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WELL- OR ILL-POSED PROBLEM OF PARAMETER IDENTIFICATION – THE METHOD OF ANALYSIS APPLIED IN A CASSETTE RECORDER DRIVE MODELING

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In the paper, the method of error analysis of physical parameter estimates identified by the least-squares fitting of the dynamical model characteristics to the corresponding experimental data points, measured (observed) with systematic errors, is discussed. The quantity measures of global sensitivity of parameter estimate to the observation errors are introduced. The formulas for computing the derivatives of parameter estimates with respect to the observation errors are derived. The advantages of the method are illustrated by the example of application of the method in the modeling process of a tape recorder drive at the stage of experiment design.

1. INTRODUCTION

The method and algorithm of sensitivity analysis of physical parameter estimates to the measurement errors of frequency characteristics were developed on the basis of a very particular problem, i.e. the problem of modeling the cassette recorder drive [3], but it can be easily generalized to time-domain dynamical characteristics, and applied to other linear dynamical systems. The objective of the modeling was to develop the physical model of the system. By a physical model we do not mean here any material, laboratory device, but a mathematical model, the structure of which reflects the structure of the actual system, and its parameters are of some physical meaning. For a designer of any mechanical or electro-mechanical system, the physical model is cognitively more valuable than the functional one, e.g. in the form of a complex, rational function, which from the control standpoint could be sufficient.

The structure of the mathematical model of the recorder drive, in the form of a set of ordinary linear differential equations with constant coefficients, was adopted on the basis of careful analysis of physical phenomena. Some physical parameters were measured directly, others – indirectly. The estimates of several

selected parameters were identified by fitting the amplitude-frequency characteristic of the model to the corresponding experimental characteristic by the least-squares method.

The experimental amplitude-frequency characteristic was recorded by means of well-known and commonly used Bruel&Kjaer analogue measuring arrangement for measuring the transfer functions of dynamical systems. In such an arrangement, the output signal passes through the narrow-band filter – being tuned automatically to the frequency of harmonic excitation – and then its level is averaged and recorded by the analogue level recorder. Due to narrow-band filtering and level averaging, various disturbances, in particular the random disturbances, are eliminated to a high degree. This is a great advantage of the frequency analysis in comparison with transient analysis, and therefore the frequency analysis is still willingly used, if only harmonic excitation is technically possible.

The only disadvantage of using the experimental frequency characteristic for parameter identification is the fact that very little is known about the systematic measurement errors, which dominate over the random ones. In cases when parameter estimates are calculated from the measured values of a process running in time, it is often possible to assume that average values of the measurement errors equal zero and also to estimate the statistical properties of measurement errors, and this in turn enables a more sophisticated analysis of statistical properties of parameter estimates [5, 6]. This is probably one of the reasons why a vast literature on identification of dynamical systems concerns almost exclusively the methods involving time domain analysis.

The lack of knowledge of mean values of the measurement errors and their statistical properties is a drawback of the method of identification from the amplitude-frequency characteristics measured in the way described above. But this cannot be a reason to give up the method which, in many respects, is better than other possible methods. Of course, some kind of analysis of parameter estimates errors is absolutely necessary. In this paper, the quantity measure of global sensitivity of parameter estimate with respect to measurement errors is introduced. This measure is analogous to a maximum error commonly used in the analysis of systematic or global error of indirect measurement. We do not know any general theory of systematic errors of indirect measurement [7]. The same concerns the systematic errors in parameter identification. So the proposed approach to the errors analysis – based on linearisation of implicit relationships between the parameter estimates and measurements – seems to be one of a very few options. Many numerical experiments performed with the model of the recorder drive [3], and other dynamical systems, showed that such an analysis can be useful and valuable in the modeling process. Other possibilities of error analysis of estimates will be discussed very briefly in Final Remarks.

2. Global sensitivity of parameter estimate to measurement errors; definitions and computation formulas

2.1. Notation

In the sequel we will use the following notation:

- m number of parameters to be identified,
- $\mathbf{p}$ vector of parameters to be identified, $\mathbf{p} = (p_1, \ldots, p_m)^T$,
- $\hat{\mathbf{p}}$ vector of parameter estimates; the sign $\hat{}$ over the symbol $\mathbf{p}$ or p_j is used, when necessary, in order to distinguish the estimates from the real values,
- f argument of dynamical characteristic, selected as a basis for parameter identification,
- $h(f, \mathbf{p})$ dynamical characteristic adopted as a basis for parameter identification, obtained from the mathematical model,
- $h_i(\mathbf{p})$ dynamical characteristic $h(f, \mathbf{p})$ for $f = f_i$,
- n number of observations, i.e. the observed (measured) values of experimental dynamical characteristic corresponding to the theoretical characteristic $h(f, \mathbf{p})$,
- $\mathbf{z}$ vector of observations, $\mathbf{z} = (z_1, \ldots, z_n)^T$, $z_i > 0$,
- $\mathbf{v}$ vector of observation errors including external noise, measurement errors and also modeling errors, $\mathbf{v} = (v_1, \ldots, v_n)^T$.

