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Research Paper

A Shear Deformation Theory of Beams with Bisymmetrical Cross-Sections Based on the Zhuravsky Shear Stress Formula

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This paper is devoted to simply supported beams with bisymmetrical cross-sections under a generalized load. Based on the Zhuravsky shear stress formula, the shear deformation theory of a planar beam cross-section is formulated. The deflections and the shear stresses of exemplary beams are determined. Moreover, the numerical-FEM computations of these beams are carried out. The results of the research are shown in figures and tables.

Key words: shear deformation theory; shear stress; beam deflection; shear effect.

1. Introduction

The problem of shear stress in bending of beams was recognized and described by D.I. Zhuravsky in the 19th century. Gere and Timoshenko [1] described in detail Zhuravsky's shear stress formula. S.P. Timoshenko in 1921 introduced an important problem consisting of considering the shearing effect in bending of beams by the generalized Euler-Bernoulli beam theory. He assumed that the normal straight line before bending remains a straight line after bending, being no longer perpendicular to the bent beam line. The theory of normal straight line deformation of bent beams and plates was intensively developed in the second half of the 20th century. Rychter [2] presented the shear-deformation beam-bending theories and extended them to two-dimensional displacement and stress fields. The relationships between accuracy, shear factors, and physical interpretations of the theory have been determined, showing that the Bernoulli-Euler type theory is a special case, providing accuracy equal to that of the other shear-deformation theories. KATHNELSON [3] considered small static deformation of isotropic linear elastic beams subjected to bending under arbitrary transverse load varying along the axial coordinate. A three-dimensional variational equation was used to formulate the governing equations of the Timoshenko type, allowing to derive the formulae for the displacements and stresses in the cross-sectional area. WANG *et al.* [4] presented the theories of normal straight line deformation taking into account the shearing effect for beams and plates, with consideration of the papers published in past decades. HUTCHINSON [5] derived a new formula for the shear coefficient. In the case of a circular cross-section, the resulting shear coefficient was perfectly compatible with the values obtained by most of the publications. The shear coefficients were found for a number of various beam cross-sections. KADOLI *et al.* [6] studied the static behaviour of functionally graded metal-ceramic beams using the higher-order shear deformation theory. The principle of stationary potential energy was used in order to derive the static equilibrium equation of the beam in the finite element form. It was shown that the beam behaviour depends on whether the load is applied to the beam's ceramic rich face or metal rich face. JUN and HONGXING [7] dealt with a uniform laminated composite beam using the trigonometric shear deformation theory, providing for the sinusoidal variation of the axial displacement over the cross-section. The solution of the governing differential equations of motion enabled to formulate the dynamic stiffness matrix. REDDY [8] used the differential constitutive relations of Eringen and the von Kármán nonlinear strains to improve the classical and shear deformation beam and plate theories. This allowed to develop a new finite element model. Such an approach enabled to determine the effect of geometric nonlinearity and nonlocal constitutive relations on the bending response of these structures. GHUGAL and SHARMA [9] analysed static bending of thick isotropic beams with consideration of the transverse shear deformation effects. The authors applied the hyperbolic shear deformation theory to avoid using the shear correction factor. The principle of virtual work allowed to formulate the governing differential equations and boundary conditions of the beam. Accuracy of the theory was verified by comparison to the results obtained with the help of other refined shear deformation theories. THAI and VO [10] dealt with bending, buckling, and vibration of nanobeams using a nonlocal sinusoidal shear deformation beam theory, thus avoiding the need to use the shear correction factors. The equations of motion and boundary conditions of the beam were derived with the use of Hamilton's principle. The results obtained in this way confirmed the accuracy of the approach. AKGÖZ and CIVALEK [11] developed a new higher-order shear deformation model of a beam using the modified strain gradient theory without the need of any shear correction factors. The authors studied static bending and free vibration of simply supported microbeams. The effects of thickness-to-material length ratio, slenderness ratio and shear deformation on deflections and natural frequencies of microbeams were analysed. It was shown that the effect of shear deformation becomes more significant for less slender beams. Sawant and Dahake [12] worked out a new hyperbolic shear deformation theory allowing for analysis of a thick beam bending. This theory enabled calculating the transverse shear stresses directly from the constitutive relations, using no shear correction factors, and was verified on the example of a cantilever isotropic beam subjected to varying load. The results were compared with those based on other theories. Bourada *et al.* [13] considered bending and vibration of functionally graded beams. The authors developed a simple and refined trigonometric higher-order beam theory for this purpose, distinguished by fewer unknowns and equations than the other theories. The equations of motion were derived from the Hamilton principle, and the solutions were compared with those obtained based on other theories, demonstrating the effect of the inclusion of transverse normal strain on the deflections and stresses. Polizzotto [14] considered a series of shear deformable beams using various approaches – from the Euler-Bernoulli to Timoshenko theory. Three deformation variables, namely two curvatures and a shear angle, were applied to determine two bending moments and the shear force. The principle of virtual power allowed to derive the equilibrium equations and boundary conditions. In order to compute the normal stress, the classical Navier expression was used, while the shear stress was calculated with the use of the Zhuravsky formula. The results were illustrated on the example of a simple cantilever beam. Endo [15] presented a historical survey on the bending and shearing deformation concepts of beams and plates, compared with the corrected classical models. The survey shows that according to the corrected classical models the bending and shearing deflections are considered as separate physical entities. The method developed by the author enabled to carry out the frequency analysis of isotropic plates and carbon fiber reinforced plastic laminated composite beams. The deformation concept enabled to develop a practical finite element formulation that is free from shear locking. Endo [16] used the Hamilton principle to derive his own alternative beam and plate theories. The author assumed that the total deflection is a sum of the bending and shearing deflections. It was shown that according to these theories, only half of the shear deformations could be expressed explicitly. Such an approach is efficient in the case of moderately thick structural models supplemented with the total deflection concept. Endo [17] formulated exact frequency relationships between the classical Kirchhoff plate and the "one-half order" shear deformation plate or beam theory of the Mindlin plate theory in case of simply supported plate edges or beam ends. Additionally, an approximate frequency relationship was determined, having a very simple form and nearly enough accuracy for purposes of practical use. Kharlab [18] proposed a complement to the theory of flexural shear stresses, being a generalization of the Zhuravsky theory. The improved approach takes into account the deplanation of the beam cross-section and allows to develop a simple formula for the potential energy of its deformation. GENOVESE and ELISHAKOFF [19] presented a discussion of the role of the principle of virtual work and the energy balance for the formulation of planar static rod theories, in the range of large deformations, with consideration of the transverse shear. The discussion also included the differences between the Haringx and Engesser approaches to the problem. MAGNUCKI *et al.* [20] considered the simply supported beams subjected to non-uniformly distributed load. The model of the beams was formulated based on the nonlinear hypothesis of the deformation of the beam planar cross-section and with consideration of the shear effect. Numerical-FEM models of the beams were developed to compare the analytical and numerical results for an exemplary beam family. MAGNUCKI *et al.* [21] analytically studied the bending problem of the beams with consideration of a seventh-order shear deformation theory.

