Research Paper

Analysis of Euler-Bernoulli Beams with Arbitrary Boundary Conditions on Winkler Foundation Using a B-Spline Collocation Method

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Structural beams are important parts of engineering projects. The structural analysis of beams is required to ensure that they provide the specifics needed to prevent and withstand failure. Therefore, the numerical solution to analyze an Euler-Bernoulli beam with arbitrary boundary conditions using sextic B-spline method is presented in this paper. A direct modeling technique is applied for modeling the Euler-Bernoulli beam with arbitrary boundary conditions on an elastic Winkler foundation. For this purpose, the effect of the translational along with rotational support, the type of beam supports and the elastic coefficient of Winkler foundation are assessed. Finally, some numerical examples are shown to present the efficiency of the sextic B-spline collocation method. To validate the analysis of the Euler-Bernoulli beam with the presented method, the results of B-spline collocation method are compared with the results of the analytical method and the integrated finite element analysis of structures (SAP2000).

Key words: Euler-Bernoulli beam, arbitrary boundary conditions, Winkler foundation, B-spline collocation method.

1. Introduction

Many geotechnical engineering problems can be studied by analyzing beams on foundations. The various foundation models such as Winkler, Pasternak, Kerr, Vlasov, Hetenyi and viscoelastic are applied in the analysis of structures on elastic foundations [1]. Among these models, the Winkler foundation model is the most common model used in such analyses. However, the modelling of soil using the Winkler approach is inadequate in the handling of the various problems [2]. The main weakness of the Winkler model lies in the fact that it neglects the shear interaction between the spring elements [3].

Analysis of statically indeterminate beams is an important problem in civil engineering. But this analysis is sometimes difficult or impossible if the degree of static indeterminacy in the beam is high. Analysis of a beam is used to determine the values of deflection, slope, shear force and bending moment. The fourth-
(or fifth-) order differential equations must be solved to obtain the displacement. The differential equations of the Euler-Bernoulli beam on the uniform elastic foundation are as follows:

- for uniformly distributed load:
  \[
  y^{(4)}(x) + \frac{K}{EI} y(x) - \frac{q(x)}{EI} = 0, \quad x \in [a, b],
  \]

- for linearly distributed load:
  \[
  y^{(5)}(x) + \frac{K}{EI} y'(x) - \frac{dq(x)}{dx} \cdot \frac{1}{EI} = 0, \quad x \in [ab],
  \]

where \(y(x)\) is the transverse deflection of the mid-surface of the Euler-Bernoulli beam and \(q(x)\) is the external force function on the beam. In addition, \(I\), \(E\) and \(K\) are the second moment of area, the Young’s modulus of elasticity, and the elastic coefficient of Winkler foundation, respectively. In the Euler-Bernoulli beam theory, the boundary conditions are given below:

\[
\forall t \exists x = a : M(a) = K_{RL} \theta(a), \quad Q(a) = -K_{TL} w(a),
\]

\[
\forall t \exists x = b : M(b) = -K_{RR} \theta(b), \quad Q(b) = K_{TR} w(b),
\]

where \(M\) and \(Q\) are the bending moment and the shear force, respectively (Fig. 1) [4]. \(K_{TL}, K_{TR}, K_{RL}\) and \(K_{RR}\) are the transverse and rotational elastic coefficients at the supports at the left and right boundary ends, respectively. For example, the boundary condition of the simple supports on both sides associated with a uniformly distributed load can be defined as

\[
y(a) = 0, \quad y(b) = 0, \quad y''(a) = 0, \quad y''(b) = 0, \quad y^{(5)}(a) + \frac{K}{EI} y'(a) = 0.
\]

\[
\begin{align*}
&\text{Fig. 1. Sign convention for shear forces, bending moments and slopes of the Euler-Bernoulli beam.}
\end{align*}
\]

Also, the boundary condition of the simple supports on both sides associated with a linearly distributed load is defined as

\[
y(a) = 0, \quad y(b) = 0, \quad y''(a) = 0, \quad y''(b) = 0, \quad y^{(4)}(a) + \frac{K}{EI} y(x) = q(a).
\]

In this paper, the collocation method based on sextic \(B\)-spline is applied to analyze the Euler-Bernoulli beam with arbitrary boundary conditions. A spline
function is the piecewise polynomial function of degree \( n \). This function is the composite of several internal points. On the other hand, the number points must equal or be greater than \((k-1)\) degree. The differential equations with \( k \) degree are solved by \( B \)-spline functions of \((k+1)\) degree [5]. Over the years, the spline method has been used for solving the differential system of equations with different boundary conditions. For example, the sextic spline function for the solution of second-order boundary value problems associated with unilateral, obstacle and contact problems is presented by Rashidiania et al. [6]. Their results show that the approximate solutions obtained using the present method are better than spline and finite difference methods. A quintic non-polynomial spline method is investigated by Ramadan et al. for the numerical solution of the fourth-order two-point boundary value problems [7]. Based on their findings, the quintic non-polynomial spline method presents better approximations and generalizes all the existing polynomial spline methods up to fourth order. The natural frequencies of the non-uniform Euler-Bernoulli beam on elastic foundation are obtained using the spline collocation method by Hsu [8]. The Kuramoto-Sivashinsky equation is solved using septic \( B \)-spline collocation method by Zarebnia and Parvaz [9]. The solution is approximated as the linear combination of the septic \( B \)-spline functions. It is shown that this method is unconditionally stable by applying the von-Neumann stability analysis technique. Zarebnia and Parvaz presented the cubic \( B \)-spline collocation method for the numerical solution of the problem arising from chemical reactor theory [10]. Mohammadi developed a numerical method based on sextic \( B \)-spline to solve the fourth-order time-dependent partial differential equations [11]. In this paper, the convergence analysis of the sextic \( B \)-spline approximation for the Euler-Bernoulli beams with fixed and cantilever boundary conditions is discussed in detail. Reali and Gomez introduced an isogeometric analysis collocation method for the solution of the Bernoulli-Euler beam and Kirchhoff plate [12]. Akram also used the sextic spline method for solving a system of fifth-order boundary value problems [13].