We will also use the additional superscript 0 in order to stress that the symbols $\hat{\mathbf{p}}^0$, $\hat{p}_j^0$, $\mathbf{z}^0$, z_i^0 stand for the concrete values of estimates and observations, whereas $\hat{\mathbf{p}}$, $\hat{p}_j$, $\mathbf{z}$, z_i denote the parameter estimates and observations regarded as variables.

2.2. Definitions

An indirect measurement of a quantity p_i consists in measuring directly the quantities $z_1, \ldots, z_n$, and then calculating the value of p_i from the formula given in an explicit form

$$p_i = w_i(z_1, \ldots, z_n). \tag{2.1}$$

The estimate of the systematic or global error of p_i is usually computed by the formula [7]

$$s_{p_i} = \sum_{k=1}^{n} \left| \frac{\partial w_i}{\partial z_k} \right| v_k \,, \tag{2.2}$$

where $v_1, \ldots, v_n$ denote the estimates of the measurement errors of $z_1, \ldots, z_n$.

In the particular case, when for every k the the percentage error defined as 100% v_k/z_k equals d, we can derive from (2.2) the formula for the maximum percentage error of parameter p_i in the form

$$e_{p_i} = d \sum_{k=1}^{n} \left| \frac{\partial w_i}{\partial z_k} \right| \frac{z_k^0}{w_i^0}. \tag{2.3}$$

One can observe that parameter identification is similar to indirect measurement in the meaning that the values of parameters $p_i, \ldots, p_m$ are calculated from the measured values $z_1, \ldots, z_n$. So if the formulas (2.2), (2.3) are appropriate in the analysis of indirect measurement errors, they could also be useful in the analysis of parameter identification errors. The important difference between indirect measurement and parameter identification is that in the second case, the relationships $p_i = w_i(z_1, \ldots, z_n)$ for $i = 1, \ldots, m$ are not known in an explicit form, and we are even not certain whether such unique relationships do exist at all.

So the idea presented here is to find the way of proving that these relationships do exist, to develop the algorithm for computing the values of derivatives $\partial w_i/\partial z_j$, and to apply the formula (2.2), or (2.3) in the modeling process, both at the stage of the experiment design, and in model validation after completing the parameter identification.

In the next section, we will show, using the example of a model of tape recorder drive, how the calculation of the errors e_{p_i} for various possible vectors $\mathbf{p}$ of parameters to be identified can help us to avoid the situations in which the identification problem is ill-posed.

2.3. Deterministic identification case

Let us first consider the case of the so-called deterministic identification, when estimates of m selected parameters are calculated by solving m algebraic, generally nonlinear equations in the form

$$h(\hat{\mathbf{p}}, f_i) - z_i^0 = 0, \qquad i = 1, \ldots, m. \tag{2.4}$$

In the neighbourhood of the solution $(z_1^0, \ldots, z_n^0, \hat{p}_1^0, \ldots, \hat{p}_m^0)$, the set (2.4) can be regarded as an implicit form of the differentiable function

$$\hat{\mathbf{p}} = \mathbf{w}(\mathbf{z}) \tag{2.5}$$

if the set (2.4) satisfies the conditions sufficient for the existence of such a function. The related theorem [1] determines these conditions for the system of m equations given in the general, following form: $F_i(\mathbf{p}, \mathbf{z}) = 0$, $i = 1, \ldots, m$.

The theorem says that in the neighbourhood of the point $(z_1^0, \ldots, z_n^0, \hat{p}_1^0, \ldots, \hat{p}_m^0)$, which holds the system, the differentiable function $\mathbf{p} = \mathbf{w}(\mathbf{z})$ does exist, if the functions $F_i(\mathbf{p}, \mathbf{z})$ are differentiable and the determinant of the Jacobian matrix, the i,j-th element of which is $\partial F_i / \partial p_j$, differs from zero.

