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Influence of Exponent in Damage Evolution Equation on the Accuracy of Damage Modelling in Brittle Materials

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This paper analyses the influence of nonlinearity of the damage evolution equation that is introduced by exponent to the results obtained in the simulation of elastic-brittle material. Constitutive equation of linear-elastic medium with damages is described by the linear-tensorial function due to damage tensor. The nucleation and growth of microdamages are modelled using a two-parameter equation of damage evolution, in which the current level of damage is expressed by the principal values of Vakulenko-Kachanov and Murakami-Ohno damage tensors. The study examines a relationship between the time of the first macro crack appearance, principal values of damage tensor at the critical moment and the exponent adopted to the equation of damage evolution. The subjects of the analysis are changes in both the qualitative and quantitative variables characterizing the damage.

Key words: continuum damage mechanics, damage evolution equation, damage tensor.

1. Introduction

Modern structures and their components are exposed to impacts of high energy. These processes, during which a substantial amount of energy is produced, are accompanied by elevated temperature field variable in time, which activates creep phenomenon in engineering materials. The solutions presented in this work examine the thermal field approximated by a constant function in finite, short intervals of time. The solutions obtained with these assumptions will give some approximations of the operation of the devices and structures in real situations.

Problems associated with reducing vulnerability to accumulation of damage at elevated temperatures are mainly solved by changes in the structure of the material. These changes are achieved by the appropriate selection of alloy components and the application of technological components to the finished construction.

An important aspect of research is the theoretical estimation of the first crack time and the time interval for achieving the load-carrying capacity of the structure. Its solutions are used in the design of structures. Evidence of this is intensive development of continuum damage mechanics [1–3] with the focus on incorporating damage mechanics models into finite element procedures [4–6].

The issue related to carrying capacity of surface structures operating at high temperatures can be considered in terms of growth restriction deflection (usability condition), and in layered structures in terms of ensuring the durability of protective layers. The time of safe work of the structure is also associated with the reduction of load-carrying capacity due to the nucleation, growth and propagation of micro damage. This phenomenon mostly occurs in two stages. The first stage, starting at the initial period of work of the structure, involves the creation of hidden microdamages (before the failure criterion is fulfilled), which combine to form macro cracks (when the failure criterion is fulfilled). The second stage is the propagation of macro cracks, forming the damage front.

The basic problem of structure's design in a life time of a structure is an approach that does not permit to initiate the formation of the previously mentioned second stage of macro crack front, while in the case of appearance of the first macro cracks it allows to determine the time remaining to complete destruction of the cross section of the structure.

In this study, the sensitivity of mathematical model to the behaviour of surface structures, in terms of damage nucleation and propagation of the damage front, is analysed, In particular, the coupling of the constitutive material with decreasing stiffness is described.

Improvement of one of the existing commercial packages to add new modelling capabilities, in this case the modelling of brittle damage, makes it possible to use the newly developed computing and graphics features and to become a widely adapted method. This is in accordance with modern trends in engineering design involving extensive use of computers and CAD software. In this study, the commercial package ABAQUS [7] was selected to perform the calculations with the implemented constitutive model. Adapting ABAQUS program for the analysis of surface structures with variable material characteristics requires implementing the classical finite element method. This includes, apart from the constitutive equations, and after developing algorithms and numerical procedures, the implementation of equations describing the development of damage. Similar attempts can be found in literature. For example, the implementation of user subroutines set to perform finite element analysis connected with the identification of damage material parameter (for nickel-based superalloy INC0718 and for steel) in the MSC.Marc computer code is described in [8]. The comparison of experiments with numerical simulations to investigate the properties of damage level in the material is presented in [9]. Here, the dependency between the thermal properties and changes of material structure during the damage process was the method of assessment of the damage stage. Interesting results can also be found for the structures made of concrete material [10], e.g., the implementation of coupled rate-dependent damage-plasticity concrete model in LS-DYNA finite element code. The authors conducted a study of damage mechanisms which develop in concrete under high rate loading. Moreover, the comparison between numerical simulation and the tests carried out in laboratory is given. The development of orthotropic properties in initially isotropic materials described by the constitutive equations within the phenomenological approach of continuum damage mechanics is presented in [11].

2. Measures of damage, evoluton equation

The development of damage defects in a material is described by the damage evolution equation. In the case of scalar damage measure, mainly formulated for uniaxial stress states, this equation has an uncomplicated form such as an equation for a scalar measure introduced by Kachanov [12], where ϕ is defined as the loss of effective cross-sectional area, which is a ratio of the effective area to the undamaged area:

$$\frac{d\phi}{dt} = -A\left[\frac{\sigma}{\phi}\right]^m,$$

where A and m are material constants while σ is the maximum tensile principal stress. This equation expresses the rate of change in time of the effective stress (σ/ϕ) defining the value of the stresses per unit surface area of the undamaged material. Scalar measures can be relatively easily included in the physical model; therefore, they have practical importance and are used in engineering calculations, especially for isotropic damage [13].

