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ANALYSIS AND OPTIMIZATION OF A NONLINEAR, CONTINUOUS, AUTOPARAMETRIC SYSTEM

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In the present paper a transversely vibrating autoparametric system consisting of three non-prismatic rods is presented. The considerations refer especially to stability of the semi – trivial solution. Proper selection of the values of parameters may lead to considerable reduction of the autoparametric resonance effects or may shift the autoparametric resonance to another frequency region.

1. Introduction

Nonlinear coupled vibrating systems with many degrees of freedom are rich in many kinds of resonances. An interesting resonance is internal resonance of autoparametric nature [1–4]. More often the vibrating system can be divided into two subsystems "autoparametrically" (nonlinearly) coupled. The first subsystem (I) – "oscillator" – is periodically excited, the second one (II) is a "non-excited subsystem" and oscillates when certain conditions are satisfied. The analysed autoparametric systems are described by differential equations which admit also semi-trivial solutions, stable or unstable. When the semi-trivial solution is unstable in some range of frequency, an autoparametric resonance occurs. So we have two cases of autoparametric resonance in properly coupled systems:

a) autoparametric system is tuned to internal resonance,

b) autoparametric system has an unstable semi-trivial solution.

Many authors give their attention to autoparametric resonance. The references [5–10] concern an autoparametric resonance in prismatic and non-prismatic rod systems. In these publications the authors analysed the autoparametric systems which were tuned to internal resonance [5–8], i.e. the ratio of natural frequencies of sub-systems I and II is close to 2:1, or analysed the system tuned to the combination resonance [9, 10].

In the present paper we consider a continuous, transverse vibrating system which consists of three non-prismatic rods. The couplings of rods are realized through periodic longitudinal forces which are transverse forces at the ends of neighbouring rods. The considerations refer especially to stability of semi-trivial solution of such autoparametric system.

2. Autoparametric system – analysis of equation of motion

The present paper concerns an analysis of autoparametric resonance in a plane system of three non-prismatic rods, Fig. 1. Problems of dynamic stability of continuous non-prismatic elements interacting through internal longitudinal forces were analyzed in references [6–11]. The references deal with the analysis of the influence of the shapes of three bars constituting a plane vibrating system, upon the vibration amplitudes or upon the instability region, in the states of stationary internal and periodic combination resonances. Nonlinear damping effects or nonlinear inertia effects were considered. In this case of interaction of system elements, the resonances have an autoparametric nature. Proper selection of geometric parameters may lead to considerable reduction of the resonance amplitudes and influences the frequency region in which the resonances occur.

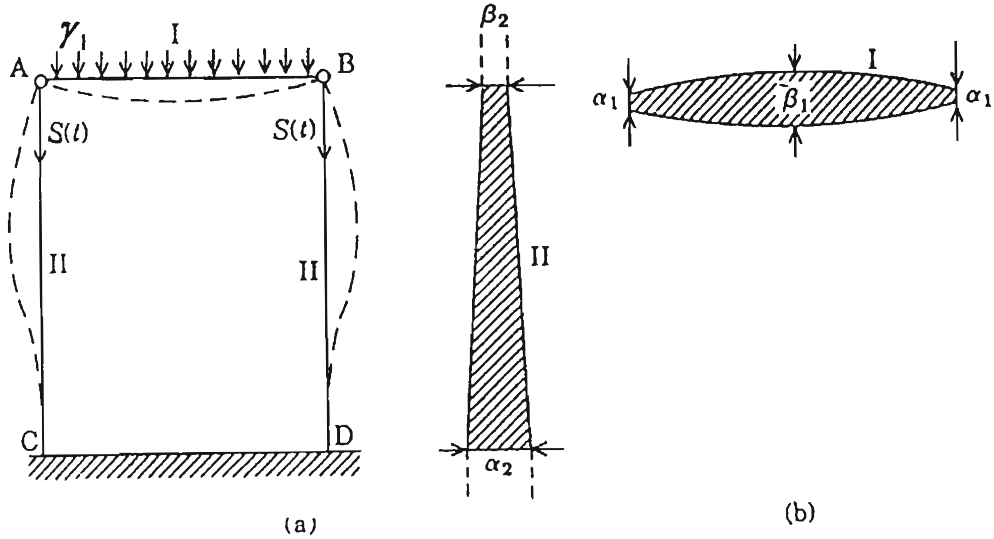

Fig. 1. Model of the vibrating system.

In the paper a similar nonlinear, symmetric system of non-prismatic rods illustrated in Fig. 1 is analysed, but considerations are devoted, especially, to stability of a semi – trivial solution, cf. [1, 2]. The rods are visco-elastic (Kelvin-

Voigt model) and of square cross-section, the vertical rods are identical, the deflections of rods are small. The external transverse, harmonic load $\gamma_1 \sin\,\omega t$ acts on the horizontal rod. We consider only transverse, symmetrical vibration in the plane of the system at rest.

In the paper [11] an approximate method of describing of such rod system was proposed. We assume that the articulated joints A and B do not move in horizontal direction. The dynamic coupling of the rods is realized through periodic longitudinal forces $S_i(t), i = 1, 2$ which are transverse forces acting at the ends of neighbouring rods. This coupling slightly influences the modes of transverse vibration of the rods. Displacements $u_i, i = 1, 2$ of the joints of the rods are taken into account; except these small displacements, the joints do not move. The generalized forces corresponding to the longitudinal forces S_i are obtained by assuming that the longitudinal forces S_i perform work on the longitudinal displacements

$$u_i = \frac{1}{2} \int\limits_0^{l_i} \left(\frac{\partial w_i}{\partial x_i}\right)^2 dx_i, \qquad i = 1, 2. \tag{2.1}$$

We make the simplifying assumption that the solution of the problem is of the separable form:

$$w_i(x_i, t) = Y_i(x_i) T_i(t), \qquad i = 1, 2. \tag{2.2}$$

Solving the proper boundary value problems, one gets

$$\begin{aligned} Y_1(x_1) &= sin(\pi x_1/l_1), \\ Y_2(x_2) &= -cos\lambda_1\{sin(\lambda_1 x_2/l_2) - sh(\lambda_1 x_2/l_2) \\ &\quad - tg\lambda_1[cos(\lambda_1 x_2/l_2) - ch(\lambda_1 x_2/l_2)]\}. \end{aligned} \tag{2.3}$$