In the previous studies, the Euler-Bernoulli beam on an arbitrary variable elastic Winkler foundation was not analyzed using the \( B \)-spline collocation method. On the other hand, these solutions can be generalized only to simple boundary conditions. In the present study, the solution using the sextic \( B \)-spline method is introduced to analyze the Euler-Bernoulli beam with arbitrary boundary conditions on the partial Winkler foundation. Furthermore, the analysis of the Euler-Bernoulli beam is written in a general form. Therefore, the objective of this paper is:

- To present a simple and practical numerical technique for determining the response of Euler-Bernoulli beams with elastically restrained boundary conditions, resting on a partial Winkler foundation.
To state numerical solutions using the sextic B-spline function for an analysis of the beam with and without the partial Winkler foundation.

This paper is structured as follows. Section 2 outlines the sextic B-spline collocation method. Then, in Sec. 3, the numerical solution of the differential equation of the Euler-Bernoulli beam on uniform foundation is developed using the B-spline method. Section 4 presents some numerical examples to illustrate the efficiency of the presented method. Finally, in Sec. 5, brief conclusions are drawn.

## 2. Definition of B-spline curve

Let \( x = (x_0, x_1, \ldots, x_N) \) be a knot vector. A B-spline function of \( k \)-degree is defined as [14]

\[
B^0_i(x) = \begin{cases} 
1 & \text{for } x \in [x_i, x_{i+1}), \\
0 & \text{otherwise,}
\end{cases}
\]

\[
B^k_i(x) = \frac{x - x_i}{x_{i+k} - x_i} B^{k-1}_i(x) + \frac{x_{i+k+1} - x}{x_{i+k+1} - x_{i+1}} B^{k-1}_{i+1}(x),
\]

where \( 0 \leq i \leq N - k - 1 \) and \( 1 \leq k \leq N - 1 \).

Sextic B-spline can be obtained by calculating the B-spline basis function up to sixth order using Eq. (2.2). Therefore, the sextic B-spline basis function \( B^6_i(x) \) is as follows:

\[
(2.3) \quad B^6_i(x) = \frac{1}{h^6} \begin{cases} 
(x - x_i + 3h)^6, & x \in [x_{i-3}, x_{i-2}], \\
(x - x_i + 3h)^6 - 7(x - x_i + 2h)^6, & x \in [x_{i-2}, x_{i-1}], \\
(x - x_i + 3h)^6 - 7(x - x_i + 2h)^6 + 21(x - x_i + h)^6, & x \in [x_{i-1}, x_i], \\
(x - x_i + 3h)^6 - 7(x - x_i + 2h)^6 + 21(x - x_i + h)^6 - 35(x - x_i)^6, & x \in [x_i, x_{i+1}], \\
(x - x_i - 4h)^6 - 7(x - x_i - 3h)^6 + 21(x - x_i - 2h)^6, & x \in [x_{i+1}, x_{i+2}], \\
(x - x_i - 4h)^6 - 7(x - x_i - 3h)^6, & x \in [x_{i+2}, x_{i+3}], \\
(x - x_i - 4h)^6, & x \in [x_{i+3}, x_{i+4}], \\
0, & \text{otherwise.}
\end{cases}
\]
In this paper, the solution domain \( a \leq x \leq b \) is divided into \( N \) segments with a uniform length of \( h = \frac{b-a}{N} \) at the knots \( x_i \) where \( i = 0, 1, 2, \ldots, N \) and \( x_{i+1} = x_i + h \) such that \( a = x_0 < x_1 < \ldots < x_N = b \). In the sextic \( B \)-spline, basis function is defined as follows:

\[
y(x) = \sum_{i=0}^{N+5} c_i B_i(x),
\]

where \( B_0(x), \ldots, B_{N+5}(x) \) are the sextic \( B \)-splines functions at the knots and are given by Eq. (2.3). \( c_0, \ldots, c_{N+5} \) are unknown real coefficients that are determined by satisfying the boundary conditions at each end of the beam and the continuity conditions of displacement, slope and moment along with the shear force and the collocation form of the differential Eqs. (1.1) and (1.2). Also, first, second, third, fourth and fifth derivatives of \( B_i \) with respect to variable \( x \) are used to solve the fifth-order differential equation. Values of \( B_i \) and its derivatives at the nodal points are given in Table 1.

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>( x_{i+1} )</th>
<th>( x_{i+2} )</th>
<th>( x_{i+3} )</th>
<th>( x_{i+4} )</th>
<th>( x_{i+5} )</th>
<th>( x_{i+6} )</th>
<th>( x_{i+7} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_i )</td>
<td>0</td>
<td>( \frac{1}{720} )</td>
<td>( \frac{57}{720} )</td>
<td>( \frac{302}{720} )</td>
<td>( \frac{302}{720} )</td>
<td>( \frac{57}{720} )</td>
<td>( \frac{1}{720} )</td>
</tr>
<tr>
<td>( B'_i )</td>
<td>0</td>
<td>( \frac{6}{720h} )</td>
<td>( \frac{150}{720h} )</td>
<td>( \frac{240}{720h} )</td>
<td>( \frac{240}{720h} )</td>
<td>( \frac{150}{720h} )</td>
<td>( \frac{6}{720h} )</td>
</tr>
<tr>
<td>( B''_i )</td>
<td>0</td>
<td>( \frac{30}{720h^2} )</td>
<td>( \frac{270}{720h^2} )</td>
<td>( \frac{-300}{720h^2} )</td>
<td>( \frac{-300}{720h^2} )</td>
<td>( \frac{270}{720h^2} )</td>
<td>( \frac{30}{720h^2} )</td>
</tr>
<tr>
<td>( B'''_i )</td>
<td>0</td>
<td>( \frac{120}{720h^3} )</td>
<td>( \frac{120}{720h^3} )</td>
<td>( \frac{-960}{720h^3} )</td>
<td>( \frac{-960}{720h^3} )</td>
<td>( \frac{-120}{720h^3} )</td>
<td>( \frac{-120}{720h^3} )</td>
</tr>
<tr>
<td>( B^{(4)}_i )</td>
<td>0</td>
<td>( \frac{360}{720h^4} )</td>
<td>( \frac{-1080}{720h^4} )</td>
<td>( \frac{720}{720h^4} )</td>
<td>( \frac{720}{720h^4} )</td>
<td>( \frac{-1080}{720h^4} )</td>
<td>( \frac{360}{720h^4} )</td>
</tr>
<tr>
<td>( B^{(5)}_i )</td>
<td>0</td>
<td>( \frac{720}{720h^5} )</td>
<td>( \frac{-3600}{720h^5} )</td>
<td>( \frac{7200}{720h^5} )</td>
<td>( \frac{-7200}{720h^5} )</td>
<td>( \frac{3600}{720h^5} )</td>
<td>( \frac{-720}{720h^5} )</td>
</tr>
</tbody>
</table>