When considering the development of defects in real structures, working in a multi-axis stress state, more realistic results can be obtained using the tensor measurement. In this paper, it is assumed that the material damage state is represented by the tensor damage measure denoted by symbol $\mathbf{\Omega}$ and presented by Murakami and Ohno in [14] and by equivalent $\mathbf{D}$ introduced by Vakulenko and Kachanov in [15].

The damage tensor $\mathbf{\Omega}$ is defined as the ratio between the elementary surface dA plane with the normal unit $\mathbf{n}$ lying in the damaged configuration (current) and the elementary area dA^* in the same plane with normal $\mathbf{n}^*$ lying in the configuration without damage (ideal) with reduced cross section area

$$\mathbf{n}^* dA^* = (\mathbf{I} - \mathbf{\Omega})\mathbf{n}\, dA,$$

where **I** is a unit tensor of the second order. The principal values of $\mathbf{\Omega}$ and **D** tensors are related by the following formula:

$$\varOmega_i = D_i/(1 + D_i), \qquad i = 1, 2, 3.$$

With such damage measure assumption the main components of effective stress are defined by the relationship

$$\sigma_i^* = \frac{\sigma_i}{(I_i - \varOmega_i)}, \qquad i = 1, 2, 3, \tag{2.1}$$

where σ^* is a tension stress in the ideal configuration and σ is the stress in a damaged configuration. From the definition it follows that the theoretical values $\varOmega_i$ are in the interval [0, 1). In accordance with the experimental results in the critical time of macro cracks appearance, $\varOmega_i$ is smaller than a unity.

In the general formulation of the constitutive model the damage tensor is an internal variable whose current value should be determined on the basis of the equation of evolution. For many internal parameters describing processes in the material at the level of microstructure, the evolution of internal parameters $\beta_i(t)$ frequently depends on the state of all the internal variables $\beta_j(t)$ and the state of macroscopic variables [14, 16]

$$\frac{d\beta_i}{dt} = f_i(\beta_j; \boldsymbol{\sigma}, t, h),$$

where h stands for the hardening parameter, t for the time and $\boldsymbol{\sigma}$ is the stress tensor.

Leaving aside the hardening parameter, we obtain the following form of the equation of evolution [17, 18]:

$$\frac{d\mathbf{\Omega}}{dt} = f(\mathbf{\Omega}; \boldsymbol{\sigma}). \tag{2.2}$$

This is the tensor function of two independent symmetric tensor variables $\mathbf{\Omega}$ and $\boldsymbol{\sigma}$. Based on the theory of tensor function representation (for a symmetric second-order tensor), the right side of Eq. (2.2) is the product of the stress and damage tensors to the powers of no more than two and the function of tensor invariants.

The assumption of proportionality of rate increase in the damage tensor principal values to the positive principal values of the stress tensor is due to the lack of evolution of the damage in compressive stress directions. Also according to the experimental results, it is assumed that the principal directions of stress tensors and damage coincide and the factor determining the evolution of the damage, in addition to the positive principal stresses, is the second invariant of the stress deviator and the first invariant of the stress tensor.

According to the proposal set out in the paper of Litewka [17], the evolution equation was adopted in the form of

$$\frac{d\mathbf{\Omega}}{dt} = B_1 \Phi_e^m \mathbf{I} + \frac{B_2}{(2E)^n} \Phi_e^n \boldsymbol{\sigma} H(\boldsymbol{\sigma}), \tag{2.3}$$

where H is the Heaviside function eliminating damage to the directions of the negative stress, B_1 and B_2 are the material constants, and m and n are the material parameters. The symbol Φ_e in Eq. (2.3) is the elastic energy density expressed by the following formula:

$$\Phi_e = \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{\varepsilon}),$$

where $\boldsymbol{\varepsilon}$ is the strain tensor.

The first part of Eq. (2.3) corresponds to the development of isotropic damage (i.e., it does not show privileged directions), which in the case of metal working at elevated temperatures does not confirm the results of experiments; therefore, further considerations take into account only the second element responsible for the development of anisotropy [17]

$$\frac{d\mathbf{\Omega}}{dt} = k \Phi_e^n \boldsymbol{\sigma} H(\boldsymbol{\sigma}), \tag{2.4}$$

where $k = B_2/(2E)^n$.

The problem of selection of the coefficient n was described by Litewka [19], where the author considered using the exponent 1, 2 and 3, depending on the type of material used.