Therefore

$$\begin{aligned} S_1(t) &= E_2 I_2 Y_2'''(l_2) T_2(t), \\ S_2(t) &= -E_1 I_1 \; Y_1'''(0) T_1(t), \end{aligned} \tag{2.4}$$

and the generalized forces are

$$\begin{aligned} Q_{S1} &= E_2 I_2 Y_2'''(l_2) T_1(t) T_2(t) \int\limits_0^{l_1} [Y_1'(x_1)]^2 dx_1, \\ Q_{S2} &= -E_1 I_1 \; Y_1'''(0) T_1(t) T_2(t) \int\limits_0^{l_2} [Y_2'(x_2)]^2 dx_2. \end{aligned} \tag{2.5}$$

Together with the internal friction connected with the transverse vibrations of the rods themselves, also the damping forces $P_i, i = 1, 2$ associated with the rates of displacements $\dot{u}_i : P_i = -k_i\dot{u}_i$ are taken into account.

The following notation is used: l_1, l_2 – lengths of rods; E_1, E_2 – Young's moduli; I_1, I_2 – cross-sectional moments of inertia; m_1, m_2 – linear mass densities; w_1, w_2 – transverse displacements; u_1, u_2 – longitudinal displacements; S_1, S_2 – internal coupling forces; η_1, η_2 – viscosities and k_1, k_2 – coefficients of nonlinear damping.

From the Lagrange equations the problem can be transformed into discrete system of two degrees of freedom, described by the equations (cf. [6, 7, 10])

$$\begin{aligned} &\ddot{T}_1 + (B/A)T_1 = (C/A)T_2T_1 - (D/A)\dot{T}_1 - (E/A)\dot{T}_1T_1^2 + (\Gamma_1/A)\sin\omega t, \\ &\ddot{T}_2 + (\bar{B}/\bar{A})T_2 = -(\bar{C}/\bar{A})T_1T_2 - (\bar{D}/\bar{A})\dot{T}_2 - (\bar{E}/\bar{A})\dot{T}_2T_2^2, \end{aligned} \tag{2.6}$$

where the coefficients are (cf. [6]):

$$\begin{aligned} A &= \int_0^{l_1} m_1(x_1)Y_1^2(x_1)dx_1, \qquad \bar{A} = \int_0^{l_2} m_2(x_2)Y_2^2(x_2)dx_2, \\ B &= E_1\int_0^{l_1} I_1(x_1)[Y_1''(x_1)]^2dx_1, \qquad \bar{B} = E_2\int_0^{l_2} I_2(x_2)[Y_2''(x_2)]^2dx_2, \\ C &= \frac{\partial}{\partial x_2}[E_2I_2(x_2)Y_2''(x_2)]_{x_2=l_2}\int_0^{l_1}[Y_1'(x_1)]^2dx_1, \\ \bar{C} &= \frac{\partial}{\partial x_1}[E_1I_1(x_1)Y_1''(x_1)]_{x_1=0}\int_0^{l_2}[Y_2'(x_2)]^2dx_2, \\ D &= \eta_1\int_0^{l_1} I_1(x_1)[Y_1''(x_1)]^2dx_1, \qquad \bar{D} = \eta_2\int_0^{l_2} I_2(x_2)[Y_2''(x_2)]^2dx_2, \\ E &= k_1\left\{\int_0^{l_1}[Y_1'(x_1)]^2dx_1\right\}^2, \qquad \bar{E} = k_2\left\{\int_0^{l_2}[Y_2'(x_2)]^2dx_2\right\}^2, \\ \Gamma_1 &= \int_0^{l_1}\gamma_1Y_1(x_1)dx_1. \end{aligned} \tag{2.7}$$

All possible loadings, internal longitudinal forces, forces connected with damping and nonlinear forces are taken into account by means of the corresponding generalized forces. The first term on the right-hand side of equations (3.1) stands for coupling, the second one describes linear damping, the third one is connected with nonlinearity of geometrical nature (nonlinear damping). The last term of the first equation corresponds to external harmonic loading. Coefficients of the equations are functions of the shape parameters of rods.

The transverse dimension a_1 of element I changes as a quadratic function of x_1, so the cross-sectional area and the moment of inertia are (cf. Fig. 1 and papers [6, 10])

$$A_1 = a_1^2 = \alpha_1^2\phi_1^2, \qquad I_1 = \frac{1}{12}\alpha_1^4\phi_1^4, \tag{2.8}$$

where

$$\phi_1(x_1, \kappa_1) = 4\kappa_1\{(x_1/l_1)^2 + x_1/l_1\} + 1, \tag{2.9}$$

and the parameter κ_1 which defines the shape of the rod I (Fig. 1) takes the form

$$\begin{aligned} &\kappa_1 = (\alpha_1 - \beta_1)/\alpha_1, \qquad && a_1(0) = a_1(l_1) = \alpha_1, \\ &a_1(l_1/2) = \beta_1, && \kappa_1 \in (-\infty, 1]. \end{aligned} \tag{2.10}$$

The vertical rods are identical and their transverse dimension is a linear function of x_2. Therefore the cross-sectional area and the moment of inertia are

$$A_2 = a_2^2 = \alpha_2^2\phi_2^2, \quad I_2 = \frac{1}{12}\alpha_2^4\phi_2^4, \tag{2.11}$$

where

$$\phi_2(x_2, \kappa_2) = 1 - \kappa_2(x_2/l_2), \tag{2.12}$$

the parameter κ_2 defines the shape of the rods II and takes the form

$$\begin{aligned} &\kappa_2 = (\alpha_2 - \beta_2)/\alpha_2, \qquad && a_2(0) = \alpha_2, \\ &a_2(l_2) = \beta_2, && \kappa_2 \in (-\infty, 1]. \end{aligned} \tag{2.13}$$