3. Construction of the proposed solution

By substituting Eq. (2.4) into Eqs. (1.1) and (1.2), equations yield as follows:

\[
\sum_{i=0}^{N+5} c_i B^{(4)}_i(x_j) + \frac{K}{EI} \sum_{i=0}^{N+5} c_i B_i(x) - \frac{q(x)}{EI} = 0, \quad \text{for} \ j = 0, 1, \ldots, N,
\]
• for linearly distributed load:

\[
(3.2) \quad \sum_{i=0}^{N+5} c_i B_i^{(5)}(x_j) + \frac{K}{EI} \sum_{i=0}^{N+5} c_i B_i'(x) - \frac{dq(x)}{dx} \cdot \frac{1}{EI} = 0, \quad \text{for } j = 0, 1, \ldots, N.
\]

From Table 1 and Eq. (2.4), \(y_i\), \(y_i'\), \(y_{i''}\), \(y_{i}^{(4)}\) and \(y_{i}^{(5)}\) are obtained as follows:

\[
(3.3) \quad y_i = \frac{1}{720} \left( c_i + 57c_{i+1} + 302c_{i+2} + 302c_{i+3} + 57c_{i+4} + c_{i+5} \right),
\]

\[
(3.4) \quad y_i' = \frac{1}{720h} \left( 6c_i + 150c_{i+1} + 240c_{i+2} - 240c_{i+3} - 150c_{i+4} - 6c_{i+5} \right),
\]

\[
(3.5) \quad y_{i}'' = \frac{1}{720h^2} \left( 30c_i + 270c_{i+1} - 300c_{i+2} - 300c_{i+3} + 270c_{i+4} + 30c_{i+5} \right),
\]

\[
(3.6) \quad y_{i}''' = \frac{1}{720h^3} \left( 120c_i + 120c_{i+1} - 960c_{i+2} + 960c_{i+3} - 120c_{i+4} - 120c_{i+5} \right),
\]

\[
(3.7) \quad y_{i}^{(4)} = \frac{1}{720h^4} \left( 360c_i - 1080c_{i+1} + 720c_{i+2} + 720c_{i+3} - 1080c_{i+4} + 360c_{i+5} \right),
\]

\[
(3.8) \quad y_{i}^{(5)} = \frac{1}{720h^5} \left( 720c_i - 3600c_{i+1} + 7200c_{i+2} - 7200c_{i+3} + 3600c_{i+4} - 720c_{i+5} \right).
\]

\(y_i\), \(y_i'\), \(EIy_i''\), and \(EIy_i'''\) can be stated as displacement, slope, bending moment and shear force in the beam, respectively. Substituting Eq. (3.7) into Eq. (3.1), for uniformly distributed load, results in:

\[
(3.9) \quad \frac{1}{720h^4} \left( 360c_j - 1080c_{j+1} + 720c_{j+2} + 720c_{j+3} - 1080c_{j+4} + 360c_{j+5} \right) \\
+ \frac{K}{720EI} \left( c_i + 57c_{i+1} + 302c_{i+2} + 302c_{i+3} + 57c_{i+4} + c_{i+5} \right) - \frac{q(x)}{EI} = 0, \quad i \text{ or } j = 0, \ldots, N.
\]

The above solution for uniformly distributed load can be written in the form of:

\[
(3.10) \quad \frac{1}{720} \left( \left( \frac{360}{h^4} + \frac{k}{EI} \right) c_j + \left( - \frac{1080}{h^4} + \frac{57k}{EI} \right) c_{j+1} + \left( \frac{720}{h^4} + \frac{302k}{EI} \right) c_{j+2} \right) \\
+ \left( \frac{720}{h^4} + \frac{302k}{EI} \right) c_{j+3} + \left( - \frac{1080}{h^4} + \frac{57k}{EI} \right) c_{j+4} \\
+ \left( \frac{360}{h^4} + \frac{k}{EI} \right) c_{j+5} \right) - \frac{q(x)}{EI} = 0, \quad i \text{ or } j = 0, \ldots, N.
\]
Similarly, it is possible to develop the solution for the linearly distributed load by substituting Eq. (3.8) into Eq. (3.2):

\[
\frac{1}{720h^5} (720c_j - 3600c_{j+1} + 7200c_{j+2} - 7200c_{j+3} + 3600c_{j+4} - 720c_{j+5}) \\
+ \frac{K}{720hEI} (6c_i + 150c_{i+1} + 240c_{i+2} - 240c_{i+3} - 150c_{i+4} - 6c_{i+5}) \\
- \frac{dq(x)}{dx} \cdot \frac{1}{EI} = 0, \quad i \text{ or } j = 0, 1, \ldots, N.
\]

By simplifying the above solution, the solution for linearly distributed load can be rewritten as follows:

\[(3.11) \quad \frac{1}{720} \left( \left( \frac{720}{h^5} + \frac{6k}{hEI} \right) c_j + \left( -\frac{3600}{h^5} + \frac{150k}{hEI} \right) c_{j+1} \right) \\
+ \left( \frac{7200}{h^5} + \frac{240k}{hEI} \right) c_{j+2} - \left( \frac{7200}{h^5} + \frac{240k}{hEI} \right) c_{j+3} + \left( \frac{3600}{h^5} - \frac{150k}{hEI} \right) c_{j+4} \\
- \left( \frac{720}{h^5} + \frac{6k}{hEI} \right) c_{j+5} ) - \frac{dq(x)}{dx} \cdot \frac{1}{EI} = 0, \quad i \text{ or } j = 0, 1, \ldots, N.
\]