The set of equations that model the adopted material also complements also constitutive equation for a damaged solid [19]:

$$\begin{aligned} \varepsilon_{ij} &= A_{ijkl}(D_{mn}) \cdot \sigma_{kl}, \\ A_{ijkl} &= -\frac{\nu}{E}\delta_{ij}\delta_{kl} + \frac{1+\nu}{2E}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ &\quad + \frac{D_1}{4(1+D_1)E}(\delta_{ik}D_{jl} + \delta_{il}D_{jk} + \delta_{jk}D_{il} + \delta_{jl}D_{ik}), \end{aligned} \tag{2.5}$$

where the compliance matrix A_{ijkl} is the function of the components of the damage tensor $\mathbf{D}$ and the constants of the undamaged material at current temperature (Young's modulus E and Poisson's ratio ν). The critical combination of damage tensor components is calculated from the failure criterion [17]

$$[C_1 \quad C_2 \quad C_3]\left[\ \mathrm{tr}^2\boldsymbol{\sigma} \ \vdots\ \mathrm{tr}\mathbf{S}^2 \ \vdots\ \mathrm{tr}(\boldsymbol{\sigma}^2\mathbf{D})\ \right]^T - \sigma_u^2 = 0, \tag{2.6}$$

where **S** is the stress deviator and σ_u is the time-dependent ultimate strength of the undamaged material. The application of Eq. (2.6) to three different states of the stress: uniaxial tension in the perpendicular directions (coaxial with principal directions of the damage tensor) and biaxial tension in the same directions leads to the following three algebraic equations set in respect to the components of **C** vector:

$$\begin{bmatrix} (1-\Omega_1)^2 & \frac{2}{3}(1-\Omega_1)^2 & (1-\Omega_1)^2\Omega_1 \\ (1-r_2\Omega_1)^2 & \frac{2}{3}(1-r_2\Omega_1)^2 & (1-r_2\Omega_1)r_2\Omega_1 \\ 4(1-\Omega_1)^2 & \frac{2}{3}(1-\Omega_1)^2 & 2(1-\Omega_1)\Omega_1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \mathbf{I},$$

$$r_i = \sigma_i H(\sigma_i), \qquad i = 1, 2, 3.$$

where Ω_1 stands for the maximum principal value of the damage tensor $\mathbf{\Omega}$.

3. Numerical examples

Numerical calculations were carried out using the standard ABAQUS FEA program [7], in which the mathematical model consisting of constitutive equation (25), the damage evolution equation (24) and the failure criterion (26) is defined using the procedure UMAT written by the user [20]. The iterative procedure adopted here assumes that the structure is made of a material with stiffness varying over time, due to progressive degradation. For a given load-induced stress level, the evolution equation is solved and the criterion of damage is checked. In the second and subsequent steps, solutions are obtained taking into account the distribution of stresses caused by the occurrence of microdamages and macro cracks.

With the assumption of a constant value of the exponent n (n = const) the equation of damage evolution (2.4) is replaced by a one-parameter formula, only dependent on the material constant k. The impact of parameter n on the process of damage evolution and the time of macrocracks were analysed in numerical studies. The calculations were performed for four values of the parameter $n = 1, 2, 3$ and 4.

The focus was on determining the time interval from structure loads to the appearance of the first macro cracks.

In the numerical analysis the assumed dimensionless magnitudes of material data are referred to carbon steel at the temperature of 811 K with dimensionless Young's modulus $E/\sigma_u = 417$, Poisson's ratio $\nu = 0.47$ and ultimate strength $\sigma_u = 288$ MPa [19]. The results are illustrated for four different coupling pairs of material parameters n and k: ($n = 1$, $k = 3.42$), ($n = 2$, $k = 8.21\text{E}1$), ($n = 3$, $k = 1.97\text{E}7$) and ($n = 4$, $k = 4.73\text{E}12$). The analysis is performed for middle-thick plate with all hard clamped edges and the surface uniform load $q = q/\sigma_u = 4.2 \cdot 10^{-2}$ (Fig. 1).

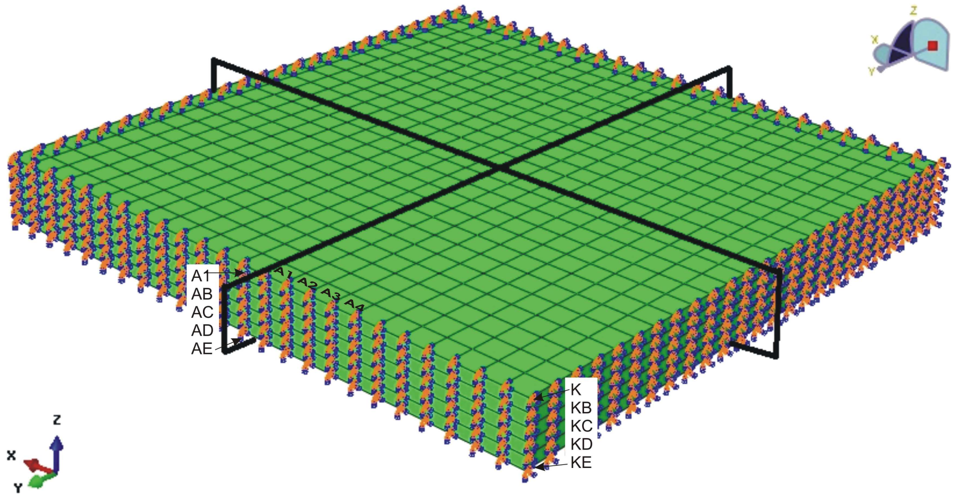

FIG. 1.