On the basis of the relations (2.3), (2.7) – (2.12), one obtains the coefficients of equations (3.1) in the following form (cf. [6 – 10]):

$$
(2.14)\quad
\begin{aligned}
&A(\alpha_1,\kappa_1) = \rho_1 l_1 \alpha_1^2 f_{\mathrm{A}}(\kappa_1), && \bar{A}(\alpha_2,\kappa_2) = \rho_2 l_2 \alpha_2^2 f_{\bar{A}}(\kappa_2),\\
&B(\alpha_1,\kappa_1) = E_1 \frac{\alpha_1^4}{l_1^3} f_B(\kappa_1), && \bar{B}(\alpha_2,\kappa_2) = E\frac{\alpha_2^4}{l_2^3} f_{\bar{B}}(\kappa_1),\\
&C(\alpha_2) = E_2 \frac{\alpha_2^4}{l_1 l_2^3} f_C, && \bar{C}(\alpha_1) = E_1 \frac{\alpha_1^4}{l_2 l_1^3} f_{\bar{C}},\\
&D = \frac{\eta_1}{E_1} B, && \bar{D} = \frac{\eta_2}{E_2}\bar{B},\\
&E = \frac{k_1}{l_1^2} f_E, && \bar{E} = \frac{k_2}{l_2^2} f_{\bar{E}},\\
&\Gamma_1 = \frac{2l_1}{\pi}\gamma_1,
\end{aligned}
$$

where

$$
(2.15)\quad
\begin{aligned}
&f_{\mathrm{A}}(\kappa_1) = 0.391\kappa_1^2 - 0.896\kappa_1 + 0.500,\\
&f_{\mathrm{B}}(\kappa_1) = 2.701\kappa_1^4 - 11.64\kappa_1^3 + 19.01\kappa_1^2 - 14.12\kappa_1 + 4.058,\\
&f_{\bar{A}}(\kappa_1) = 0.1747\kappa_1^2 - 0.5680\kappa_1 + 0.499,\\
&f_{\bar{B}}(\kappa_1) = 1.253\kappa_1^4 - 7.153\kappa_1^3 + 15.94\kappa_1^2 - 17.06\kappa_1 + 9.890,\\
&f_{\mathrm{C}} = 24.20, \qquad f_{\mathrm{D}} = f_{\mathrm{B}}, \qquad f_{\mathrm{E}} = 24.35,\\
&f_{\bar{C}} = -14.40, \qquad f_{\bar{D}} = f_{\bar{B}}, \qquad f_{\bar{E}} = 33.09.
\end{aligned}
$$

2.1. Semi-trivial solution

We consider an autoparametric resonance in the system (Fig. 1) which is not tuned to resonance. The system can be divided into two subsystems. The first one (horizontal element) – the oscillator, is periodically excited and the second one (vertical elements) is not excited and can oscillate if some conditions are satisfied. The system of equations (3.1) admits the semi-trivial solution, i.e. the trivial solution: $T_2 = 0$ and the non-trivial solution $T_1 = a \cos \omega t + b \sin \omega t$. So the amplitude $\tilde{A}$ of element I is determined by the relation

$$
(2.16)\quad \omega_{1,2}^2 = \omega_{01}^2 - \frac{1}{2}\left(2\delta_1 + \frac{1}{4}F\tilde{A}^2\right)^2 \mp \sqrt{\frac{1}{4}\left(2\delta_1 + \frac{1}{4}F\tilde{A}^2\right)^2\left[\left(2\delta_1 + \frac{1}{4}F\tilde{A}^2\right)^2 - 4\omega_{01}^2\right] + \frac{\gamma^2}{\tilde{A}^2}},
$$

where

$$\omega_{01}^2 = B/A, \qquad 2\delta_1 = D/A, \qquad F = E/A, \qquad \gamma = \Gamma_1/A.$$

The stability of the solution can be examined by means of the procedure used in the case of a small disturbance. The variational equations are:

$$\begin{aligned} &\ddot{u} + (2\delta_1 + FT_1^2)\dot{u} + (\omega_{01}^2 + 2FT_1\dot{T}_1)u - GT_1v = 0, \\ &\ddot{v} + 2\delta_2\dot{v} + (\omega_{02}^2 + \bar{G}T_1)v = 0, \end{aligned} \tag{2.17}$$

where:

$$\tilde{T}_1 = T_1 + u, \quad \tilde{T}_2 = T_2 + v, \quad \omega_{02}^2 = \frac{\bar{B}}{\bar{A}},$$

$$2\delta_2 = \frac{\bar{D}}{\bar{A}}, \qquad G = \frac{C}{A}, \qquad \bar{G} = \frac{\bar{C}}{\bar{A}}.$$

Only the second equation (3.4) is not coupled and has a form of the Mathieu equation, so the stability can be determined by analysing these equations separately. On the boundaries of the first, most important instability region, the solution is periodic of the form: $v = \alpha \cos \omega t/2 + \beta \sin \omega t/2$. So the boundaries of the first instability region are determined by the equation

$$R^2 = a^2 + b^2 = \frac{4}{\bar{G}^2}\left[\left(-\frac{\omega^2}{4} + \omega_{02}^2\right)^2 + \delta_2^2\omega^2\right]. \tag{2.18}$$

In the second example we analyse the semi-trivial solution for nonlinear inertia. The equations of motion take the following form [7]:

$$\begin{aligned} \ddot{T}_1 + (B/A)T_1 &= (C/A)T_2T_1 - (D/A)\dot{T}_1 - (E/A)\dot{T}_1T_1^2 \\ &\quad - (F_M/A)(T_1\dot{T}_1^2 + T_1^2\ddot{T}_1) + (\Gamma_1/A)\sin\omega t, \\ \ddot{T}_2 + (\bar{B}/\bar{A})T_2 &= -(\bar{C}/\bar{A})T_1T_2 - (\bar{D}/\bar{A})\dot{T}_2 - (\bar{E}/\bar{A})\dot{T}_2T_2^2 \\ &\quad + (\bar{F}_M/\bar{A})(T_2\dot{T}_2^2 + T_2^2\ddot{T}_2). \end{aligned} \tag{2.19}$$