The shear effect in the beams, described in the publications mentioned above, is taken into account with the use of the first-, second- and third-order beam theory. Thus, the main purpose of this paper is to present an individual theory of planar cross-section deformations of the beams formulated based on the Zhuravsky shear stress formula. The performance of this theory is demonstrated for bending of the example beams with various bisymmetrical cross-sections. The paper is a continuation of the studies presented by MAGNUCKI *et al.* in [20, 21].

2. Bisymmetrical cross section of the beam – Zhuravsky shear stress

The object of the study is the beam with a bisymmetrical cross-section of depth h and length L (Fig. 1). The beam is situated in the Cartesian coordinate system xyz.

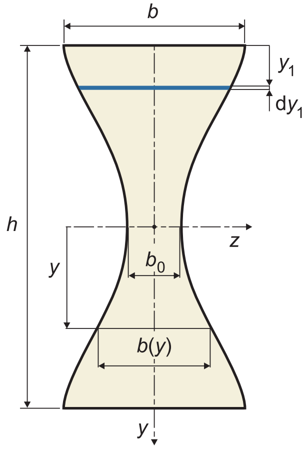

FIG. 1. Scheme of the bisymmetrical cross-section of the beam.

The symmetrically varying width of the cross-section in the depth direction is assumed in the following form:

$$b(y) = b \cdot \widetilde{b}(\eta), \tag{2.1}$$

where the dimensionless width

$$\widetilde{b}(\eta) = \beta_0 + (1 - \beta_0)(8\eta^2 - 16\eta^4)^{k_c}, \tag{2.2}$$

and $\beta_0 = b_0/b$ – parameter, k_c – exponent-real number, $\eta = y/h$ – dimensionless coordinate.

The Zhuravsky shear stress formula (Gere and Timoshenko [1]) is as follows:

$$\tau_{xy}^{(Zh)}(x, y) = \frac{V(x)}{J_z} \frac{Q_z(y)}{b(y)}, \tag{2.3}$$

where $V(x)$ – shear force, the second moment of the cross-section about the z-axis is

$$J_z = \int_{-h/2}^{h/2} b(y) y^2 \, \mathrm{d}y = b h^3 \widetilde{J}_z, \tag{2.4}$$

the dimensionless second moment

$$\widetilde{J}_z = \int_{-1/2}^{1/2} \widetilde{b}(\eta) \eta^2 \, \mathrm{d}\eta, \tag{2.5}$$

the first moment of the part of the cross-section about the z-axis

$$Q_z(y) = -\int_{-h/2}^{y} b\,(y_1) y_1 \, \mathrm{d}y_1 = b h^2 \widetilde{Q}_z\,(\eta)\,, \tag{2.6}$$

the dimensionless first moment

$$\widetilde{Q}_z(\eta) = -\int_{-1/2}^{\eta} \widetilde{b}(\eta_1) \eta_1 \, \mathrm{d}\eta_1, \tag{2.7}$$

and $\eta_1 = y_1/h$ – dimensionless coordinate $(-1/2 \leq \eta_1 \leq \eta)$, $b < h$ – condition.