The systems (3.10) and (3.11) consist of \(N + 1\) equations in the \(N + 6\) unknowns \(\{c_0, c_j, \ldots, c_{N+5}\}\). Thus, the five equations are needed at this stage. Therefore, the boundary conditions are used to obtain these extra equations. Four extra equations are explicitly obtained using two boundary conditions at each end of the beam depending on the type of end support and one extra equation for uniformly distributed load is given below:

\[
\frac{1}{720h^5} (720c_0 - 3600c_1 + 7200c_2 - 7200c_3 + 3600c_4 - 720c_5) \\
+ \frac{K}{720hEI} (6c_0 + 150c_1 + 240c_2 - 240c_3 - 150c_4 - 6c_5) = 0.
\]

The above solution for uniformly distributed load can be rewritten as

\[(3.12) \quad \frac{1}{720} \left( \left( \frac{720}{h^5} + \frac{6k}{hEI} \right) c_0 + \left( -\frac{3600}{h^5} + \frac{150k}{hEI} \right) c_1 + \left( \frac{7200}{h^5} + \frac{240k}{hEI} \right) c_2 \right) \\
- \left( \frac{720}{h^5} + \frac{240k}{hEI} \right) c_3 + \left( \frac{3600}{h^5} - \frac{150k}{hEI} \right) c_4 - \left( \frac{720}{h^5} + \frac{6k}{hEI} \right) c_5 ) = 0.
\]

Also, one extra equation for linearly distributed load is obtained as

\[
\frac{1}{720h^4} (360c_0 - 1080c_1 + 720c_2 + 720c_3 - 1080c_4 + 360c_5) \\
+ \frac{K}{720EI} y(x) (c_i + 57c_{i+1} + 302c_{i+2} + 302c_{i+3} + 57c_{i+4} + c_{i+5}) = q(a).
\]
By simplifying the above solution, the solution for linearly distributed load can be given as follows:

\[
\begin{align*}
(3.13) \quad & \frac{1}{720} \left( \left( \frac{360}{h^4} + \frac{k}{EI} \right) c_0 + \left( -\frac{1080}{h^4} + \frac{57k}{EI} \right) c_1 + \left( \frac{720}{h^4} + \frac{302k}{EI} \right) c_2 
\right) \\
& + \left( \frac{720}{h^4} + \frac{302k}{EI} \right) c_3 + \left( -\frac{1080}{h^4} + \frac{57k}{EI} \right) c_4 + \left( \frac{360}{h^4} + \frac{k}{EI} \right) c_5 
\right) - \frac{q(x)}{EI} = 0.
\end{align*}
\]

In addition, the continuity conditions of displacement, slope and moment along with the shear force in the vicinities of the different segment connections are defined as [4]

\[
\begin{align*}
(3.14) \quad & Y(a) = y(a), \\
& \theta(a) = y'(a), \\
& M(a) = EIy''(a), \\
& V(a) = EIy'''(a).
\end{align*}
\]

By applying the relationships between the individual physical quantities and the \(B\)-spline function, the continuity conditions at the first and last knot (the end knots) can be rewritten as follows:

\[
\begin{align*}
(3.15) \quad & Y(a) = \frac{1}{720} \left( c_0 + 57c_1 + 302c_2 + 302c_3 + 57c_4 + c_5 \right), \\
& \theta(a) = \frac{1}{720h} \left( 6c_0 + 150c_1 + 240c_2 - 240c_3 - 150c_4 - 6c_5 \right), \\
& M(a) = \frac{EI}{720h^2} \left( 30c_0 + 270c_1 - 300c_2 - 300c_3 + 270c_4 + 30c_5 \right), \\
& V(a) = \frac{EI}{720h^3} \left( 120c_0 + 120c_1 - 960c_2 + 960c_3 - 120c_4 - 120c_5 \right),
\end{align*}
\]

and

\[
\begin{align*}
(3.16) \quad & Y(b) = \frac{1}{720} \left( c_N + 57c_{N+1} + 302c_{N+2} + 302c_{N+3} + 57c_{N+4} + c_{N+5} \right), \\
& \theta(b) = \frac{1}{720h} \left( 6c_N + 150c_{N+1} + 240c_{N+2} - 240c_{N+3} - 150c_{N+4} - 6c_{N+5} \right), \\
& M(b) = \frac{EI}{720h^2} \left( 30c_N + 270c_{N+1} - 300c_{N+2} - 300c_{N+3} + 270c_{N+4} + 30c_{N+5} \right), \\
& V(b) = \frac{EI}{720h^3} \left( 120c_N + 120c_{N+1} - 960c_{N+2} + 960c_{N+3} - 120c_{N+4} - 120c_{N+5} \right),
\end{align*}
\]
where \( Y, \theta, M, \) and \( V \) are displacement, slope, bending moment, and shear force of the Euler-Bernoulli beam, respectively. Finally, the matrix equation is given as

\[
(A) \times [C] = [F],
\]

where the coefficient matrix \( A \), matrix \( C \) and the load matrix \( F \) are cited in the appendix.