The first rupture occurs simultaneously in the middle of the clamped edge (symmetry point A1) of the upper surface of the plate in each case of n parameter. The more detailed analysis of the obtained results is conducted in the area where the first rupture appears. The histories of the first principle values of the functions of damage tensor $\mathbf{\Omega}_1$ (Fig. 2 for $n = 1$, Fig. 3 for $n = 2$, Fig. 4 for $n = 3$ and Fig. 5 for $n = 4$), the nominal stresses S_1 (Fig. 6) and the effective stresses N_1 (Fig. 7) (T denotes the dimensionless time) in the neighbourhood

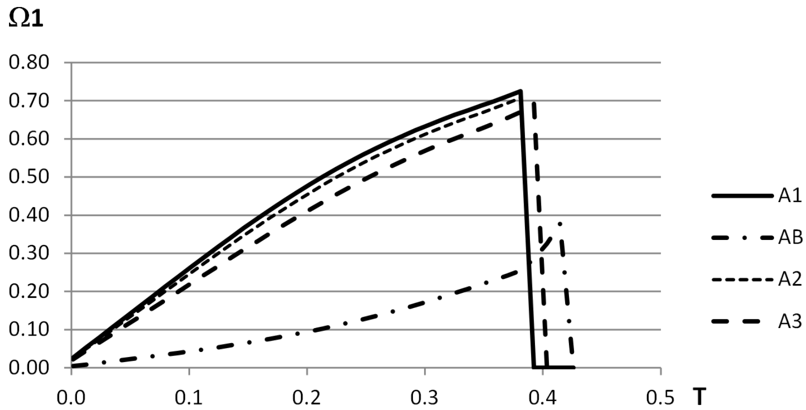

FIG. 2.

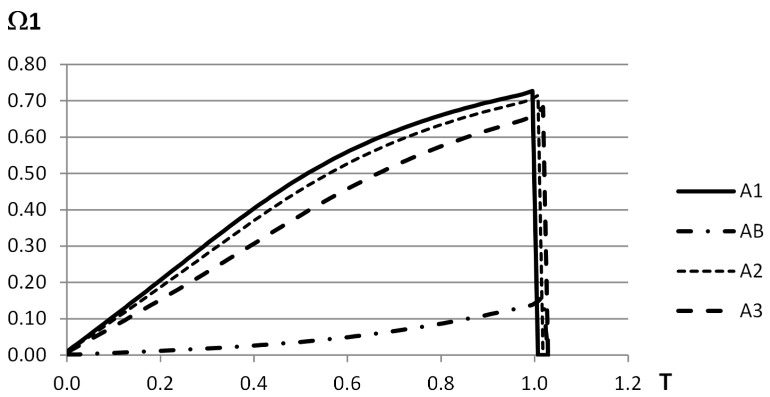

Fig. 3.

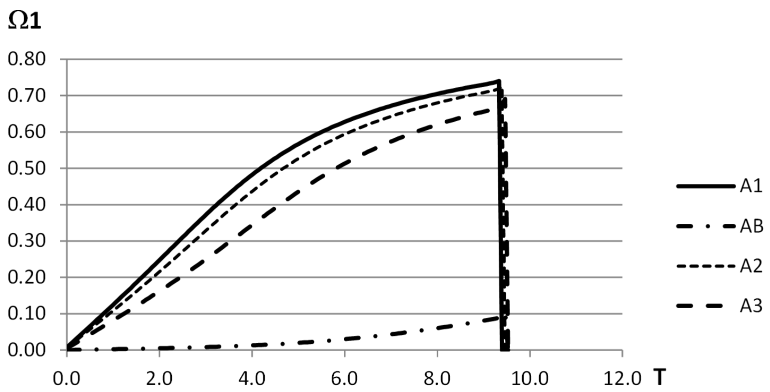

Fig. 4.

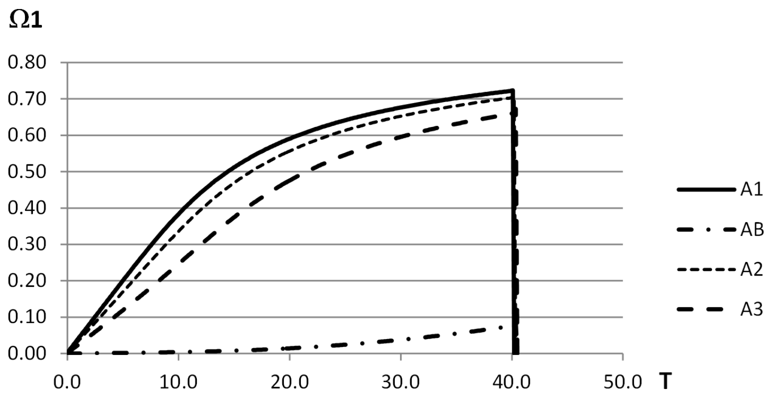

Fig. 5.

of the first macro crack place in the plate are shown below. The dimensionless principal values of the stress tensor are related to the ultimate strength σ_u in the virgin state.