Thus, the Zhuravsky shear stress formula (2.3) takes the following form:

$$\tau_{xy}^{(Zh)}(x, \eta) = \frac{V(x)}{b h \widetilde{J}_z} \frac{\widetilde{Q}_z(\eta)}{\widetilde{b}(\eta)}. \tag{2.8}$$

The formula (2.8) is a basis of the formulation of the individual theory of planar cross-section deformations of the beams.

3. Analytical formulation of the shear deformation theory of bending beams

A planar cross-section before bending does not remain planar after the bending of the beam (Fig. 2).

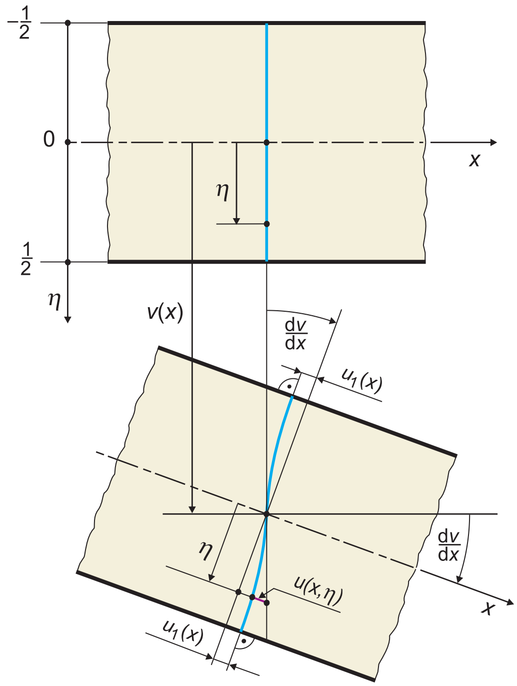

Fig. 2. Scheme of the deformation of the planar cross-sections of the beam.

The longitudinal displacement in accordance with Fig. 2 is as follows:

$$u(x,\eta) = -h\left[\eta\frac{\mathrm{d}v}{\mathrm{d}x} - f_d(\eta)\psi(x)\right], \tag{3.1}$$

where $\psi(x) = u_1(x)/h$ – dimensionless displacement function, $f_d(\eta)$ – deformation function of the planar cross-section of the beam, $v(x)$ – deflection.

Therefore, the longitudinal and shear strains are in the following form:

$$\begin{aligned} \varepsilon_x(x,\eta) &= \frac{\partial u}{\partial x} = -h\left[\eta\frac{\mathrm{d}^2v}{\mathrm{d}x^2} - f_d(\eta)\frac{\mathrm{d}\psi}{\mathrm{d}x}\right], \\ \gamma_{xy}(x,\eta) &= \frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\partial u}{\partial y} = \frac{\mathrm{d}f_d}{\mathrm{d}\eta}\psi(x). \end{aligned} \tag{3.2}$$

Consequently, the normal and shear stresses in accordance with Hooke's law take the following form:

$$
\begin{aligned}
&\sigma_x(x,\eta) = -Eh\left[\eta\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - f_d(\eta)\frac{\mathrm{d}\psi}{\mathrm{d}x}\right],\\
&\tau_{xy}^{(An)}(x,\eta) = \frac{E}{2(1+\nu)}\frac{\mathrm{d}f_d}{\mathrm{d}\eta}\psi(x),
\end{aligned}
\tag{3.3}
$$

where E – Young's modulus, ν – Poisson's ratio.

Equating the Zhuravsky shear stress formula (2.8) to the above shear stress expression $(3.3)_2$ $\tau_{xy}^{(Zh)}(x,\eta) = \tau_{xy}^{(An)}(x,\eta)$, after simple transformations one obtains:

- the derivative and deformation function

$$
\begin{aligned}
&\frac{\mathrm{d}f_d}{\mathrm{d}\eta} = \frac{1}{C_0}\frac{\left(1-4\eta^2\right)\beta_0 - 8\left(1-\beta_0\right)J_1(\eta)}{\widetilde{b}(\eta)},\\
&f_d(\eta) = \frac{1}{C_0}\int\frac{\left(1-4\eta^2\right)\beta_0 - 8\left(1-\beta_0\right)J_1(\eta)}{\widetilde{b}(\eta)}\,\mathrm{d}\eta,
\end{aligned}
\tag{3.4}
$$

where

$$
J_1(\eta) = \int\limits_{-1/2}^{\eta}\left(8\eta_1^2 - 16\eta_1^4\right)^{k_c}\eta_1\,\mathrm{d}\eta_1,
$$

and

$$
C_0 = \int\limits_{0}^{1/2}\frac{\left(1-4\eta^2\right)\beta_0 - 8\left(1-\beta_0\right)J_1(\eta)}{\widetilde{b}(\eta)}\,\mathrm{d}\eta,
$$

- the dimensionless displacement function

$$
\psi(x) = \frac{1}{4}(1+\nu)\frac{C_0}{\widetilde{J}_z}\frac{V(x)}{Ebh}.
\tag{3.5}
$$

The derivative and the deformation function (3.4) of the planar cross-section of the beam satisfy the conditions: $\mathrm{d}f_d/\mathrm{d}\eta|_{1/2} = 0$ and $f_d(\mp 1/2) = \mp 1$.