In our case the equation $F_i(\mathbf{p}, \mathbf{z}) = 0$ has the form (2.4), and the element g_{ij} of the Jacobian matrix equals $\partial h_i / \partial p_j$. In order to get the vector of the partial derivatives $\partial \mathbf{w}(\mathbf{z}) / \partial z_j$ we will use the general rules for differentiating the implicit functions, i.e., we will differentiate the system (2.4) with respect to z_j. As the result we obtain a new system of m equations. Systems of equations obtained by differentiating equations (2.4) with respect to z_j, for $j = 1, \ldots, n$ differ from each other only by their right-hand sides. In a matrix form all these systems can be written as one equation

$$\mathbf{G}\,\mathbf{U} = \mathbf{I}. \tag{2.6}$$

The developed form of (2.6) is

$$\begin{bmatrix} \frac{\partial h_1}{\partial p_1} & \cdots & \frac{\partial h_1}{\partial p_i} & \cdots & \frac{\partial h_1}{\partial p_m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial h_j}{\partial p_1} & \cdots & \frac{\partial h_j}{\partial p_i} & \cdots & \frac{\partial h_j}{\partial p_m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial h_m}{\partial p_1} & \cdots & \frac{\partial h_m}{\partial p_i} & \cdots & \frac{\partial h_m}{\partial p_m} \end{bmatrix} \begin{bmatrix} \frac{\partial \hat{p}_1}{\partial z_1} & \cdots & \frac{\partial \hat{p}_1}{\partial z_j} & \cdots & \frac{\partial \hat{p}_1}{\partial z_m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \hat{p}_i}{\partial z_1} & \cdots & \frac{\partial \hat{p}_i}{\partial z_j} & \cdots & \frac{\partial \hat{p}_i}{\partial z_m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \hat{p}_m}{\partial z_1} & \cdots & \frac{\partial \hat{p}_m}{\partial z_j} & \cdots & \frac{\partial \hat{p}_m}{\partial z_m} \end{bmatrix} = \mathbf{I}, \tag{2.7}$$

where $\mathbf{I}$ denotes the identity matrix. From Eq. (2.7) one can easily conclude that the columns of matrix $\mathbf{G}$ should not be linearly dependent, otherwise the Jacobian matrix $\mathbf{G}$ would be singular and the unique solution would not exist. In other words, the particular parameter is identifiable if it influences the observations not only significantly but also in a different way than others.

2.4. The least-squares identification case

Now let us consider the case of parameter identification by the least-squares method. Generally the reasoning is the same, and it leads to the matrix equation

$$\mathbf{G}\,\mathbf{U} = \mathbf{M} \tag{2.8}$$

as well, only the formulas for calculating the elements g_{ij} $(i = 1, \ldots, m,\ j = 1, \ldots, m)$ and m_{ij} $(i = 1, \ldots, m,\ j = 1, \ldots, n)$ of the matrix $\mathbf{G}$ and $\mathbf{M}$, respectively, are more complicated.

In the case of identification of dynamical model parameters by the least-squares method, the estimates of m parameters $p_1, \ldots, p_m$ are calculated from n observed data points: (f_k, z_k), $k = 1, \ldots, n$, and the number of observations n is usually much greater than m. The values of parameters which minimize the weighted sum of squared residuals (computed minus the observed values) in the form

$$K(\mathbf{p}) = \sum_{k=1}^{n} (h(f_k, \mathbf{p}) - z_k)^2 r_k \tag{2.9}$$

are accepted as the estimates of parameters. In the formula (2.9), r_k denotes the weighting coefficient of the k-th observation.

The condition necessary for existence of a minimum at the point $\mathbf{p}^0 = (p_1^0, \ldots, p_m^0)$ is that the partial derivatives $\partial K/\partial p_i$, $i = 1, \ldots, m$ are at this point equal to zero. This condition yields the system of m equations, the i-th equation of which has the form

$$\sum_{k=1}^{n} (h_k(\mathbf{p}) - z_k) r_k \frac{\partial h_k(\mathbf{p})}{\partial p_i} = 0, \qquad i = 1, \ldots, m. \tag{2.10}$$

Now we can apply again, this time to the set of equations (2.10), the theorem on the existence of differentiable implicit function $\mathbf{w}(\mathbf{z})$ in the neighbourhood of the point $(z_1^0, \ldots, z_n^0, \hat{p}_1^0, \ldots, \hat{p}_m^0)$ which holds the system. In this case, the element g_{ij} $(i = 1, \ldots, m,\ j = 1, \ldots, m)$ of the Jacobian matrix $\mathbf{G}$, which cannot be singular, is determined by the formula

$$g_{ij} = \sum_{k=1}^{n} \frac{\partial h_k}{\partial p_j} \frac{\partial h_k}{\partial p_i} r_k + \sum_{k=1}^{n} (h_k - z_k) r_k \frac{\partial^2 h_k}{\partial p_i \partial p_j}. \tag{2.11}$$