4. Numerical examples

To validate the sextic \( B \)-spline method, the results of different examples are presented. First, the high computational efficiency of the method is shown and then it is examined for the feedback with arbitrary boundary conditions. In all the examples, \( E \) and \( I \) are assumed as

\[
E = 2038901.91 \text{ kg} \text{ cm}^2, \quad I = 6572.4175 \text{ cm}^4.
\]

4.1. Euler-Bernoulli beam under uniformly distributed load

For the purpose of verification of the presented method, the Euler-Bernoulli beam with translational restraint supported under a uniformly distributed load is considered (Fig. 2). The beam is assumed to have the following characteristics:

\[
q = 15 \text{ kg cm}^{-1}, \quad k = 2500 \text{ kg cm}^{-1}, \quad L = 500 \text{ cm}.
\]

![Fig. 2. Euler-Bernoulli beam under with translational restraint supported under uniformly distributed load.](image)

Analytical solution of the displacement, slope, shear force, and bending moment of the Euler-Bernoulli beam with the translational restraint supported under a uniformly distributed load can be determined as
\[ Y(x) = \frac{1}{EI} (-0.625X^4 + 625X^3 - 78125000x - 1.5EI), \]
\[ \theta(x) = \frac{1}{EI} (-2.5X^3 + 1875X^2 - 78125000), \]
\[ V(x) = 3750 - 15x, \quad M(x) = -7.5x^2 + 3750x. \]

Table 2 compares the values of the displacement, slope, bending moment, shear force of the Euler-Bernoulli beam with translational restraint supported under a uniformly distributed load. It can be seen that the results are fairly close. The maximum difference of the obtained results is approximately 0.0004%.

<table>
<thead>
<tr>
<th>Location [cm]</th>
<th>Displacement [cm]</th>
<th>Slope [rad]</th>
<th>Shear force [kg]</th>
<th>Bending moment [kg/cm]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Analytical solution</td>
<td>B-spline function</td>
<td>Analytical solution</td>
<td>B-spline function</td>
</tr>
<tr>
<td>0</td>
<td>-1.50000</td>
<td>-1.50000</td>
<td>-0.00583</td>
<td>-0.00583</td>
</tr>
<tr>
<td>50</td>
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<td>-1.78596</td>
<td>-0.00550</td>
<td>-0.00550</td>
</tr>
<tr>
<td>100</td>
<td>-2.04102</td>
<td>-2.04102</td>
<td>-0.00462</td>
<td>-0.00462</td>
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<tr>
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<td>-2.24070</td>
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<td>-0.00331</td>
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<tr>
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<td>-2.36750</td>
<td>-2.36750</td>
<td>-0.00173</td>
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<td>-2.36750</td>
<td>0.00173</td>
<td>0.00173</td>
</tr>
<tr>
<td>350</td>
<td>-2.24070</td>
<td>-2.24070</td>
<td>0.00331</td>
<td>0.00331</td>
</tr>
<tr>
<td>400</td>
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<td>-2.04102</td>
<td>0.00462</td>
<td>0.00462</td>
</tr>
<tr>
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<td>-1.78596</td>
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<td>0.00550</td>
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<td>-1.50000</td>
<td>0.00583</td>
<td>0.00583</td>
</tr>
</tbody>
</table>

4.2. Euler-Bernoulli beam under linearly and uniformly distributed load

In order to illustrate the accuracy of the presented method, the Euler-Bernoulli beam with arbitrary boundary conditions under uniformly and linearly distributed load is considered (Fig. 3). Analytical solution of the displace-
ment, slope, shear force, and bending moment of the Euler-Bernoulli beam with translational restraint supported under a uniformly distributed load can be determined as

\[
y(x) = \frac{1}{EI} \begin{cases} 
-1.25x^4 + \frac{80875}{18}x^3 - 4864583.334x^2 & 0 \leq x \leq 500 \text{ cm}, \\
-1.25x^4 + \frac{80875}{18}x^3 - 4864583.334x^2 \\
+ 0.30100763EIx - 150.503816EI & 500 \leq x \leq 850 \text{ cm}, \\
1173958.334x^2 + 0.03255167EI \cdot x \\
+ 90.67052EI & 850 \leq x \leq 1350 \text{ cm}, \\
- \frac{1}{1050}x^5 + \frac{905}{168}x^4 - \frac{82125}{7}x^3 + 13215476.19x^2 \\
- 0.382741EI \cdot x - 246.48704EI & 1350 \leq x \leq 1700 \text{ cm}, \\
\end{cases}
\]

\[
\theta(x) = \frac{1}{EI} \begin{cases} 
-5x^3 + \frac{80875}{6}x^2 - 9729166.667x & 0 \leq x \leq 500 \text{ cm}, \\
-5x^3 + \frac{80875}{6}x^2 - 9729166.667x \\
+ 0.30100763EI & 500 \leq x \leq 850 \text{ cm}, \\
2347916.667x + 0.03255167EI \\
- \frac{1}{210}x^4 + \frac{905}{42}x^3 - \frac{492750}{14}x^2 \\
+ 26430952.381x - 0.382741EI & 850 \leq x \leq 1350 \text{ cm}, \\
\end{cases}
\]

\[
M(x) = \begin{cases} 
-15x^2 + \frac{80875}{3}x - 9729166.667 & 0 \leq x \leq 850 \text{ cm}, \\
2347916.667 \\
- \frac{2}{105}x^3 + \frac{905}{14}x^2 - \frac{492750}{7}x + 26430952.381 & 850 \leq x \leq 1350 \text{ cm}, \\
\end{cases}
\]

\[
V(x) = \begin{cases} 
-30x + \frac{80875}{3} & 0 \leq x \leq 850 \text{ cm}, \\
0 & 850 \leq x \leq 1350 \text{ cm}, \\
- \frac{2}{35}x^2 + \frac{905}{7}x - \frac{492750}{7} & 1350 \leq x \leq 1700 \text{ cm}.
\end{cases}
\]
Table 3 presents the values of the displacement, slope, bending moment, shear force of the Euler-Bernoulli beam with arbitrary boundary conditions under uniformly and linearly distributed load. It can be seen that the results are fairly close. The maximum difference of the obtained results is approximately 0.00005%.

Table 3. The values of the displacement, slope, bending moment, shear force of the Euler-Bernoulli beam under uniformly and linear distributed load – arbitrary boundary conditions case.