The example cross-sections of the beams (**CS-1**: $\beta_0 = 2/10$, $k_c = 2$, **CS-2 (I-100)**: $\beta_0 = 4.5/50$, $k_c = 11.155$, **CS-3 (I-200)**: $\beta_0 = 7.5/90$, $k_c = 16.397$) and corresponding graphs of the deformation function $(3.4)_2$ are shown in Fig. 3.

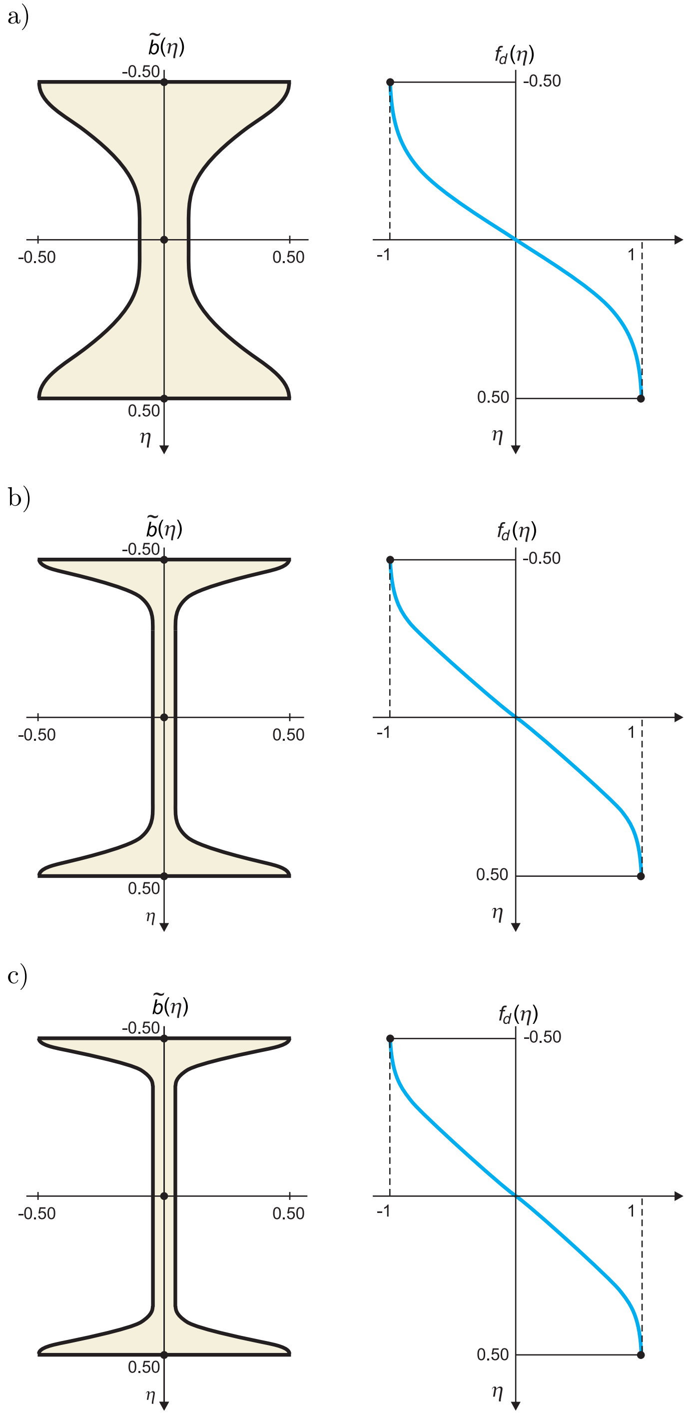

Fig. 3. Three example cross-sections and corresponding graphs of the function $(3.4)_2$: a) CS-1, b) CS-2 (I-100), c) CS-3 (I-200).

In the particular case for rectangular cross-section ($\beta_0 = 1$, $\widetilde{b}(\eta) = 1$, $\widetilde{J}_z = 1/12$, $C_0 = 1/3$), the derivative and the deformation function (3.4) of the planar cross-section are as follows:

$$\frac{\mathrm{d}f_d}{\mathrm{d}\eta} = 3\left(1 - 4\eta^2\right), \qquad f_d(\eta) = \left(3 - 4\eta^2\right)\eta. \tag{3.6}$$

The above expressions (3.6) are consistent with the shear deformation theory presented, for example, by Wang *et al.* [4].