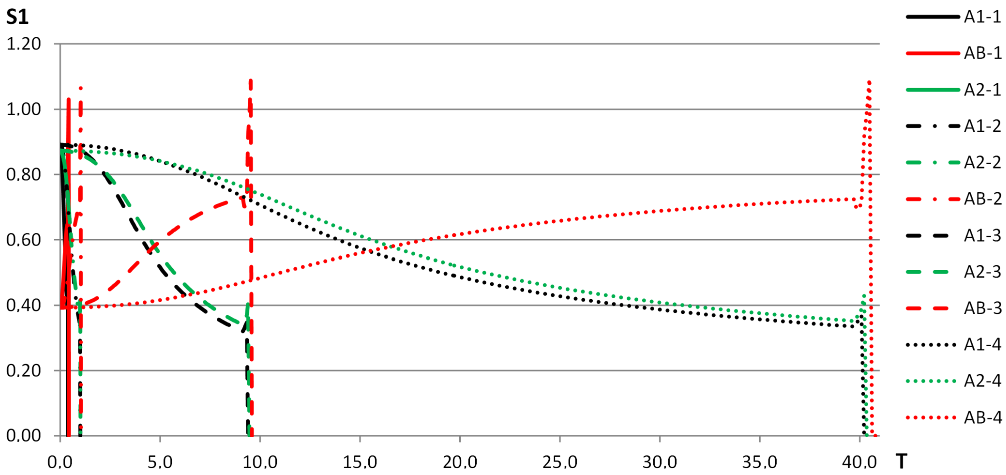

FIG. 6.

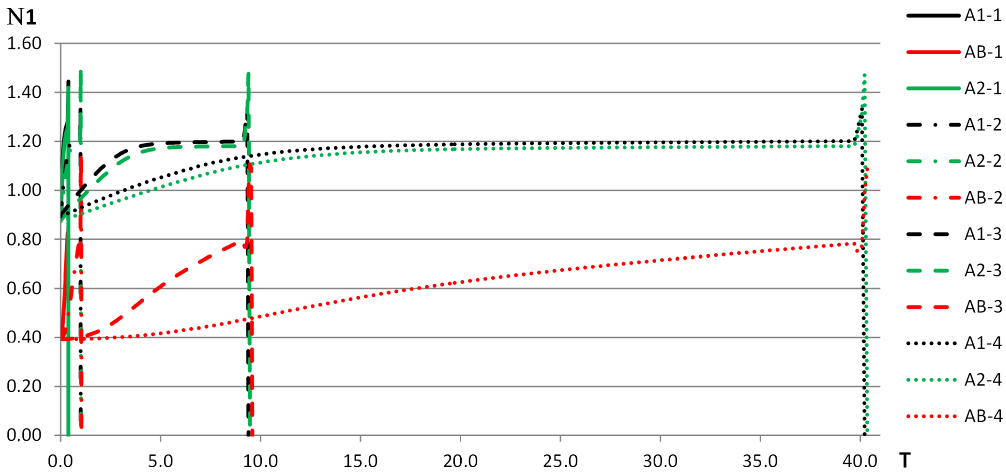

FIG. 7.

The names in the legend correspond to the element numbers in Fig. 1. In all the cases, the second macro crack is located in the neighbourhood of element A2 in the same top layer. The process develops almost simultaneously in the second layer of the plate containing point AB. As far as the level of material deterioration is concerned, we notice almost identical values of critical Ω_1 in points of the top layer A1 for each n parameter. The differences appear for point AB of the second layer; in this case we notice a decrease of level of material deterioration accompanied by greater values of n. The important difference can be noticed, however, for time periods before the first rupture appears. The vertical lines indicate the rupture time in the elements. For comparison, dimensionless time was normalized with respect to time of the first macro crack, which was obtained assuming the exponent $n = 2$. Adoption of the exponent $n = 2$ is done according to research carried out for the samples made of metal working at high temperatures, which in these conditions exhibit characteristics of brittle materials. For such materials, assuming a state of uniaxial tension, compliance with the experimental results was obtained by adopting the exponent equal to $n = 2$. For $n = 1$ the first time interval is equal to 0.4, for $n = 3$ this time is almost 10 times longer and for $n = 4$ it is equal to 40. After the appearance of first macro cracks in the upper layer, the process becomes avalanche-like with successive elements cracking in both the upper layer (toward the corner) and the cross-sectional direction (point K).

Material deterioration, determined by elastic energy density and directions of principal stresses, decreases the stiffness of a structure [21]. The course of the maximum principal value of stress tensor S_1 (Fig. 6, the number after the symbol of element symbolizes the value of n index) in the first macro crack point A1 is similar in each case, i.e., it increases what makes the process develop. In the neighbouring elements (AB and A2) the opposite phenomenon is noticed, the stresses decrease till the moment of the first macro crack appearance.

The effective stresses (2.1) take into account the current state of damage accumulation making therefore the strength of the structure objective. Despite a decrease in the effective stress N in each element, damage accumulation process develops under the influence of load (Fig. 7).