<table>
<thead>
<tr>
<th>Location [cm]</th>
<th>Displacement [cm]</th>
<th>Angle [rad]</th>
<th>Shear [kg]</th>
<th>Bending moment [kg/cm]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Analytical solution</td>
<td>B-spline function</td>
<td>Analytical solution</td>
<td>B-spline function</td>
</tr>
<tr>
<td>0</td>
<td>0.00</td>
<td>5.92E-15</td>
<td>0.00000</td>
<td>1.4E-13</td>
</tr>
<tr>
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<td>26958.33</td>
<td>26958.33</td>
<td>23958.33</td>
<td>23958.34</td>
</tr>
<tr>
<td>100</td>
<td>−3.30418</td>
<td>−3.30418</td>
<td>−0.06292</td>
<td>−0.06292</td>
</tr>
<tr>
<td>200</td>
<td>−11.98752</td>
<td>−11.98750</td>
<td>−0.10796</td>
<td>−0.10796</td>
</tr>
<tr>
<td>300</td>
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<td>−24.37410</td>
<td>−0.13735</td>
<td>−0.13735</td>
</tr>
<tr>
<td>400</td>
<td>−39.01177</td>
<td>−39.01180</td>
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<td>−0.15335</td>
</tr>
<tr>
<td>500</td>
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<td>−54.67240</td>
<td>−0.15819</td>
<td>−0.15819</td>
</tr>
<tr>
<td>600</td>
<td>−60.25101</td>
<td>−60.25100</td>
<td>0.14691</td>
<td>0.14691</td>
</tr>
<tr>
<td>700</td>
<td>−25.06780</td>
<td>−25.06780</td>
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</tr>
<tr>
<td>800</td>
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<td>−8.56622</td>
<td>0.17290</td>
<td>0.17290</td>
</tr>
<tr>
<td>850</td>
<td>0.29166</td>
<td>0.29167</td>
<td>0.18148</td>
<td>0.18148</td>
</tr>
<tr>
<td>900</td>
<td>9.586411</td>
<td>9.58473</td>
<td>0.19024</td>
<td>0.19024</td>
</tr>
<tr>
<td>1000</td>
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<td>−29.48493</td>
<td>0.20776</td>
<td>0.20776</td>
</tr>
<tr>
<td>1100</td>
<td>51.13893</td>
<td>51.13725</td>
<td>0.22528</td>
<td>0.22528</td>
</tr>
<tr>
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<td>74.54167</td>
<td>0.24280</td>
<td>0.24281</td>
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<tr>
<td>1300</td>
<td>99.69899</td>
<td>99.69820</td>
<td>0.26033</td>
<td>0.26033</td>
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<tr>
<td>1350</td>
<td>112.93520</td>
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</tr>
<tr>
<td>1350</td>
<td>−103.3721</td>
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<td>0.26909</td>
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<tr>
<td>1400</td>
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<tr>
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<td>−30.99830</td>
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<td>0.30664</td>
</tr>
<tr>
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<td>0.00</td>
<td>2.06E-16</td>
<td>0.31175</td>
<td>0.31175</td>
</tr>
</tbody>
</table>

4.3. Indeterminate beam under uniformly and linearly distributed load

The indeterminate beam under uniformly and linearly distributed load with spring supports is evaluated. The beam characteristics are shown in Fig. 4. Analytical solution of the displacement, slope, shear force, and bending moment
of the indeterminate beam under uniformly and linearly distributed load with springs supports can be determined as

\[
Y(x) = \frac{1}{EI} \begin{cases} 
-\frac{5}{6}x^4 + 988.77103x^3 \\
-262393786.2x - 2.96631309EI 
\end{cases} \quad 0 \leq x \leq 800 \text{ cm},
\]

\[
\theta(x) = \frac{1}{EI} \begin{cases} 
-\frac{10}{3}x^3 + 2966.31309x^2 - 262393786.2 \\
-\frac{1}{1280}x^4 - \frac{5}{6}x^3 + 11033.68691x^2 \\
-16107798.112x + 0.4851101EI 
\end{cases} \quad 800 \leq x \leq 1600 \text{ cm},
\]

\[
M(x) = \begin{cases} 
5932.62618x - 10x^2 \\
-0.003125x^3 - 2.5x^2 + 22067.37382x \\
-16107798.112 
\end{cases} \quad 800 \leq x \leq 1600 \text{ cm},
\]

\[
V(x) = \begin{cases} 
5932.62618 - 20x \\
-0.009375x^2 - 5x + 22067.37382 
\end{cases} \quad 800 \leq x \leq 1600 \text{ cm}.
\]

Table 4 presents the values of the displacement, slope, bending moment, shear force of the beam under uniformly and linearly distributed load.
Table 4. The values of the displacement, slope, bending moment, shear force of the indeterminate beam under uniformly and linearly distributed load.

<table>
<thead>
<tr>
<th>Location [cm]</th>
<th>Displacement [cm]</th>
<th>Angle [rad]</th>
<th>Shear [kg]</th>
<th>Bending moment [kg/cm]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Analytical solution</td>
<td>B-spline function</td>
<td>Analytical solution</td>
<td>B-spline function</td>
</tr>
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<td>-0.0195809</td>
<td>-0.0195809</td>
</tr>
<tr>
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<td>-4.85683</td>
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<td>-0.0176160</td>
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<tr>
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<td>-0.0127165</td>
</tr>
<tr>
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<td>-7.35206</td>
<td>-0.0063748</td>
<td>-0.0063748</td>
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<tr>
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<tr>
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<tr>
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<tr>
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<td>-2.48316</td>
<td>0.0329206</td>
<td>0.0329206</td>
</tr>
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</table>
4.4. Euler-Bernoulli beam on Winkler foundation under uniformly distributed load

Now the Euler-Bernoulli beam on the uniform Winkler foundation under a uniformly distributed load with spring supports is considered. The beam characteristics are shown in Fig. 5. Figure 6 presents the displacement, slope, bending moment, shear force of the beam on the Winkler foundation under uniformly distributed load.

Fig. 5. Euler-Bernoulli beam on Winkler foundation under uniformly distributed load.

Fig. 6. Displacement, slope, bending moment, shear force of Euler-Bernoulli beam on Winkler foundation under uniformly distributed load.
4.5. Beam with the translational and rotational support on Winkler foundation under uniformly distributed load

The Euler-Bernoulli beam with the translational and rotational support on the uniform Winkler foundation under uniformly distributed load is considered in this section. The beam characteristics are shown in Fig. 7. Table 5 compares the values of the displacement, slope, bending moment, shear force of the Euler-Bernoulli beam using the $B$-spline collocation method along with the integrated finite element analysis of structures (SAP2000) [14]. It can be seen that the

![Euler-Bernoulli beam with the translational and rotational support on Winkler foundation under uniformly distributed load](image)

**Fig. 7.** Euler-Bernoulli beam with the translational and rotational support on Winkler foundation under uniformly distributed load.