4. Deflection of the beam with consideration of the shear deformation theory

The bending moment in accordance with the definition is as follows:

$$M_b(x) = \int_{-h/2}^{h/2} hy\sigma_x(x, y)b(y)\,\mathrm{d}y. \tag{4.1}$$

Substituting the expressions for the dimensionless width Eq. (2.2) and normal stress Eq. $(3.3)_1$ into the above expression (4.1) and integrating, one obtains the equation

$$\widetilde{J}_z \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - C_{v\psi}\frac{\mathrm{d}\psi}{\mathrm{d}x} = -\frac{M_b(x)}{Ebh^3}, \tag{4.2}$$

where $C_{v\psi} = \displaystyle\int_{-1/2}^{1/2} f_d(\eta)\widetilde{b}(\eta)\eta\,\mathrm{d}\eta$ is dimensionless coefficient.

By considering the findings of Magnucki *et al.* [20], the generalized load of the simply supported beam is assumed (Fig. 4). The intensity of the non-uniformly distributed load is in the following form:

$$q(\xi) = \frac{k}{2\tanh(k/2)}\frac{1}{\cosh^2\left[k(0.5-\xi)\right]}\frac{F}{L}, \tag{4.3}$$

where k is parameter $k \in \langle 0, \infty)$, $\xi = x/L$ is dimensionless coordinate $\xi \in \langle 0, 1\rangle$, F is total load of the beam, and L is length of the beam.

Consequently, the shear force is as follows:

$$V(\xi) = \frac{1}{2\tanh\left(k/2\right)}\tanh\left[k(0.5-\xi)\right]F, \tag{4.4}$$

and the bending moment

$$M_b(\xi) = \frac{1}{2k\tanh\left(k/2\right)}\ln\frac{\cosh\left(k/2\right)}{\cosh\left[k\left(0.5-\xi\right)\right]}FL. \tag{4.5}$$

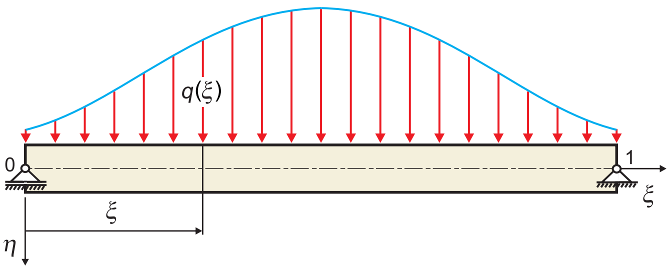

FIG. 4. Scheme of the simply supported beam under the generalized load.

The above expressions enable to formulate two particular cases of the load: the first one – *the uniformly distributed load* for $k \to 0$, and the second one – *three-point bending* for $k \to \infty$ (Fig. 5).

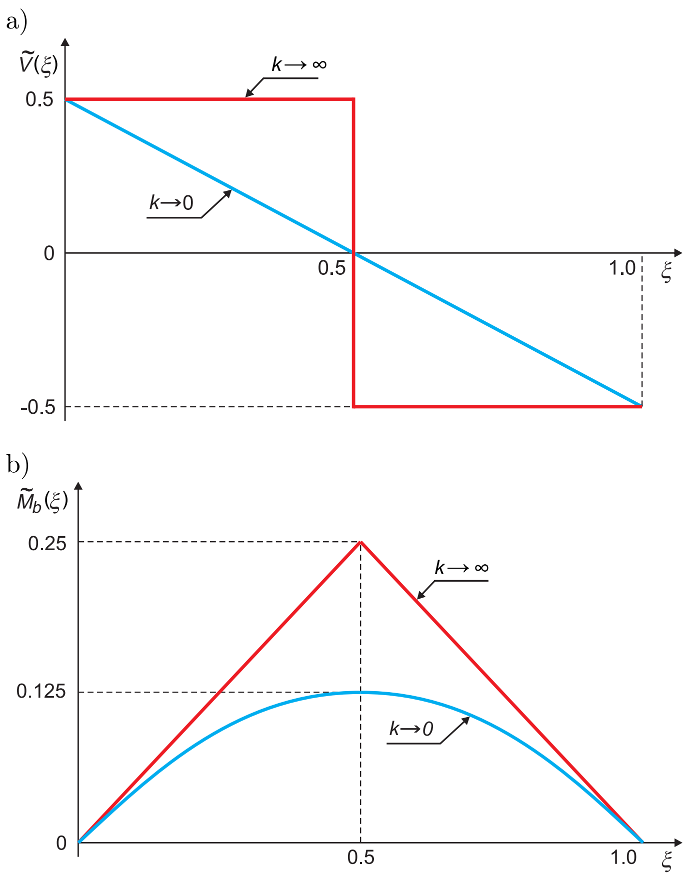

FIG. 5. Graphs of the dimensionless shear forces $\widetilde{V}(\xi) = \dfrac{V(\xi)}{F}$ (a) and dimensionless bending moments $\widetilde{M}_b(\xi) = \dfrac{M_b(\xi)}{FL}$ (b).