The elements of the matrices $\mathbf{U}$ and $\mathbf{M}$ in the matrix equation (2.8) are

$$u_{ij} = \frac{\partial \hat{p}_i}{\partial z_j}, \qquad i = 1, \ldots, m, \quad j = 1, \ldots, n, \tag{2.12}$$

$$m_{ij} = \frac{\partial h_j}{\partial p_i} r_j, \qquad i = 1, \ldots, m, \quad j = 1, \ldots, n. \tag{2.13}$$

The formulas (2.6)–(2.13) were derived assuming only that the function $h(f, \mathbf{p})$ is differentiable (once or twice) with respect to p_i $(i = 1, \ldots, m)$ for all discrete arguments f_j $(j = 1, \ldots, n)$. The function $h(f, \mathbf{p})$ is not necessarily the amplitude-frequency characteristic, it can be the transient response in terms of time as well.

2.5. The sensitivity functions of dynamical characteristics to the model parameters

In order to determine the derivatives $\partial \hat{p}_i/\partial z_j$, the derivatives $\partial h_k/\partial p_i$, $\partial^2 h_k/\partial p_i \partial p_j$ have to be computed first. For both mentioned types of dynamical characteristics, it is easy to compute the values of their sensitivity to the model parameters, i.e. the values of the derivatives $\partial h_k/\partial p_i$, $\partial^2 h_k/\partial p_i \partial p_j$ without using unreliable, approximate numerical methods based on the notion of divided differences. We neglect here the formulas for computing the derivatives, which are quite easy to derive [3] by using the following basic hints.

To get the sensitivity functions $\partial \mathbf{x}(t, \mathbf{p})/\partial p_i$ of the state variables transients $\mathbf{x}(t, \mathbf{p})$ of the linear dynamical system in the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{p})\mathbf{x} + \mathbf{B}(\mathbf{p})\,\mathbf{u}(t), \tag{2.14}$$

where $\mathbf{u}(t)$ denotes an external excitation, e.g. the step function at a chosen, single input, both sides of the state equation (2.14) should be differentiated with respect to p_i. The solutions of the new set of differential equation for zero initial conditions: $\dfrac{\partial \mathbf{x}(0, \mathbf{p})}{\partial p_i} = \mathbf{0}$, $\dfrac{d}{dt}\left(\dfrac{\partial \mathbf{x}(0, \mathbf{p})}{\partial p_i}\right) = \mathbf{0}$ are the sensitivity functions $\partial x_l(t, \mathbf{p})/\partial p_i$ of the transients $x_l(t, \mathbf{p})$ to the parameters; $l = 1, \ldots, m$, where m denotes the number of state variables.

To get the sensitivity functions $\partial \mathbf{X}(s, \mathbf{p})/\partial p_i$ of the transfer functions $\mathbf{X}(s, \mathbf{p})$ between the chosen, single input $u_j(t)$ and the state variables $\mathbf{x}(t, \mathbf{p})$, one must first differentiate the system of algebraic equations

$$s\mathbf{X} = \mathbf{A}\,\mathbf{X} + \mathbf{B}\,\mathbf{U} \tag{2.15}$$

resulting from the Laplace transformation of the system (2.14) at zero initial conditions and for the Laplace transform of $u_j(t)$ equal to $\mathbf{1}(s)$.

The second important hint for computing the sensitivity functions $\dfrac{\partial |\mathbf{X}(s, \mathbf{p})|}{\partial p_i}$ of the amplitude-frequency characteristic is the relationship

$$\operatorname{Re}\left(\left.\frac{\partial \ln X}{\partial \ln p_i}\right|_{\mathbf{p}^0}\right) = \left.\frac{\partial \ln |X|}{\partial \ln p_i}\right|_{\mathbf{p}^0} = \left.\frac{\partial |X|}{\partial p_i}\right|_{\mathbf{p}^0} \frac{|p_i^0|}{|X^0|}. \tag{2.16}$$

To obtain the second derivatives of dynamical characteristics to the parameter, the corresponding equations should be differentiated twice.

3. Design of the experiment for identification of the cassette recorder drive

3.1. Mathematical model of the recorder drive

Modeling of the cassette-recorder drive is not a subject of the paper. We use here the model only to illustrate how the formulas presented in the first section can be applied in the modeling process at the stage of experiment design in order to formulate the identification problem properly.

The recorder drive consists of a magneto-electrical motor which drives a flying wheel through a viscous-elastic belt. The capstan, driving a magnetic tape, is fixed on the axis of the flying wheel.