**Table 5.** The values of the displacement, slope, bending moment, shear force of the beam with the translational and rotational support on Winkler foundation under uniformly distributed load.

<table>
<thead>
<tr>
<th>Location [cm]</th>
<th>Displacement [cm]</th>
<th>Slope [rad]</th>
<th>Shear force [kg]</th>
<th>Bending moment [kg/cm]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SAP2000 $B$-spline function</td>
<td>SAP2000 $B$-spline function</td>
<td>SAP2000 $B$-spline function</td>
<td>SAP2000 $B$-spline function</td>
</tr>
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</tr>
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</tr>
<tr>
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</tr>
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</tr>
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<td>-0.0001016</td>
</tr>
<tr>
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<td>-2.59373</td>
<td>0.00147</td>
<td>0.00145</td>
</tr>
<tr>
<td>600</td>
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<td>-2.40044</td>
<td>0.00228</td>
<td>0.00227</td>
</tr>
<tr>
<td>700</td>
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<td>0.00197</td>
<td>0.00196</td>
</tr>
<tr>
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<tr>
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<td>-0.002265</td>
</tr>
<tr>
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<td>-2.59367</td>
<td>-0.001469</td>
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</tr>
<tr>
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<tr>
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<td>0.00632</td>
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<tr>
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<td>-1.00064</td>
<td>0.00727</td>
<td>0.00721</td>
</tr>
</tbody>
</table>
results are close. Table 5 shows that the maximum difference of obtained results is approximately 9.57%.

4.6. Euler-Bernoulli beam with general boundary conditions partially supported on Winkler foundation under uniformly and linearly distributed load

In this section, the Euler-Bernoulli beam partially supported on the Winkler foundation under uniformly and linearly distributed load is assumed with general boundary conditions. The beam characteristics are shown in Fig. 8. Figure 9 presents the displacement, slope, bending moment, shear force of the Euler-

---

**Fig. 8.** Euler-Bernoulli beam with general boundary conditions partially supported on Winkler foundation under uniformly and linearly distributed load.

---

**Fig. 9.** Displacement, slope, bending moment, shear force of Euler-Bernoulli beam with general boundary conditions partially supported on Winkler foundation under uniformly and linearly distributed load.
Bernoulli beam with general boundary conditions partially supported on the Winkler foundation under uniformly and linearly distributed load.

4.7. Euler-Bernoulli beam with arbitrary boundary conditions supported on partial Winkler foundation under uniformly and linearly distributed load

The Euler-Bernoulli beam arbitrary boundary conditions supported on the partial Winkler foundation under uniformly and linearly distributed load is considered. The beam characteristics are shown in Fig. 10. Figure 11 presents the

**Fig. 10.** Euler-Bernoulli beam with general boundary conditions partially supported on Winkler foundation under uniformly, linearly distributed and point loads.

**Fig. 11.** Displacement, slope, bending moment, shear force of Euler-Bernoulli beam with general boundary conditions partially supported on Winkler foundation under uniformly, linearly distributed and point loads.
values of the displacement, slope, bending moment, shear force of the Euler-Bernoulli beam using the $B$-spline collocation method along with the integrated finite element analysis of structures (SAP2000) [14]. It can be seen that the results are close. Figure 10 shows the maximum difference of the obtained results of approximately 4.32%.

5. Conclusion

This paper presents the analysis of the Euler-Bernoulli beam with arbitrary boundary conditions partially supported on a Winkler foundation using the sextic $B$-spline collocation method. A direct modeling technique is introduced for modeling the beam with arbitrary boundary conditions. Thus, the effect of translational along with rotational support flexibilities, the type of beam support, and the elastic coefficient of foundation are assessed. Finally, some numerical examples are shown to present the efficiency of the sextic $B$-spline collocation method. To validate the analysis of the Euler-Bernoulli beam with the presented method, the results of the $B$-spline collocation method are compared with the results of the analytical method and the integrated finite element analysis of structures (SAP2000).

Appendix

The coefficient matrix $[A]$, matrix $[C]$ and the load matrix $[F]$ in Eq. (3.17) are given for all boundary condition by:

1) for uniformly distributed load:

$$
F = \begin{bmatrix}
q/EI \\
q/EI \\
q/EI \\
\vdots \\
q/EI \\
q/EI \\
qEI \\
qEI \\
0 \\
m_1 \\
m_2 \\
m_3 \\
m_4
\end{bmatrix}, \quad C = \begin{bmatrix}
c_0 \\
c_1 \\
c_2 \\
\vdots \\
c_N \\
c_{N+1} \\
c_{N+2} \\
c_{N+3} \\
c_{N+4} \\
c_{N+5}
\end{bmatrix},
$$
where value for $m_i$ is dependent on boundary conditions

$$A_1 = \frac{360}{h^4} + \frac{k}{EI}, \quad A_2 = -\frac{1080}{h^4} + \frac{57k}{EI}, \quad A_3 = \frac{720}{h^4} + \frac{302k}{EI},$$

$$A_4 = \frac{720}{h^5} + \frac{6k}{hEI}, \quad A_5 = -\frac{3600}{h^5} + \frac{150k}{hEI}, \quad A_6 = \frac{7200}{h^5} + \frac{240k}{hEI},$$

$$A = \frac{1}{720} \begin{bmatrix} A_1 & A_2 & A_3 & A_2 & A_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & -A_6 & -A_5 & -A_4 \\ 0 & 0 & A_1 & A_2 & A_3 & A_2 & A_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & u_{13} & u_{14} & u_{15} & u_{16} & u_{17} & u_{18} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix},$$

where $u_i$ and $m_i$ depend on the boundary conditions that are determined by the kind of support and toggle.