The differential Eq. (4.2) in dimensionless coordinate with consideration of the expression (4.5) takes the following form:

$$\widetilde{J}_z \frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2} = C_{vv} L \frac{\mathrm{d}\psi}{\mathrm{d}\xi} - \frac{\lambda^3}{2k\tanh(k/2)} \ln \frac{\cosh(k/2)}{\cosh[k(0.5-\xi)]} \frac{F}{Eb}, \tag{4.6}$$

where $\lambda = L/h$ is relative length of the beam.

Integrating Eq. (4.6) and taking into account the functions (3.5) and the condition $\mathrm{d}v\,\mathrm{d}\xi|_{1/2} = 0$, one obtains the equation

$$\frac{\mathrm{d}v}{\mathrm{d}\xi} = \frac{\lambda^3}{2k\tanh(k/2)} \varPhi_1(\xi) \frac{1}{\widetilde{J}_z} \frac{F}{Eb}, \tag{4.7}$$

where

$$\varPhi_1(\xi) = \frac{1}{4}(1+\nu) \frac{C_0 C_{v\psi}}{\widetilde{J}_z \lambda^2} k \tanh[k(0.5-\xi)] + J_{v1} - \int \ln \frac{\cosh(k/2)}{\cosh[k(0.5-\xi)]} \mathrm{d}\xi, \tag{4.8}$$

and

$$J_{v1} = \int_0^{1/2} \ln \frac{\cosh(k/2)}{\cosh[k(0.5-\xi)]} \mathrm{d}\xi.$$

Consequently, integrating Eq. (4.7) and taking into account the condition $v(0) = 0$, one obtains the deflection curve of the beam

$$v(\xi) = \frac{\lambda^3}{2k\tanh(k/2)} \varPhi_2(\xi) \frac{1}{\widetilde{J}_z} \frac{F}{Ebh}, \tag{4.9}$$

where

$$\varPhi_2(\xi) = \frac{1}{4}(1+\nu) \frac{C_0 C_{v\psi}}{\widetilde{J}_z \lambda^2} \ln \frac{\cosh(k/2)}{\cosh[k(0.5-\xi)]} + J_{v1}\xi - \iint \ln \frac{\cosh(k/2)}{\cosh[k(0.5-\xi)]} \mathrm{d}\xi^2. \tag{4.10}$$

Thus, the maximum deflection $\left(v_{\max}^{(An)} = v(1/2)\right)$ of the beam is in the following form:

$$v_{\max}^{(An)} = \widetilde{v}_{\max}^{(An)} \frac{F}{Eb}, \tag{4.11}$$

where the dimensionless maximum deflection is

$$\tilde{v}_{\max}^{(An)} = \left(1 + \frac{C_s}{\lambda^2}\right) C_v \frac{\lambda^3}{\tilde{J}_z}, \tag{4.12}$$

the shear and bending coefficients are

$$\begin{aligned} C_s &= \frac{1}{2}(1+\nu)\frac{C_0 C_{v\psi}}{\tilde{J}_z}\frac{\ln[\cosh(k/2)]}{J_{v1} - 2J_{v2}}, \\ C_v &= \frac{J_{v1} - 2J_{v2}}{4k\tanh(k/2)}, \end{aligned} \tag{4.13}$$

and $J_{v2} = \int\limits_0^{1/2} \int \ln \frac{\cosh(k/2)}{\cosh[k(0.5 - \xi)]} \, d\xi^2$.

The expressions (4.12) and (4.13) make a basis for example studies the beams' bending with consideration of the shear effect.

5. Analytical studies of beams bending

The exemplary analytical studies of beams bending are carried out for the beams of relative length $\lambda = 10$ with three selected bisymmetrical cross-sections (Fig. 3): **CS-1**: $\beta_0 = 2/10$, $k_c = 2$, **CS-2 (I-100)**: $\beta_0 = 4.5/50$, $k_c = 11.155$, **CS-3 (I-200)**: $\beta_0 = 7.5/90$, $k_c = 16.397$), and material constant – Poisson's ratio $\nu = 0.3$. The dimensionless second moment and coefficients of these selected cross-sections are specified in Table 1.

Table 1. The geometrical characteristic of the selected cross-section of beams.

Cross-section	**CS-1**	**CS-2 (I-100)**	**CS-3 (I-200)**
$\tilde{J}_z$	0.060245	0.03420	0.02972
C_0	0.830938	1.290076	1.253780
$C_{v\psi}$	0.1457584	0.0762374	0.0653415

The studies are carried out for two particular typical cases of the load (Fig. 5):

- *the uniformly distributed load of intensity* q $(k \to 0)$, then, the values of the shear and bending coefficients (4.13) are as follows:

$$\begin{aligned} C_s^{(q)} &= \frac{1}{2}(1+\nu)\frac{C_0 C_{v\psi}}{\tilde{J}_z}\lim_{k\to 0}\frac{\ln[\cosh(k/2)]}{J_{v1} - 2J_{v2}} = \frac{12}{5}(1+\nu)\frac{C_0 C_{v\psi}}{\tilde{J}_z}, \\ C_v^{(q)} &= \lim_{k\to 0}\frac{J_{v1} - 2J_{v2}}{4k\tanh(k/2)} = \frac{5}{384}. \end{aligned} \tag{5.1}$$

The values of the above shear coefficient and dimensionless maximum deflection (4.12) of the example beams are specified in Table 2.