The fourth terms on the right-hand side of Eqs (2.19) are related to nonlinear inertia. The effects of masses, concentrated at the articulated joints, on amplitude are analysed. Now the amplitude $\tilde{A}$ of element I is determined by the relation

$$\omega_{1,2}^2 = \frac{1}{(1+\frac{1}{2}H_1\tilde{A}^2)^2}\left[\left(1+\frac{1}{2}H_1\tilde{A}^2\right)\omega_{01}^2 - \frac{1}{2}\left(2\delta_1+\frac{1}{4}F\tilde{A}^2\right)^2 \mp \frac{1}{2}\sqrt{\Delta}\right], \tag{2.20}$$

where

$$\sqrt{\Delta} = \sqrt{\left(2\delta_1+\frac{1}{4}F\tilde{A}^2\right)^2\left[\left(2\delta_1+\frac{1}{4}F\tilde{A}^2\right)^2 - 4\left(1+\frac{1}{2}H_1\tilde{A}^2\right)\omega_{01}^2\right] + 4\left(1+\frac{1}{2}H_1\tilde{A}^2\right)^2\frac{\gamma^2}{\tilde{A}^2}}, \tag{2.21}$$

$$H_1 = \frac{F_M}{A}.$$

The variational equations are:

$$(1+H_1T_1^2)\ddot{u} + (2\delta_1 + FT_1^2 + 2H_1T_1\dot{T}_1)\dot{u} + (\omega_{01}^2 + 2FT_1\dot{T}_1 + H_1\dot{T}_1^2 + 2H_1T_1\ddot{T}_1)u - GT_1v = 0 \tag{2.22}$$

$$v + 2\delta_2\dot{v} + (\omega_{02}^2 + \bar{G}T_1)v = 0.$$

The results are presented in Figs. 2, 3 where the amplitude $\tilde{A}$ of element I (semi-trivial) and instability region $R(\omega)$ of small disturbance v for prismatic bars are illustrated, for nonlinear damping and nonlinear inertia. In Fig. 2 we present the graph of amplitude $\tilde{A}$ of element I of a semi-trivial solution and boundaries $R(\omega)$ of instability regions of element II versus the excitation angular frequency ω for nonlinear damping and for different values of the shape parameter κ_2. The results presented in Fig. 3 are similar but in the equation of motion, a nonlinear inertia is taken into account. The points P and Q of intersection of curves $\tilde{A}$ and R determine the stability boundary points on the amplitude $\tilde{A}$ of a semi-trivial solution. The part of the curve $\tilde{A}$ lying between these points (P, Q) corresponds to unstable semi-trivial solution, and in this region an autoparametric resonance occurs. For semi-trivial solution the change of values of geometric parameters shifts the boundaries of instability region of non-excited subsystem in comparison with the resonance curve $\tilde{A}$ of the excited subsystem.

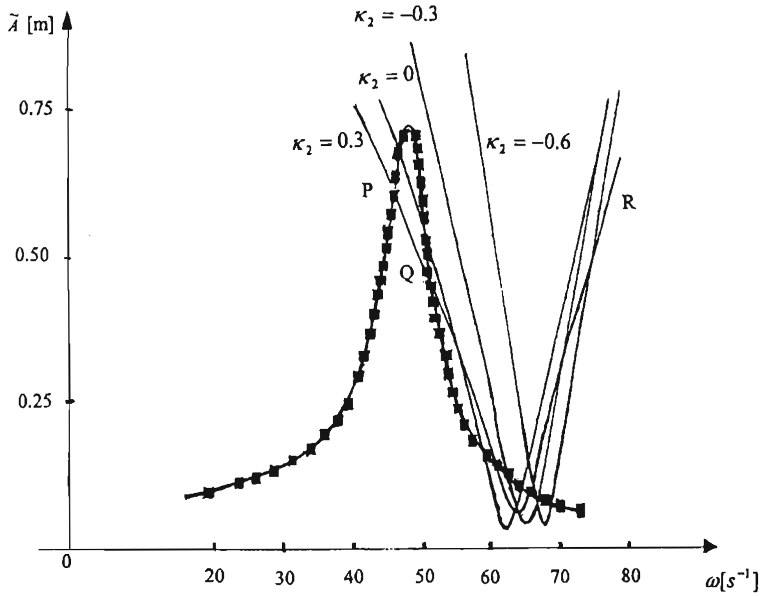

FIG. 2. Amplitude $\tilde{A}(\omega)$ and boundary of the instability region $R(\omega)$ versus frequency of external excitation for nonlinear damping.

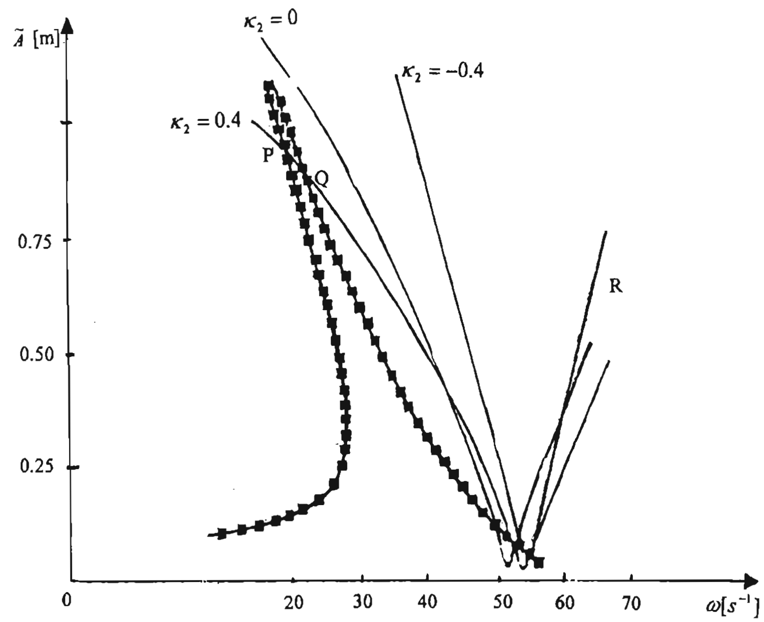

FIG. 3. Amplitude $\tilde{A}(\omega)$ and boundary of the instability region $R(\omega)$ versus frequency of external excitation for nonlinear inertia.

Next we have analysed our system in a three-dimensional space. Figures 4–6 illustrate the boundary surfaces of instability regions. Above the surfaces the system is unstable, below them the system is stable. The graphs of values of $\gamma = \frac{\Gamma_1}{A}$, proportional to amplitude γ_1 of the external excitation (cf.(3.3)) versus angular frequency ω of the external excitation, and natural angular frequency ω_{02} of the non-excited element II for different values of shape parameter $\kappa_2 \in (-0.4; 0; 0.4)$ of element II are shown.