The structure of the mathematical model – intended to describe the small oscillations disturbing the steady-state motion – has been determined on the basis of a careful analysis of physical phenomena and has the following form:

$$
\begin{aligned}
B_1 \frac{d\omega_1}{dt} &= R_1 s + m_1 \,, \\
B_2 \frac{d\omega_2}{dt} &= Ei - R_2 s + m_2 \,, \\
\frac{ds}{dt} &= K(E_2 R_2 \omega_2 - E_1 R_1 \omega_1) + C\left(E_2 R_2 \frac{d\omega_2}{dt} - E_1 R_1 \frac{d\omega_1}{dt}\right), \\
Ri + E\omega_2 &= u,
\end{aligned}
\tag{3.1}
$$

where

- ω_1 angular velocity of the flying wheel (and the capstan),
- ω_2 angular velocity of the motor rotor,
- s longitudinal axial force in the free sections of the driving belt,
- i electrical current in the rotor coil,
- u feeding voltage,
- m_1 exciting moment about the flying wheel axis,
- m_2 exciting moment about the rotor axis,
- B_1 moment of inertia of the flying wheel,
- B_2 moment of inertia of the motor rotor,
- R_1 radius of the driven pulley,
- R_2 radius of the driving pulley,
- K parameter representing stiffness of the viscous-elastic link between the flying wheel and the rotor,
- C parameter representing damping in the viscous-elastic link between the flying wheel and the rotor,

R resistance of the coil of the rotor,
E electro-mechanical constant of the rotor,
E_1 coefficient (close to 1) depending on the average value of relative elongation of the active section of the driving belt,
E_2 coefficient (close to 1) depending on the average value of relative elongation of the passive section of the driving belt.

All the variables in the model (3.1) are actually the varying components of the corresponding variables, the average values of which are by some scores greater than the amplitudes of oscillations.

3.2. The selection of amplitude-frequency characteristic for identification purposes

In general, the problem of experiment design for parameter identification can be stated as follows: which data points of measurable dynamical characteristics should be taken as the basis for parameter identification to reach the highest, or at least satisfactory accuracy of the parameter estimates.

In the case of the recorder drive, the only amplitude-frequency characteristic which was relatively easy to measure – without introducing additional masses into the system, what would change its original dynamics – was the characteristic of the transfer function

$$H(s) = \frac{\Omega_1(s)}{U(s)}, \tag{3.2}$$

where $\Omega_1(s)$, $U(s)$ denote Laplace transforms of $\omega_1(t)$ and $u(t)$, respectively.

Therefore the problem to solve at the stage of experiment design – what in the case of the system under consideration meant literally an arrangement of an experimental stand – was as follows: which physical parameters of the model can be identified with satisfactory accuracy from the measurable characteristic, and which should be measured in different ways.

Figure 1 shows the amplitude-frequency characteristic of the transfer function $H(s)$ calculated from the model (3.1) for the values of parameters determined with precision possible at this stage. In particular, moments of inertia were calculated from geometric dimensions, in case of B_2 – not very accurately due to irregular shape of the rotor. The values of electrical parameters were taken from the motor manufacturer specifications. The stiffness K and the damping coefficient C were calculated from the measured dynamic stiffness of the driving belt; the influence of structural damping between the pulleys and the belt, and the influence of flexibility of the belt sections being in touch with the pulleys on the values of parameters K and C was at this stage neglected.

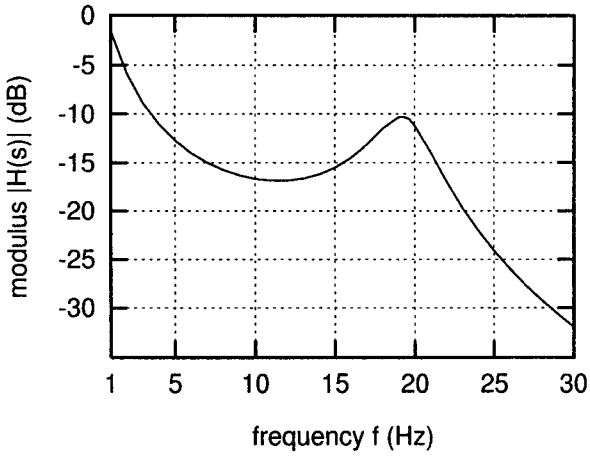

FIG. 1. Amplitude-frequency characteristic of the transfer function $H(s)$ between the input voltage $u(t)$ and the output angular velocity $\omega_1(t)$.

