is smaller and narrower, and thus it can be easily omitted in numerical calculations.

The investigation of the results for the values of parameter k indicates high compatibility between analytical and numerical calculations. Both for the minimum and maximum values of parameters k and c_1/c_0, the difference has not altered in a significant way. In the entire range of research, the difference between the results (analytical and numerical) did not exceed 3.2%, while for the parameter k the highest difference was equal to 9.5%. Sample results of the conducted investigation are presented in Tables 4–7.

Table 4. The values of critical loads for the beam on the elastic foundation with variable properties obtained with the use of analytical and numerical solution ($k = 5$).

c_1/c_0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$F_{0,CR}^{\text{analytical}}$ [kN]	43.820	42.481	40.941	39.110	36.949	34.703	32.444	29.466
$F_{0,CR}^{\text{FE}}$ [kN]	43.029	41.866	40.431	38.727	36.738	34.364	31.572	28.079
δ [%]	1.81	1.45	1.25	0.98	0.57	0.98	2.69	4.71

Table 5. The values of critical loads for the beam on the elastic foundation with variable properties obtained with the use of analytical and numerical solution ($k = 15$).

c_1/c_0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$F_{0,CR}^{\text{analytical}}$ [kN]	44.271	43.412	42.337	41.262	39.584	37.849	36.115	34.381
$F_{0,CR}^{\text{FE}}$ [kN]	43.386	42.669	41.750	40.625	38.940	36.895	34.380	31.204
δ [%]	2.0	1.71	1.39	1.54	1.63	2.52	4.8	9.24

Table 6. The values of critical loads for the beam on the elastic foundation with variable properties obtained with the use of analytical and numerical solution ($k = 30$).

c_1/c_0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$F_{0,CR}^{\text{analytical}}$ [kN]	44.509	43.892	43.190	42.400	41.054	39.614	38.173	36.733
$F_{0,CR}^{\text{FE}}$ [kN]	43.556	43.087	42.503	41.577	40.113	38.317	36.110	33.396
δ [%]	2.14	1.83	1.59	1.94	2.29	3.27	5.4	9.08

Table 7. The values of critical loads for the beam on the elastic foundation with variable properties obtained with the use of analytical and numerical solution ($k = 50$).

c_1/c_0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$F_{0,CR}^{\text{analytical}}$ [kN]	44.675	44.223	43.771	43.102	41.814	40.526	39.238	37.949
$F_{0,CR}^{\text{FE}}$ [kN]	43.664	43.351	42.977	42.092	40.813	39.252	37.363	35.092
δ [%]	2.26	1.97	1.81	2.34	2.39	3.14	4.78	7.53

Based on the conducted research, it can be concluded that the buckling modes of a homogeneous beam are similar to each other. The main objective of the numerical analysis was to find the buckling modes as well as the values of critical loads and compare them with the results obtained with the use of the analytical solution. In most cases, this mode was the second buckling load for the homogeneous beam. When it comes to investigating the evaluation of the values of critical loads, the calculation and analysis of all buckling modes help to find the weakest points of the FE model. According to this, the model can be modified to prevent buckling in a given mode.

4.2. Nonsymmetrical analysis

The second analysis concerned the nonsymmetrical properties of the elastic foundation. The function (2.3) has been supplemented with the parameter p – the offset parameter – relative to one end of the beam (the value of the parameter for symmetrical analysis was equal to $p = 0$). Sample results of conducted studies are presented in Tables 8–11.