The development of the orthotropic properties of the material causes the differentiation of the principal values of damage tensor Ω_1, Ω_2, Ω_3 at each of the directions for all the adopted parameters n (Fig. 8), revealing a significant difference in damage to the material (close to 49% between principal direction 1 and 2 and close to 75% between principal direction 1 and 3) in each of the principal directions.

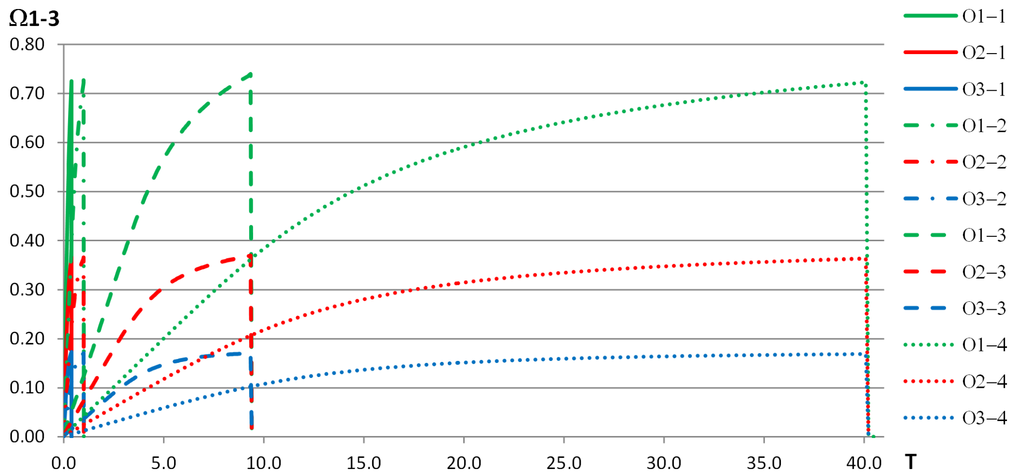

Fig. 8.

The adopted form of damage evolution Eq. (2.4) indicates the necessity of examining also the course of functions Φ and $d\Omega_1/dt$. Small changes in the energy density Φ (Figs. 9–12) in the elements forming the damage front, in the process of microdamages evolution, result from two factors: an increase in the damage

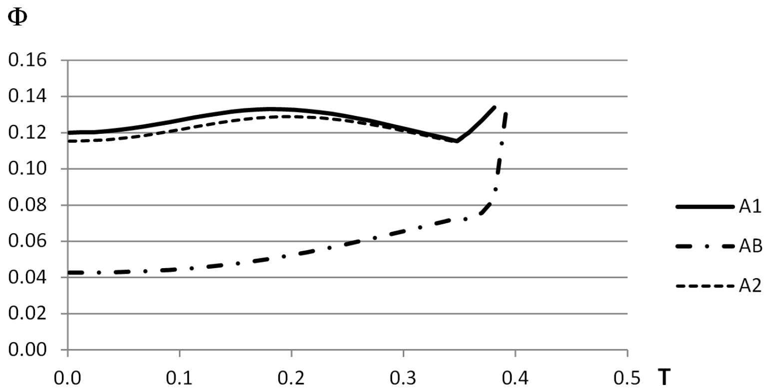

Fig. 9.

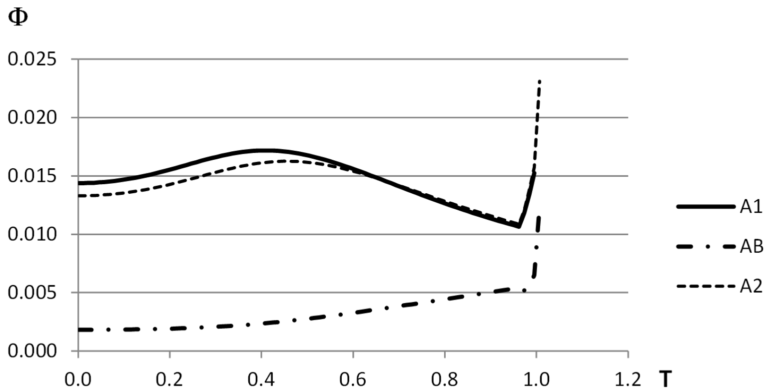

Fig. 10.

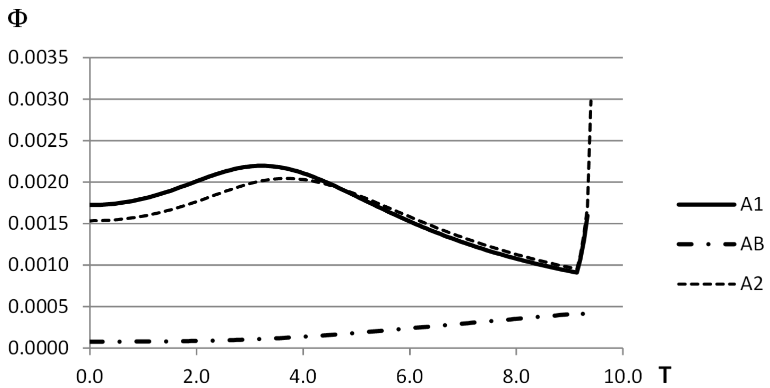

Fig. 11.