As mentioned above, in order to reach good accuracy of the parameter estimates, the parameters must influence the measured characteristics significantly. In Fig. 2 and 3, the plots of relative sensitivity of amplitude-frequency characteristic of the transfer function (3.2) to relative changes of the parameters are presented. Strictly speaking, these are the plots of the real part of the Bode sensitivity [2] denoted as $S_{p_i}^H$ and defined as follows:

$$S_{p_i}^H = \left.\frac{\partial \ln H}{\partial \ln p_i}\right|_{\mathbf{p}^0} = \left|\left.\frac{\partial H}{\partial p_i}\right|_{\mathbf{p}^0} \frac{p_i^0}{H^0}\right. . \tag{3.3}$$

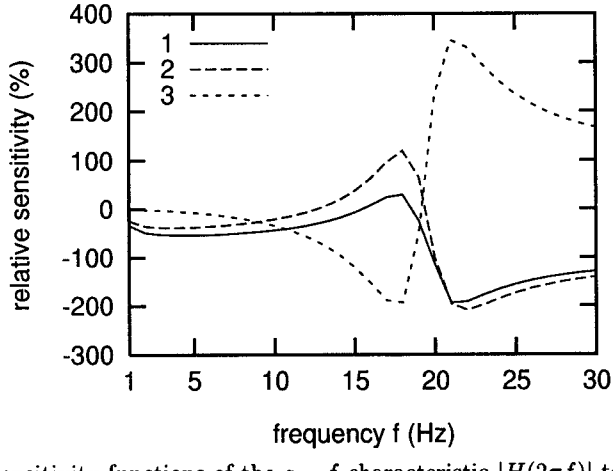

FIG. 2. Relative sensitivity functions of the $a - f$ characteristic $|H(2\pi f)|$ to the parameters: 1 - B_2, 2 - B_1, 3 - K.

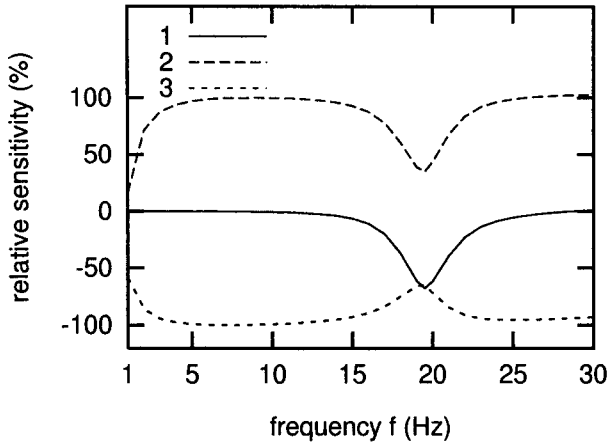

FIG. 3. Relative sensitivity functions of the $a - f$ characteristic $|H(2\pi f)|$ to the parameters: $1 - C$, $2 - E$, $3 - R$.

It was already established (see formula (2.16)) that the real part of $S_{p_i}^H$ is the relative sensitivity of the modulus $|H|$ to relative changes of the parameters.

From Figs. 2, 3 one can see that the chosen six parameters: B_1, B_2, K, C, R, E influence significantly the amplitude-frequency characteristic $|H(2\pi f)|$. The absolute values of the expression: $\dfrac{\partial|H|}{\partial p_i}\dfrac{p_i^0}{|H^0|}$ are quite large and there exists no obvious linear relationship between the sensitivity functions for different parameters. This may suggest that all these parameters can be identified simultaneously from the characteristic $|H(2\pi f)|$.

3.3. Identification accuracy versus the number m of parameters to be identified at the same time

The analysis of data given in the Table 1 leads to various conclusions. In the Table 1 are given the errors e_{p_i} calculated for 5 different vectors of parameters to be identified at the same time. Every row in the table corresponds to a particular case of the parameter vector $\mathbf{p}$ of a dimension $m = 2, \ldots, 6$. In all cases the identification criterion was in the form

$$K(\mathbf{p}) = \sum_{i=1}^{30} (h(f_i, \mathbf{p}) - z_i)^2 / z_i^2, \qquad f_i = 1, 2, \ldots, 30 \text{ (Hz)}. \tag{3.4}$$

Higher frequencies were not taken into account because the noise-to-output signal ratio increased very rapidly with the frequency and became too high for the input of a reasonable amplitude, within the range of linearity of the system.