Table 8. The values of critical loads for the beam on the elastic foundation with variable properties obtained with the use of analytical and numerical solution ($k = 1$).

c_1/c_0	0.2	0.4	0.6	p
$F_{0,CR}^{\text{analytical}}$ [kN]	41.503	37.293	32.301	
$F_{0,CR}^{\text{FE}}$ [kN]	40.549	36.321	31.092	0.1
δ [%]	2.3	2.61	3.74	
$F_{0,CR}^{\text{analytical}}$ [kN]	42.044	38.784	34.537	
$F_{0,CR}^{\text{FE}}$ [kN]	40.341	35.888	30.591	0.2
δ [%]	4.05	7.47	11.43	

Table 9. The values of critical loads for the beam on the elastic foundation with variable properties obtained with the use of analytical and numerical solution ($k = 5$).

c_1/c_0	0.2	0.4	0.6	p
$F_{0,CR}^{\text{analytical}}$ [kN]	42.721	40.118	36.579	
$F_{0,CR}^{\text{FE}}$ [kN]	41.917	38.662	34.416	0.1
δ [%]	1.88	3.63	5.91	
$F_{0,CR}^{\text{analytical}}$ [kN]	43.267	41.407	39.547	
$F_{0,CR}^{\text{FE}}$ [kN]	41.579	38.303	34.139	0.2
δ [%]	3.9	7.5	13.67	

Table 10. The values of critical loads for the beam on the elastic foundation with variable properties obtained with the use of analytical and numerical solution ($k = 10$).

c_1/c_0	0.2	0.4	0.6	p
$F_{0,CR}^{\text{analytical}}$ [kN]	43.277	41.280	38.887	
$F_{0,CR}^{\text{FE}}$ [kN]	42.341	39.823	38.990	0.1
δ [%]	2.16	3.53	0.26	
$F_{0,CR}^{\text{analytical}}$ [kN]	43.737	42.348	40.959	
$F_{0,CR}^{\text{FE}}$ [kN]	44.489	44.348	44.291	0.2
δ [%]	1.72	4.72	8.13	

Table 11. The values of critical loads for the beam on the elastic foundation with variable properties obtained with the use of analytical and numerical solution ($k = 15$).

c_1/c_0	0.2	0.4	0.6	p
$F_{0,CR}^{\text{analytical}}$ [kN]	43.567	41.916	40.092	
$F_{0,CR}^{\text{FE}}$ [kN]	45.078	45.053	45.023	0.1
δ [%]	3.47	7.48	12.3	
$F_{0,CR}^{\text{analytical}}$ [kN]	43.969	42.811	41.653	
$F_{0,CR}^{\text{FE}}$ [kN]	45.100	45.099	45.099	0.2
δ [%]	2.57	5.34	8.27	

Throughout the conducted research, the difference between the results (analytical and numerical) did not exceed 13.67%. The peak on the graph of the function $c(x)$ for $c_1/c_0 = 0.1$ is the lowest, while for $c_1/c_0 = 0.6$ the function has the highest peak (the deepest one). The form of the elastic foundation (narrow, deep peak) is the main cause of errors in numerical method (especially in the case of nonsymmetrical peak). If a calculation involves adding the high and low number, the effect of the lower number may be lost if rounding off is applied. During the nonsymmetrical analysis, the peak on the graph of the function $c(x)$ can be omitted by the software, especially for the highest values of the parameter c_1/c_0.

The problem is solved for various sizes of finite elements. A satisfactory convergence of the results is obtained, with a slight relative error in the software. In addition, the other, completely different method (the Galerkin method) allowed to obtain similar results. This fact proves the correctness of the methods used in this work as well as the results obtained in calculations.

5. Conclusions

In the presented work, an original analytical solution for a homogeneous beam on an elastic foundation was proposed. The elastic foundation was described by a mathematical function. The shape of the elastic foundation was conditioned by the form of the graph of the assumed function.

The subject of the presented investigation is the axially compressed beam on an elastic foundation. The main objective of this work was the analysis of critical loads of a homogeneous beam with variable (nonsymmetrical) mechanical properties of the foundation. The work presents the original approach to the problem of beams on elastic foundations. Original shape function, which represents the shape of the foundation as well as the function of deflection were proposed. The critical values of loads were calculated. The examples of calculations were shown. Moreover, the numerical FE analysis was performed. Good agreement between the results obtained with both methods was observed.

For a nonsymmetrical beam, the highest values of critical loads can be achieved for the highest values of parameter k and the lowest values of c_1/c_0 ratio. The value of parameter p was the variable. The values of critical loads increased with the increase of the value of the offset parameter.

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