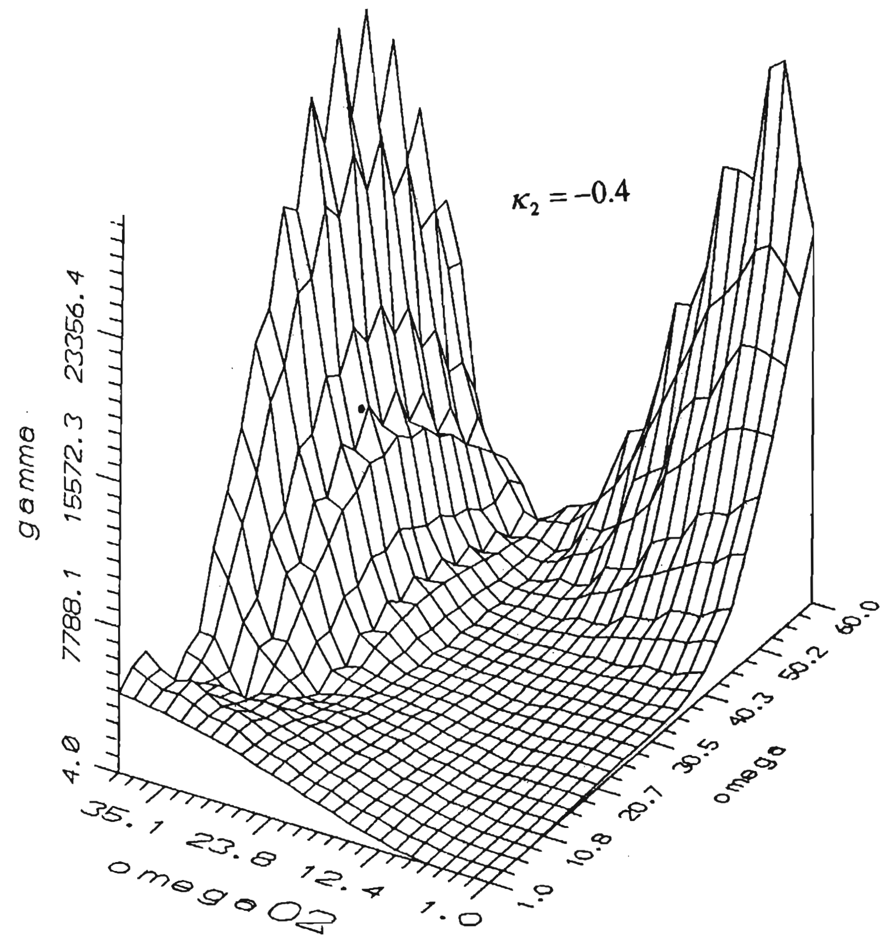

FIG. 4. Quantity γ (proportional to external excitation γ_1) versus external frequency ω and versus natural frequency ω_{02} for $\kappa_2 = -0.4$.

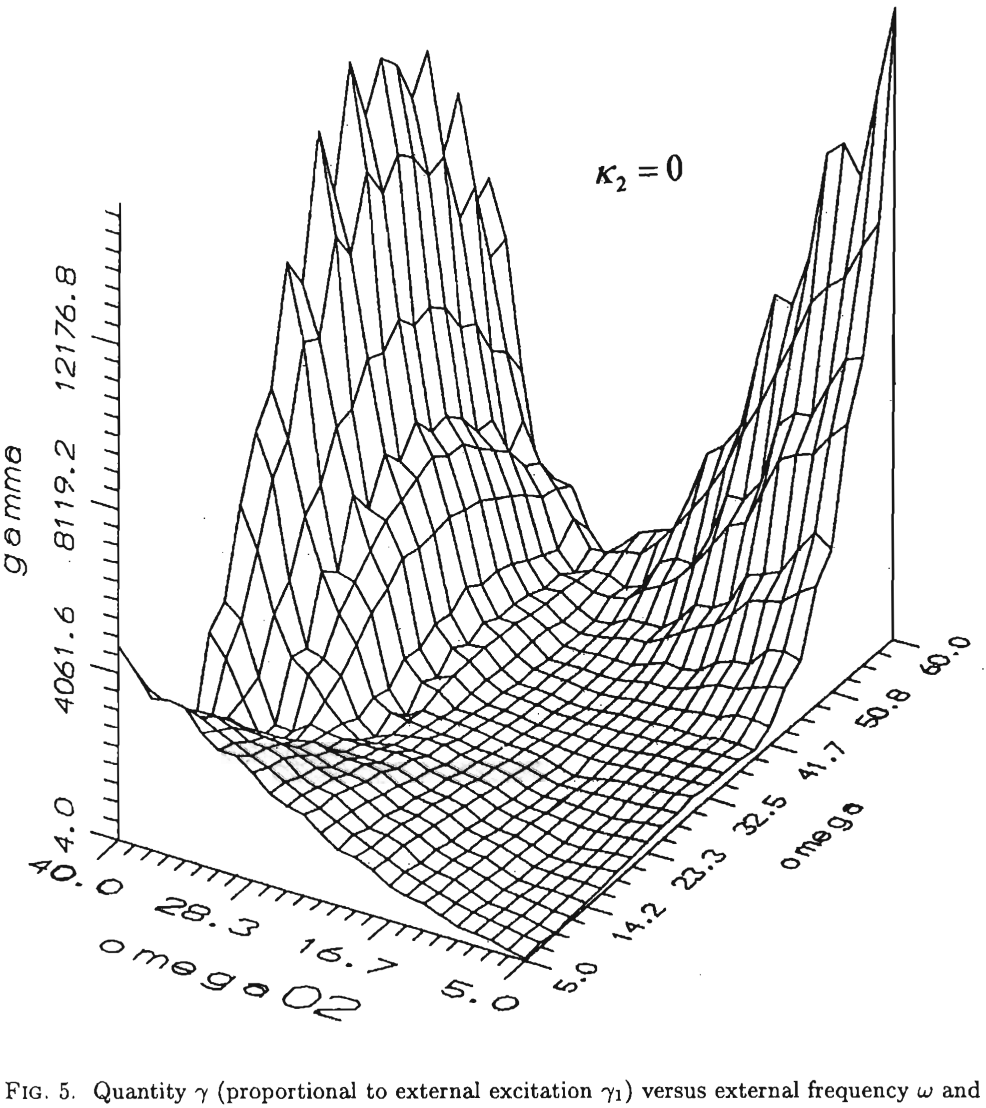

FIG. 5. Quantity γ (proportional to external excitation γ_1) versus external frequency ω and versus natural frequency ω_{02} for $\kappa_2 = 0$.