2) for linearly distributed load:

$$F = \begin{bmatrix} \frac{dq(x)}{dx} \cdot \frac{1}{EI} \\ \frac{dq(x)}{dx} \cdot \frac{1}{EI} \\ \frac{dq(x)}{dx} \cdot \frac{1}{EI} \\ \vdots \\ \frac{dq(x)}{dx} \cdot \frac{1}{EI} \\ \frac{dq(x)}{dx} \cdot \frac{1}{EI} \\ 0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix}, \quad C = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_N \\ c_{N+1} \\ c_{N+2} \\ c_{N+3} \\ c_{N+4} \\ c_{N+5} \end{bmatrix},$$
where value for \( m_4 \) is dependent on boundary conditions

\[
A = \frac{1}{720} \begin{bmatrix}
A_4 & A_5 & A_6 & -A_6 & -A_5 & -A_4 & 0 & 0 & \cdots & 0 & 0 \\
0 & A_4 & A_5 & A_6 & -A_6 & -A_5 & -A_4 & 0 & \cdots & 0 & 0 \\
0 & 0 & A_4 & A_5 & A_6 & -A_6 & -A_5 & -A_4 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & A_4 & A_5 & A_6 & -A_6 & -A_5 & -A_4 \\
A_1 & A_2 & A_3 & A_3 & A_2 & A_1 & 0 & \cdots & 0 & 0 & 0 \\
u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & 0 & \cdots & 0 & 0 & 0 \\
u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & u_{13} & u_{14} & u_{15} & u_{16} & u_{17} & u_{18} \\
0 & 0 & 0 & \cdots & 0 & u_{19} & u_{20} & u_{21} & u_{22} & u_{23} & \end{bmatrix}.
\]

**Simple spring at the first and last member of a system**

Simple spring at the first and last member:

\[
Y \cdot k_1 = V, \quad M = 0,
\]

\[
[C_3] = [C_1] = \begin{bmatrix}
k_1 - \frac{120}{h^3} EI & 57k_1 - \frac{120}{h^3} EI & 302k_1 + \frac{960}{h^3} EI & 302k_1 - \frac{960}{h^3} EI & 57k_1 + \frac{120}{h^3} EI & k_1 + \frac{120}{h^3} EI \\
\end{bmatrix},
\]

\[
[C_4] = [C_2] = \begin{bmatrix}
\frac{30}{h^2} & \frac{270}{h^2} & -\frac{300}{h^2} & -\frac{300}{h^2} & \frac{270}{h^2} & \frac{30}{h^2} \\
\end{bmatrix},
\]

\[
m_4 = m_3 = m_2 = m_1 = 0,
\]

where \( k_1 \) is a stiffness coefficient of a simple spring.

**Simple spring in the middle of a system**

Simple spring at the first member:

\[
Y \cdot k_1 = (V_{(right)} - V_{(left)}), \quad M_{(right)} = M_{(left)},
\]

\[
[C_1] = \begin{bmatrix}
k_1 - \frac{120}{h^3} EI & 57k_1 - \frac{120}{h^3} EI & 302k_1 + \frac{960}{h^3} EI & 302k_1 - \frac{960}{h^3} EI & 57k_1 + \frac{120}{h^3} EI & k_1 + \frac{120}{h^3} EI \\
\end{bmatrix},
\]

\[
[C_2] = \begin{bmatrix}
\frac{30}{h^2} & \frac{270}{h^2} & -\frac{300}{h^2} & -\frac{300}{h^2} & \frac{270}{h^2} & \frac{30}{h^2} \\
\end{bmatrix},
\]

\[
m_1 = -V_{(left \ spring)}, \quad m_2 = M_{(left \ support)}.
\]
Simple spring at the last member:

\[ Y_{\text{right spring}} = Y_{\text{left spring}}, \quad \theta_{\text{right spring}} = \theta_{\text{left spring}}, \]

\[ [C_3] = \begin{bmatrix} 1 & 57 & 302 & 302 & 57 & 1 \end{bmatrix}, \]

\[ [C_4] = \begin{bmatrix} \frac{30}{h^2} & \frac{270}{h^2} & -300 & -300 & \frac{270}{h^2} & \frac{30}{h^2} \end{bmatrix}, \]

\[ m_3 = Y_{\text{right spring}}, \quad m_4 = \theta_{\text{right spring}}. \]

**Toggle in the middle of a system**

Toggle at the first member:

\[ M_{\text{right toggle}} = 0, \quad V_{\text{right toggle}} = V_{\text{left spring}}, \]

\[ [C_1] = \begin{bmatrix} \frac{30}{h^2} & \frac{270}{h^2} & -300 & -300 & \frac{270}{h^2} & \frac{30}{h^2} \end{bmatrix}, \]

\[ [C_2] = \begin{bmatrix} \frac{120}{h^3} & \frac{120}{h^3} & -960 & 960 & -120 & -120 \end{bmatrix}, \]

\[ m_1 = 0, \quad m_1 = V_{\text{left spring}}. \]

Toggle at the last member:

\[ M_{\text{left toggle}} = 0, \quad Y_{\text{right toggle}} = Y_{\text{left toggle}}, \]

\[ [C_3] = \begin{bmatrix} 1 & 57 & 302 & 302 & 57 & 1 \end{bmatrix}, \]

\[ [C_4] = \begin{bmatrix} \frac{30}{h^2} & \frac{270}{h^2} & -300 & -300 & \frac{270}{h^2} & \frac{30}{h^2} \end{bmatrix}, \]

\[ m_3 = Y_{\text{right spring}}, \quad m_4 = 0, \]

where matrices \([C_1], [C_2], [C_3]\) and \([C_4]\) are:

\[ [C_1] = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{bmatrix}, \]

\[ [C_2] = \begin{bmatrix} u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} \end{bmatrix}, \]

\[ [C_3] = \begin{bmatrix} u_{13} & u_{14} & u_{15} & u_{16} & u_{17} & u_{18} \end{bmatrix}, \]

\[ [C_4] = \begin{bmatrix} u_{19} & u_{20} & u_{21} & u_{22} & u_{23} & u_{24} \end{bmatrix}. \]
REFERENCES


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