Table 1. Values of the error estimates e_{p_i} for the assumed observations accuracy $d = 1\%$, for five different vectors p of parameters to be identified at the same time.

	parameters					
dimension m	B_1	B_2	K	C	E	R
6	matrix **G** is singular					
5	95	120	36	140	6.0	·
4	·	12	6.7	12	5.6	·
3	·	2.8	1.8	3.3	·	·
2	·	·	0.5	2.8	·	·

In the case of 6-dimensional vector $\mathbf{p} = (B_1, B_2, K, C, E, R)^T$ the matrix **G** is singular. There is no such a point in the 6-dimensional space of parameters, in which the functional (3.4) has a unique minimum. It is easy to observe the rule: the greater is the dimension m of the parameter vector, the greater become the identification errors. One can explain this as follows: for a greater number of parameters, the possibility of a minimum to be sharp is smaller.

The same rule was observed when checking the idea to include one or more parameters of the functional model of systematic observation errors into the vector **p** of parameters to be identified. The idea seemed to be promising at the beginning because it is easy to improve in this way the goodness-of-fit of the model characteristics to the experimental data. But this fact is misleading because the errors e_{p_i} increase significantly with increasing the dimension m.

3.4. Numerical experiments

Numerical experiments – performed not only on the model of the recorder drive but also on the simple harmonic oscillator and the classical model of the vehicle suspension consisting of two masses connected by springs and dampers – has shown that the errors e_{p_i} are quite good estimates of the semi-real errors. The results of some of the experiments are given in the Table 2. In the experiments the observations errors v_j were simulated, and then the parameter estimates $\hat{\mathbf{p}}^0$ were calculated by minimizing the sum (3.4) for observations: $z_j = h(f_j, \mathbf{p}^0) + v_j$. In the Table 2, the semi-real errors, defined as $\delta_{p_i} = 100\%|\hat{p}_i^0 - p_i^0|/p_i^0$, are compared with the estimates of the errors calculated by the formula (2.3). One can conclude that the errors e_{p_i} are quite useful as the estimates of real errors. Generally the maximum errors are greater than the semi-real ones, but not excessively greater; in the given examples, no more than several times. So by calculating the errors e_{p_i} one can learn at least the orders of real errors, and to know the order is much better than to know nothing at all.

Table 2. Results of numerical experiments, performed to compare the estimates e_{p_i} of errors of the parameter estimates $\hat{p}_i$ with the semi-real errors δ_{p_i} calculated for simulated (and hence known exactly) observations errors v_j.

No	simulated observations	type of estimates errors	errors of parameter estimates %		
			B_2	K	C
1	$z_j = h_j + 0.01h_j, \quad j = 1, \ldots, 30$	δ_{p_i}	0.419	1.47	3.07
		e_{p_i}	2.7	1.8	3.3
2	$z_j = h_j + 0.05h_j, \quad j = 1, \ldots, 30$	δ_{p_i}	13.1	7.18	15.4
		e_{p_i}	14	8.8	16
3	$z_j = h_j + 0.1h_j, \quad j = 1, \ldots, 30$	δ_{p_i}	22.1	14.5	31.2
		e_{p_i}	27	18	33
4	$z_j = h_j + 0.05h_j, \quad j = 1, \ldots, 30$	δ_{p_i}	·	0.541	10.3
		e_{p_i}	·	2.5	14
5	$z_j = h_j + 0.1h_j, \quad j = 1, \ldots, 30$	δ_{p_i}	·	1.08	19.2
		e_{p_i}	·	5.0	28

At the stage of the experiment design, when design means literally an arrangement of experimental stand, the observation vector $\mathbf{z}^0$ and parameter estimates $\hat{\mathbf{p}}^0$ are not known yet, so the errors e_{p_i} can be only computed for approximate values of parameters $\mathbf{p}^s$ and of the vector of observations $\mathbf{z}^s = (h_1(\mathbf{p}^s), \ldots, h_n(\mathbf{p}^s))^T$. As the algorithm for calculating e_{p_i} is very effective, it is even possible to calculate the errors for several points in the predicted, bounded volume in the parameter space, and compare the results. In most cases there exists some kind of continuity with respect to parameters, so the introductory analysis for $\mathbf{p}^0 = \mathbf{p}^s$ and observations vector $\mathbf{z}^0 = \mathbf{z}^s$ is reliable and useful.

In the case of the recorder drive, the approximate values of parameters, before identification, did not differ much from those calculated by minimizing the identification criterion, so the qualitative conclusions drawn from the introductory analysis of identification accuracy and from the final analysis, after completing the identification process, were the same.

4. Final remarks

Let us sum up the results derived and presented in the paper.

We have introduced here the quantity measure e_{p_i} of global sensitivity of the parameter estimate $\hat{p}_i$ to the errors of all observations z_j, $j = 1, \ldots, n$. In order

to emphasize the analogy to the notion of the maximum measurement error used in the analysis of indirect measurements, and also for brevity, we have called e_{p_i} the maximum percentage error.