Table 2. The calculation results of the selected beams under uniformly distributed load.

Cross-section	**CS-1**	**CS-2 (I-100)**	**CS-3 (I-200)**
$C_s^{(q)}$	6.272	8.972	8.600
$v_{\max}^{(An-q)}$	229.69	414.88	475.80

- *three-point bending* ($k \to \infty$), then, the values of the shear and bending coefficients (4.13) are as follows

$$
\begin{aligned}
C_s^{(F)} &= \frac{1}{2}\,(1+\nu)\,\frac{C_0 C_{v\psi}}{\widetilde{J}_z}\lim_{k\to\infty}\frac{\ln\left[\cosh\left(k/2\right)\right]}{J_{v1}-2J_{v2}} = 3\,(1+\nu)\,\frac{C_0 C_{v\psi}}{\widetilde{J}_z},\\
C_v^{(F)} &= \lim_{k\to\infty}\frac{J_{v1}-2J_{v2}}{4k\tanh\left(k/2\right)} = \frac{1}{48}.
\end{aligned} \tag{5.2}
$$

The values of the above shear coefficient and dimensionless maximum deflections (4.12) of the example beams are specified in Table 3.

Table 3. The calculation results of the selected beams under three-point bending.

Cross-section	**CS-1**	**CS-2 (I-100)**	**CS-3 (I-200)**
$C_s^{(F)}$	7.840	11.216	10.750
$v_{\max}^{(An-Ft)}$	372.92	677.48	776.35

The dimensionless shear stress based on the expression $(3.3)_2$ for value $\widetilde{V}(\xi) = 1/2$ of the dimensionless shear force is as follows:

$$
\widetilde{\tau}_{xy}^{(An)}(\eta) = \frac{1}{16\widetilde{J}_z}\,\frac{\left(1-4\eta^2\right)\beta_0 - 8\,(1-\beta_0)\,J_1(\eta)}{\widetilde{b}(\eta)}. \tag{5.3}
$$

The graphs of the dimensionless shear stress (5.3) for the three selected bisymmetrical cross-sections are shown in Fig. 6.

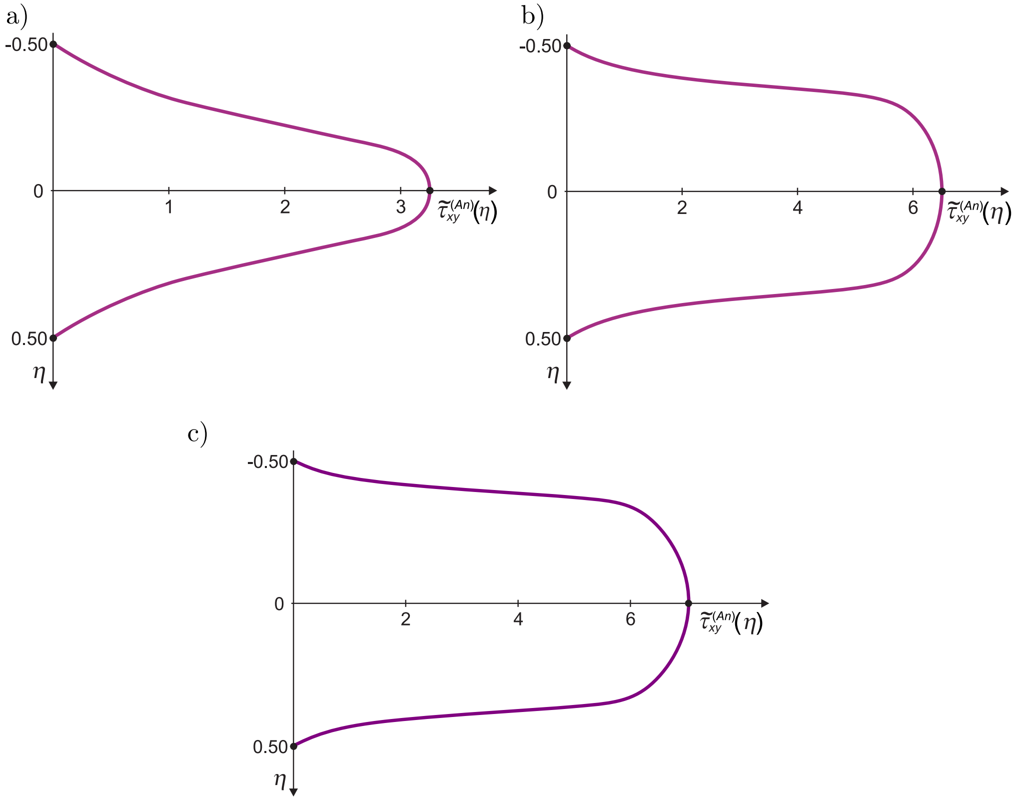

FIG. 6. Graphs of the dimensionless shear stresses in the selected bisymmetrical cross-sections: a) CS-1 ($\tilde{\tau}_{xy}^{(An)}(0) = 3.25$), b) CS-2 (I-100) ($\tilde{\tau}_{xy}^{(An)}(0) = 6.57$), c) CS-3 (I-200) ($\tilde{\tau}_{xy}^{(An)}(0) = 7.05$).