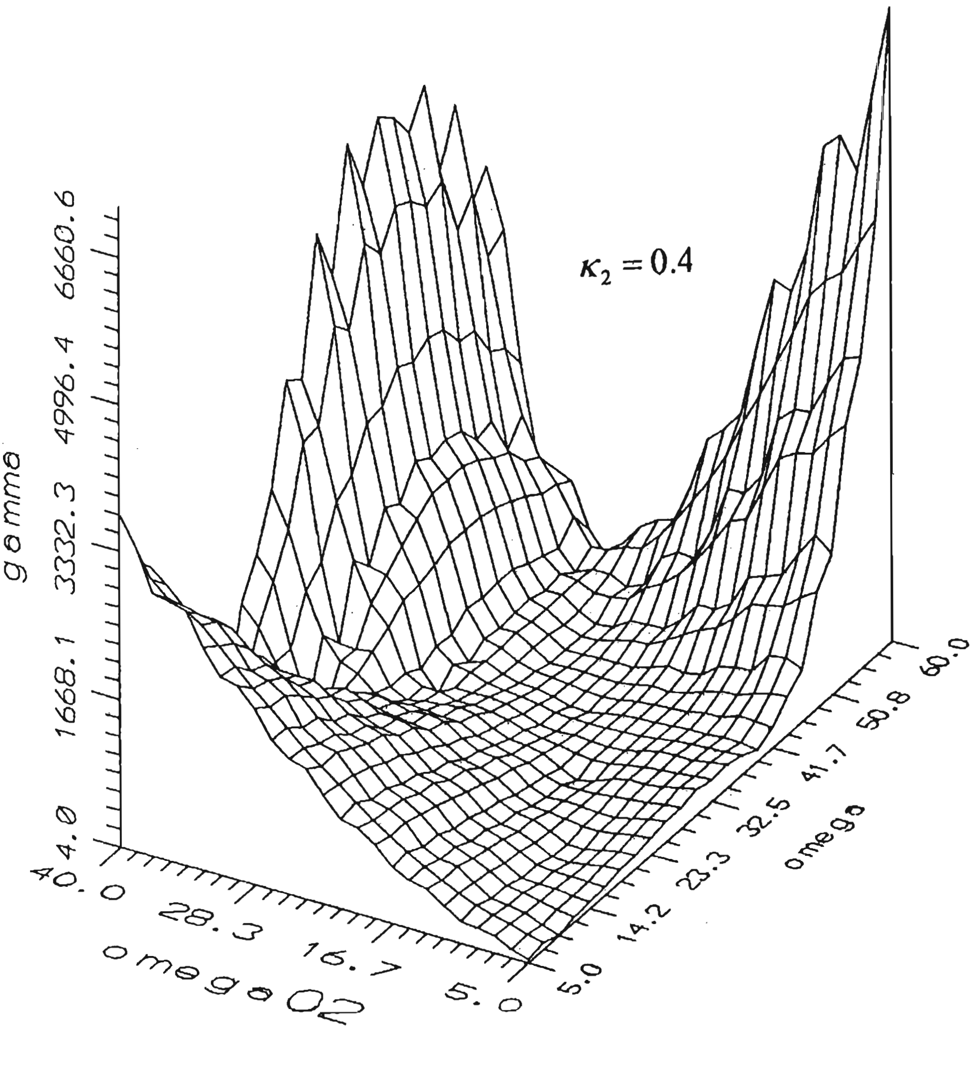

FIG. 6. Quantity γ (proportional to external excitation γ_1) versus external frequency ω and versus natural frequency ω_{02} for $\kappa_2 = 0.4$.

2.2. Autoparametric internal and combination resonances

The second part of the report is devoted to presentation of the previously obtained results (cf. papers [6–10]) for a certain system which is tuned to internal resonance, so the necessary relations are satisfied.

The papers [5–8] were devoted to the analysis of autoparametric resonance in the system described by equations (3.1) or (2.19) and tuned to the internal or combination resonance. The papers [6, 8] deal with parametric optimization of the system of three bars illustrated in Fig. 1. The system was subject to the conditions of internal resonance. The effect of geometric parameters on the resonance amplitudes was analysed for nonlinear damping and nonlinear inertia. Mathematical analysis was done by means of the modified harmonic balance method developed by A. TONDL [12]. The amplitude of parametrically excited element II is given in analytical form and it stands for an objective function. Parameters of shape: $\alpha_1, \alpha_2, \kappa_1, \kappa_2$ are the control parameters. The optimization problem is formulated as follows. We look for such values of the shape parameters which satisfy the constraints: V = const, the relations for internal resonance and

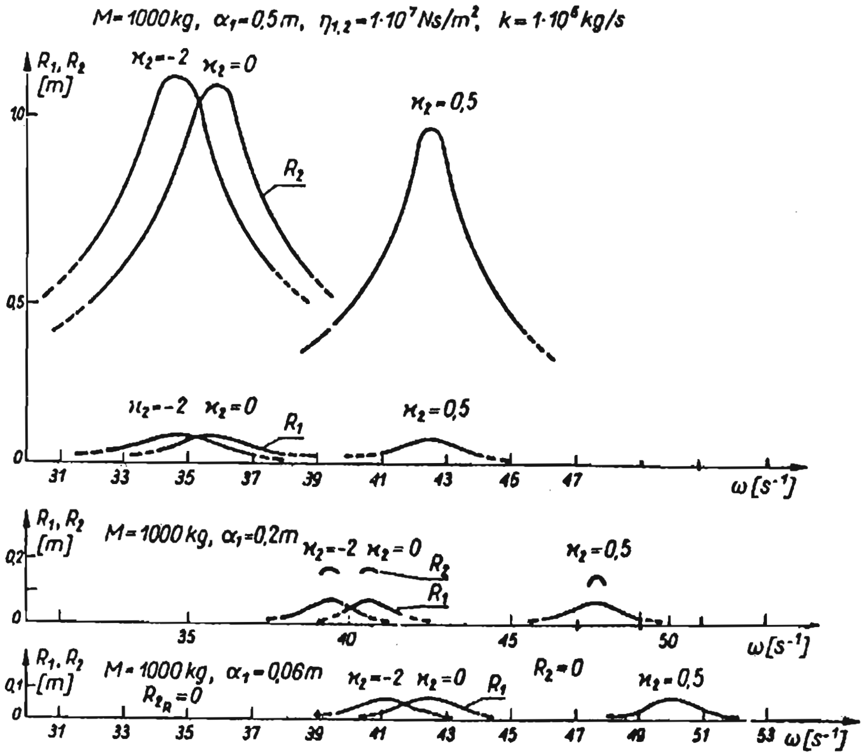

FIG. 7. Internal resonance. Resonance curves for different values of κ_2 and for nonlinear damping.