We have presented all the information needed to write very effective computer programs for computing the errors e_{p_i}, in particular – for determining the derivatives of parameter estimates with respect to the observations (or observation errors), namely the derivatives $\partial\hat{p}_i/\partial z_j$, in the case of both deterministic and least-squares identification from the measured frequency- or time-domain dynamical characteristics. The fact that the method of computing the derivatives is an accurate one – in the sense that only round-off errors may exist, the truncation errors are not involved – is very important. In this case approximate methods – based on the notion of divided differences – would be very ineffective, and what is more important, very inaccurate even for functions much easier to compute and given in an explicit form. Approximate differentiation which may seem to be very easy, is actually very risky, and should be avoided when possible, even in case of functions much easier to compute and given in explicit forms [7]. The method of calculating the derivatives is an original contribution of the author. Of course, we cannot exclude the possibility that a similar method has been published by somebody else, but the method is not described in the textbooks or well-known basic books on identification.

There is an aspect of parameter identification accuracy analysis which should be mentioned here. It was not said explicitly, but the whole reasoning presented in the paper is based on the assumption that the dynamical structure of the model is the same as the structure of the real system or rather, on more realistic assumption, that differences in the structures can be neglected or included into the observations errors. If the differences are significant, such an analysis is unreliable. Of course, the method of analysis presented here is not a panacea, but it helps to avoid situations, when the identification problem is ill-posed despite the full consistency of the model and the object structure.

The other question worth to consider: one may ask whether it is necessary to use the least-squares method rather than the deterministic method in cases when random disturbances are filtered out due to the introductory analogue signal processing. On the other hand, everybody who deals with modeling of real objects knows well that modeling is a very complex process and very often it is necessary to use different methods and compare the results to validate a model. The accuracy of deterministic identification depends very much on the choice of arguments f_i of the data points (f_i, z_i). If the number of available data points is much larger than the number of parameters to be identified, it seems very useful to take all the points into account by using the least-squares method. Thus we get the parameter estimates and also examine the flexibility

of the model, i.e. the possibility of fitting the model to the data over the whole range of frequency. Moreover, there are many sources of systematic errors of experimental data, so it is reasonable to expect that their influence on the final results of parameter identification will be, at least in part, cancelled due to the least-squares minimization.

To conclude, let us consider what are the other possibilities of answering the question, whether the identification problem is well- or ill-posed. In the latter case, the parameter estimates are very sensitive to the measurement errors.

In the computer era, one possible approach to the problem, how the measurement errors contribute to the resulting errors of parameter estimates, consists in performing numerical experiments in which different possible measurement errors are simulated. Numerical experiments described in the previous section are an example of such an approach. When random errors are simulated, the techniques used in such investigations are often referred to as the Monte Carlo methods [4]. In case of systematic errors, one could even imagine, theoretically, the following procedure: 1 – adopt the volume in n-dimensional space of measurement errors, 2 – search in this volume, by numerical optimization, for the point $(v_1^0, \ldots, v_n^0)$ at which the largest resulting error of the parameter exists. Of course, in practice such numerical optimization would be computationally very difficult and time-consuming. In comparison with that, the approach proposed in the paper is very convenient, and seems to be sufficient.

Another approach to the problem is using singular value decomposition (SVD) as a tool of analysis. At the final step of recursive computations of parameter estimates, the identification problem usually reduces to the problem of solving a set of linear algebraic equations. Such a case takes place, for example, when the identification criterion is being minimized by the Gauss–Newton method. SVD is a useful tool to decide whether a matrix is well- or ill-conditioned. If the relevant matrix is ill-conditioned, then small changes on the right-hand side of equations, caused by small changes in measurements, result in large changes in estimates, so the estimates cannot be accurate. Different possibilities of using SVD in the solution and analysis of inverse problems, in particular – the nonlinear least-squares fitting, have been explored for the last several years by several researchers. However, many practitioners still neglect any kind of accuracy analysis and they are satisfied if they get a good fitting of the model characteristics to the experimental data, but quality of a fit cannot be a measure of accuracy of the parameter estimates, so "they get what they deserve [4]".

The present author believes that the method for error analysis, presented in the paper, can be used optionally or simultaneously with the analysis involving SVD. The errors e_{p_i} may be also regarded as the indices of conditioning of the identification problem. The advantage of these indices over another indices – e.g.

the frequently used ratio of minimum to maximum of the absolute values of the eigenvalues of a matrix – is that they have also some additional meaning, familiar to everybody who has dealt with indirect measurements.

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