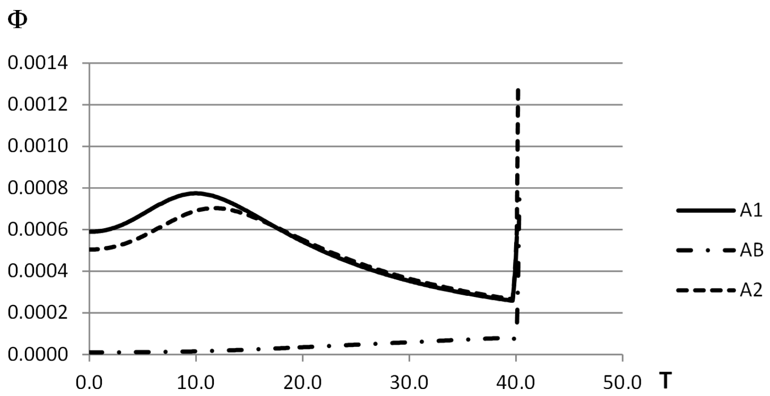

Fig. 12.

of the material (higher values of the damage tensor principal components $\mathbf{\Omega}$) and a reduction in stiffness of the structure (decrease in the stress $\mathbf{N}$).

The change of the damage rate illustrated in Figs. 13–16 reflects slowing material degradation resulting from the acceptance of a larger exponent n.

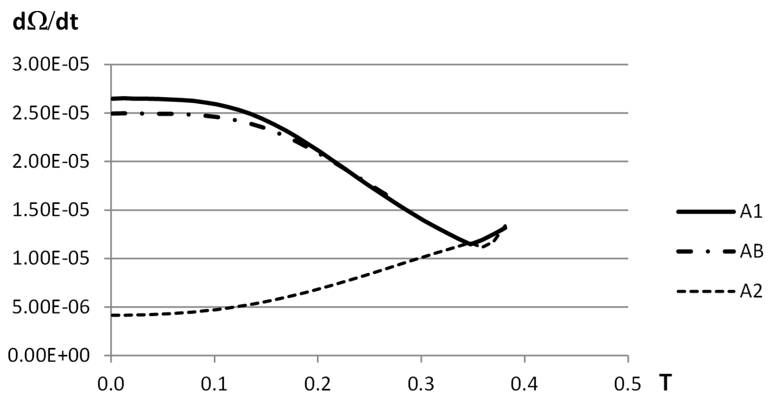

Fig. 13.

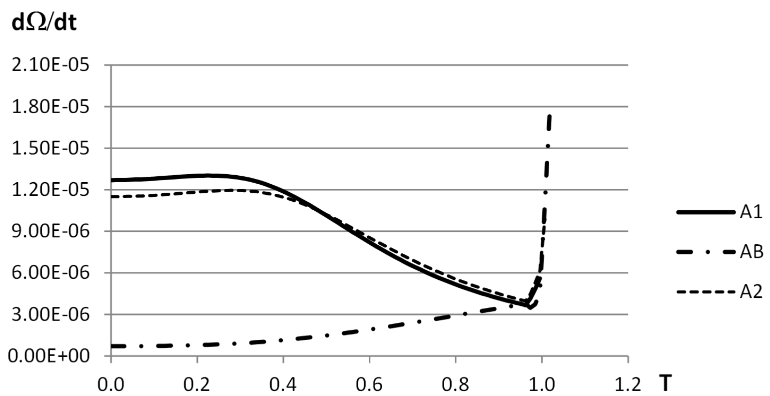

Fig. 14.

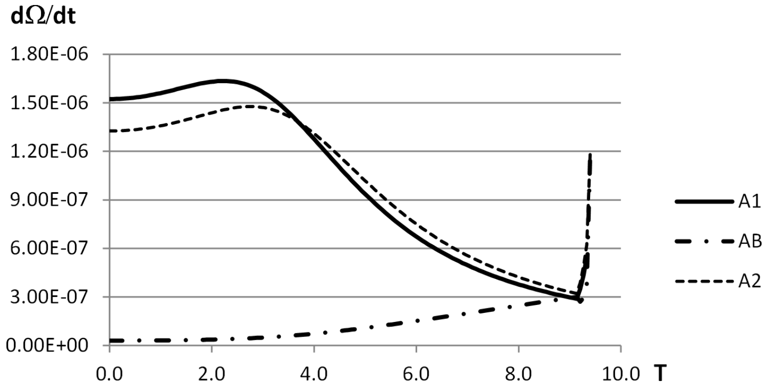

FIG. 15.