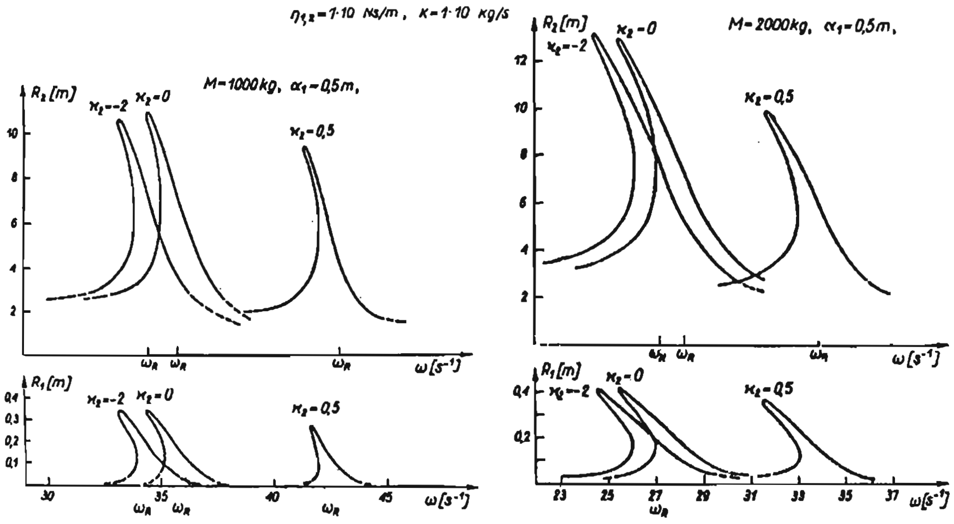

FIG. 8. Internal resonance. Resonance curves for different values of κ_2 and for nonlinear inertia.

which minimize the amplitude of vibration of element II. A part of the considerations and results are presented in Figures 7–9. It is seen from Fig. 7 that the nonlinear damping dominates the nonlinear inertia in the system. Amplitudes of vibration of elements I and II versus angular frequency of the external excitation for different parameters of shape are shown. According to the graphs in Fig. 8, the nonlinear inertia dominates the nonlinear damping [6, 7]. In Fig. 9 similar results are presented for vibrating system of rods placed on the vertically moving support [10]. In the equation of motion of such a system, two kinds of nonlinearities of geometrical nature appear. Paper [8] is also devoted to optimization of the system presented in Fig. 1, but the analysis is based on the partial differential equation for a non-prismatic rod. The parametric optimization is confined to instability regions. The paper deals with the analysis of the influence of the shapes of rods on the parameter connected with the instability regions. On the graphs presented in Fig. 10, the instability regions ($\omega(q_0)$, where q_0 is the amplitude of external excitation) for nonprismatic elements I and II are shown, for different values of κ_1 and κ_2. Changes of the shape parameters shift the instability regions to other frequency intervals. Suitable selection of the shape parameters of the elements may lead to a considerable reduction of the frequency interval in which the resonance occurs, or may lead to a total elimination of the resonance.

The considered problems may have a practical significance for the paraseismic phenomena when a weak excitation may cause great effects because of the autoparametric resonances.