6. Numerical-FEM studies of beams bending

The considered beams are modelled using the SolidWorks software package. Taking into account the symmetry of the beams and their loads, only a quarter of the whole structure is analysed (Fig. 7a). The boundary conditions imposed on the model guarantee its proper behaviour. The model of the stockiest of the beams (i.e., CS1) consists of about 572 thousand 3D tetrahedral finite elements with 4 Jacobian points. Such an approach corresponds to the simplest integration of the element area, being sufficient in the case of the fine mesh applied in the computation.

The number of nodes reaches nearly 840 000. In the case of the CS2 and CS3, variants of the number of nodes is correspondingly smaller. A part of the CS1 mesh is shown in Fig. 7b. The CS1 mesh could be coarser without harming the computation quality. Nevertheless, in order to maintain uniform mesh quality in all the considered cases, the mesh scale does not vary.

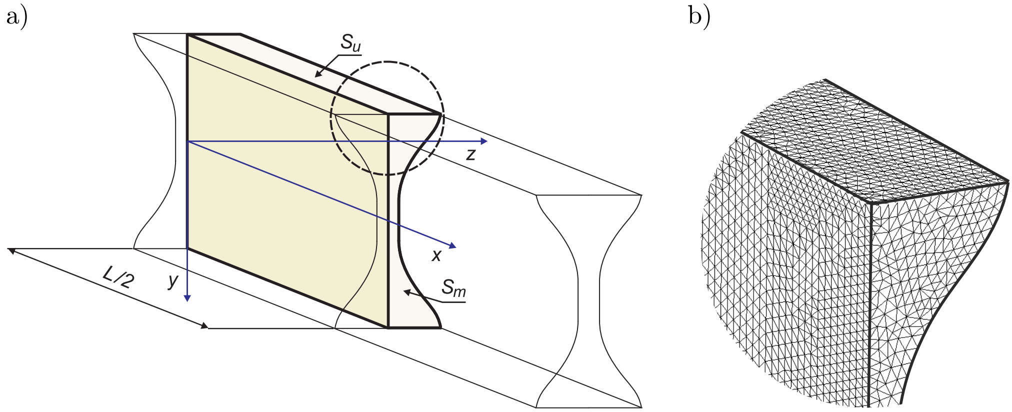

FIG. 7. a) The FEM-model of the beam, b) an example mesh (confined approximately to the part marked with the dotted circle in 7a).

The origin of the beam Cartesian coordinate system is located at the beginning of the beam neutral axis. The x-axis coincides with the neutral axis, the y-axis points down and, in consequence, the z-axis is normal to the longitudinal middle plane of the beam.

The boundary conditions specified below ensure that the one-fourth of the beam behaves the same as if it belonged to the whole beam:

- the beam model is simply supported at its edge (i.e., for $x = 0$), therefore, the y displacements of the wall lying in the yz-plane are equal to zero,
- the x displacements of the middle wall of the beam, i.e., of the middle surface S_m of the model (being the transverse plane of symmetry of the beam) are equal to zero too,
- the z displacements of the model lying in the xy-plane (for $z = 0$, the longitudinal plane of symmetry of the beam) are zero.

Two load cases are considered:

1) three-point bending, with the force $1/4F$ downward directed and applied to the S_m surface;
2) the force $1/4F$ downward directed, distributed uniformly at the S_u surface.

The results of the numerical study are specified in Table 4.

Table 4. The results of the numerical-FEM calculation of the selected beams.

Cross-section	**CS-1**	**CS-2 (I-100)**	**CS-3 (I-200)**
$v_{\max}^{(\mathrm{FEM}-q)}$	229.6	414.9	475.4
$v_{\max}^{(\mathrm{FEM}-F)}$	372.8	676.6	775.2

Comparison of the results obtained with FEM (Table 4) with data from Tables 2 and 3 shows a very good compliance of the analytical and numerical approach to the problem. The differences do not exceed 0.5%.

The graphs and values of the dimensionless shear stresses obtained with FEM are identical with those shown in Fig. 6.

7. Conclusions

The theory of planar cross-section deformations of the beams adopted based on the Zhuravsky shear stress formula well depicts the shear effect in the bent beams. The differences between the maximum deflections and shear stresses calculated analytically and numerically are insignificant (below 0.5%).

The proposed theory is novel and makes a generalization of previously known beam theories. Therefore, it may be easily used in the analytical modelling of the beams with consideration of the shear effect. It should be noted that the higher-order theories introduce additional unknowns that give rise to difficulties in analytical research, as indicated by Wang *et al.* [4].

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