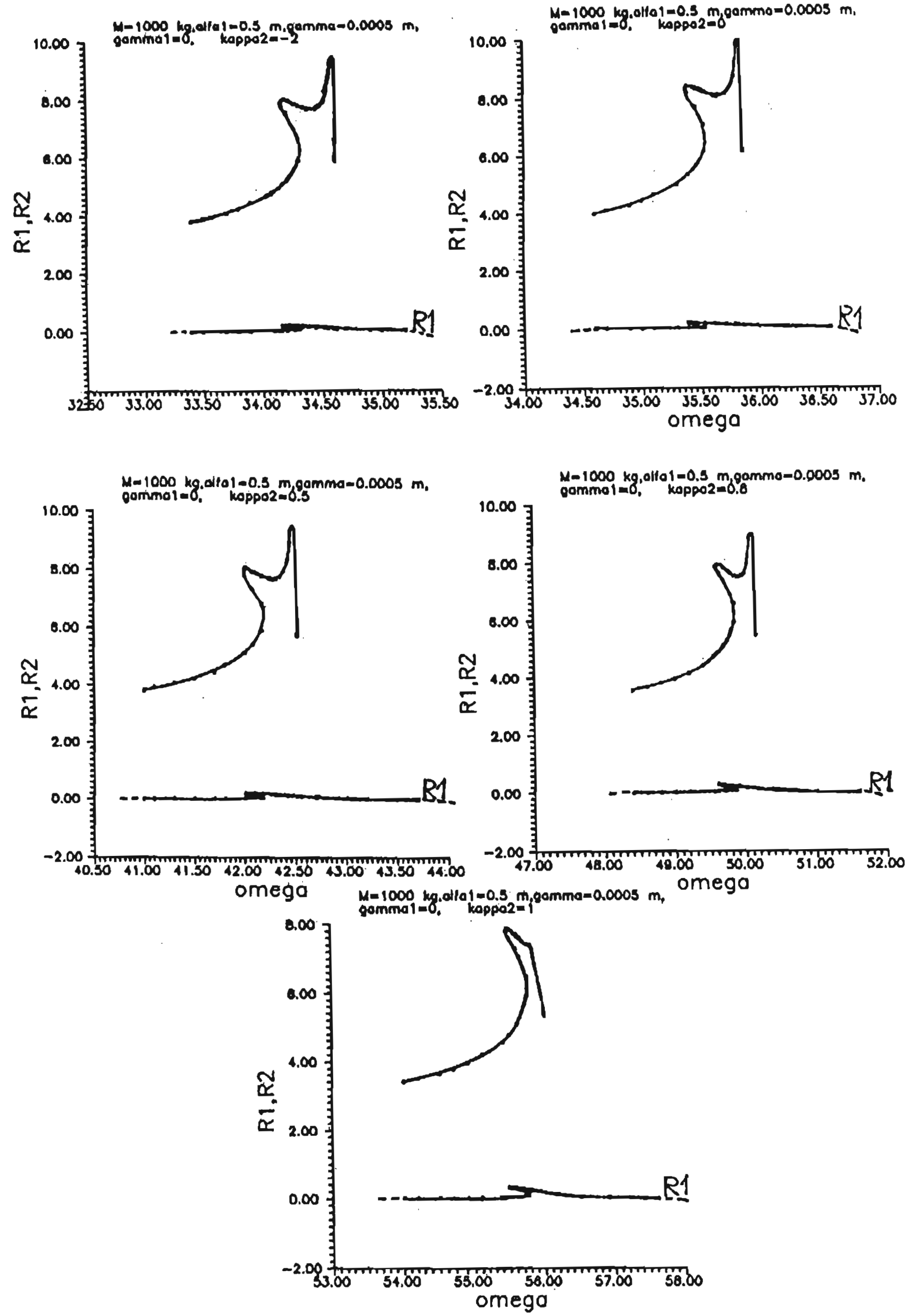

FIG. 9. Internal resonance. Amplitudes of vibration versus external excitation frequency ω for different values of κ_2 and for kinematically excited system with nonlinear damping.

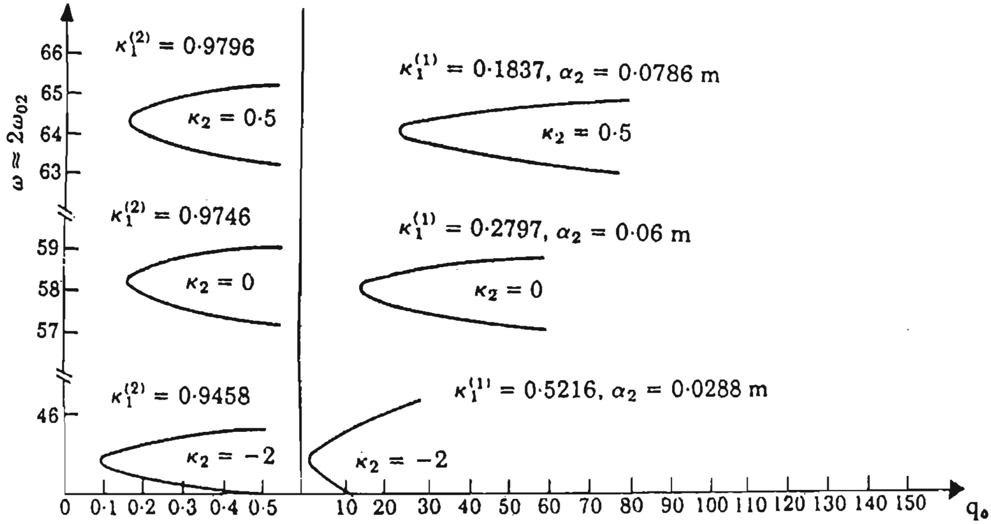

FIG. 10. Internal resonance. Instability region for different values of κ_1 and κ_2.

3. Conclusions

It results from the above analysis that it is necessary to consider two cases of autoparametric resonanse:

- autoparametrically coupled system which is tuned to internal resonace,
- analysis the stability of semi-trivial solution for a system which is not tuned.

In the second case, when unstable (semi-trivial) solutions exist, the autoparametric resonance occurs. For a semi-trivial solution, the modification of geometrical parameters shifts the instability region for disturbance of the second "non-excited subsystem II" in relation to the resonance curves of subsystem I (periodically excited – "oscillator"), and may lead to total elimination of the resonance (total elimination of unstable amplitude region).

For the system tuned to internal or combination resonance, proper selection of the parameters may lead to considerable reduction of the resonance amplitudes and may lead to reduction of the frequency region in which the resonance occurs. Modification of the geometric parameters shifts the autoparametric resonance to another frequency region.

References

1. A. Tondl, *A contribution to the analysis of autoparametric systems*, Acta Techn. CSAV **37**, 735-758, 1992.

2. A. Tondl, *To the analysis of autoparametric systems*, Z. Angew. Math. Mech. **77**, 6, 407–418, 1997.

3. F. Verhulst, *Autoparametric resonance, survey and new results*, 2-nd European Nonlinear Oscillation Conference, Prague 1996.

4. M. Cartmell, *Introduction to linear, parametric and nonlinear vibrations*, Chapman and Hall, London 1990.

5. A. Foryś, J. Nizioł, *Internal resonance in a plane system of rods*, J. Sound Vibr., **95**, 3, 361–374, 1984.

6. A. Foryś, A. Gajewski, *Analysis and optimization of a system of rods of variable cross-section under conditions of internal resonance* [in Polish], Engineering Transactions, **32**, 4, 575–598, 1984.

7. A. Foryś, A.S. Foryś, *Parametric optimization of a system of rods considering nonlinearities* [in Polish], Engineering Transactions, **34**, 4, 399–418, 1986.

8. A. Foryś, *Parametric optimization of a system of rods in view of dynamic stability*, J. Sound Vibr., **176**, 3, 341–350, 1994.

9. A. Foryś, *Periodic combination resonance in nonlinear system of non-prismatic rods*, J. Sound Vibr., **181**, 2, 355–368, 1995.

10. A. Foryś, A.S. Foryś, *Internal and combination resonances in a kinematically excited system of non-prismatic rods*, Engineering Transactions, **43**, 1–2, 113–136, 1995.

11. A. Foryś, *Vibrations and dynamical stability of selected systems of rods* [in Polish], Ph.D.Thesis, Cracow University of Technology, 1981.

12. A. Tondl, *Some problems of rotor dynamics*, Publishing House of the Czechoslovak Academy of Sciences, Prague 1965.

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