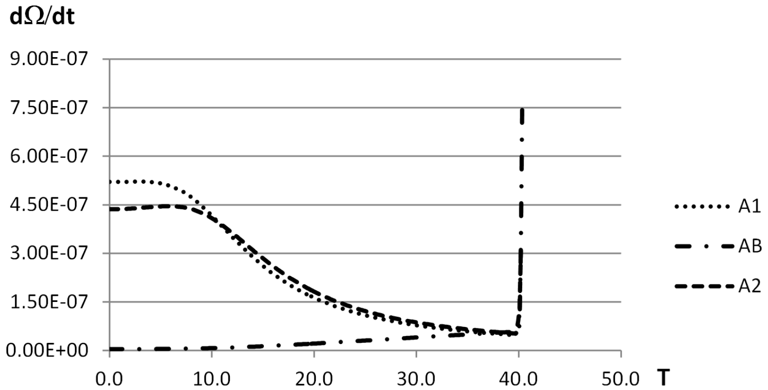

FIG. 16.

In the initial period, the formation of microdamages is accelerated. Decreasing stresses slow down this process. The smallest rate of change of the principal damage tensor value preceding the macrocrack corresponds to $n = 4$.

Summing up, the results for the first macro crack point A1 are illustrated in Figs. 17–19.

From calculations made for the plate structure it is disclosed that there is a high dependence of the critical time, understood as the time interval from the moment of structure loads to the first macro crack appearance, on the adopted exponent n (Fig. 17). An asymptotic increase of critical time for increasing n parameter is illustrated in Fig. 17, contrary to asymptotic decreasing of $d\Omega_1/dt$ (Fig. 18).

Despite the significant influence of the change of the parameter n at the critical time there was no significant influence of this parameter on the level of damage at the critical time (critical principal values of the damage tensor Ω_1 are in the range 0.7–0.75) (Fig. 19).

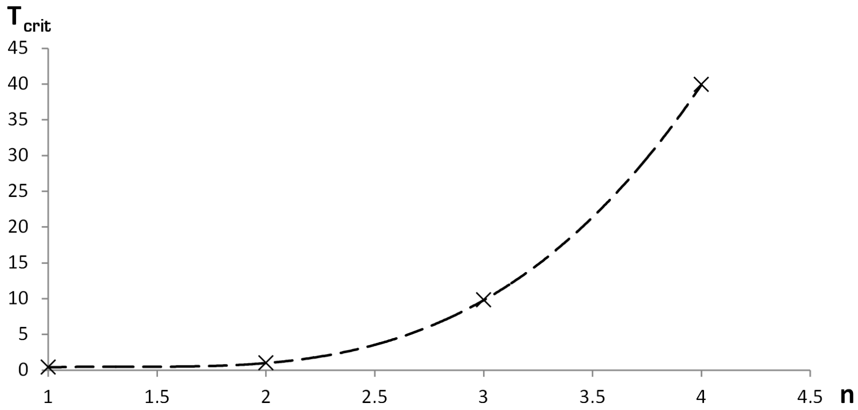

Fig. 17.

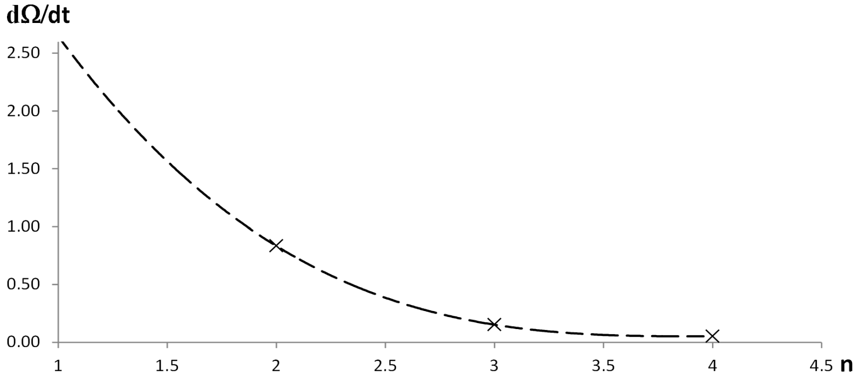

Fig. 18.

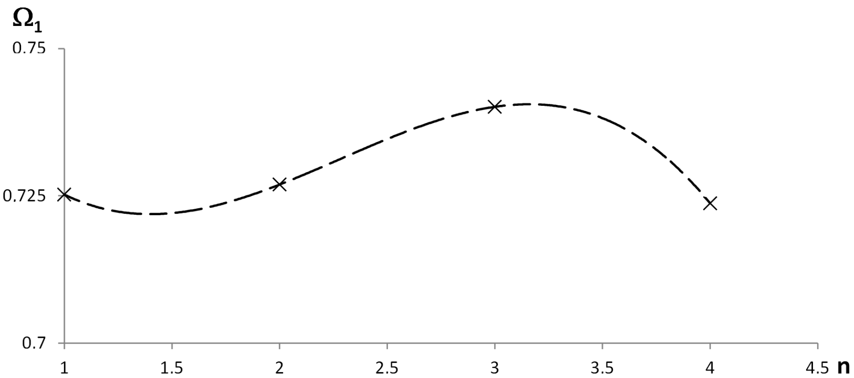

Fig. 19.

Figure 20 illustrates the variation of the normalized maximum structure deflection depending on the density distribution of finite element (FE) mesh assuming $n = 2$. The moment of the asymptotic growth of deflection indicates

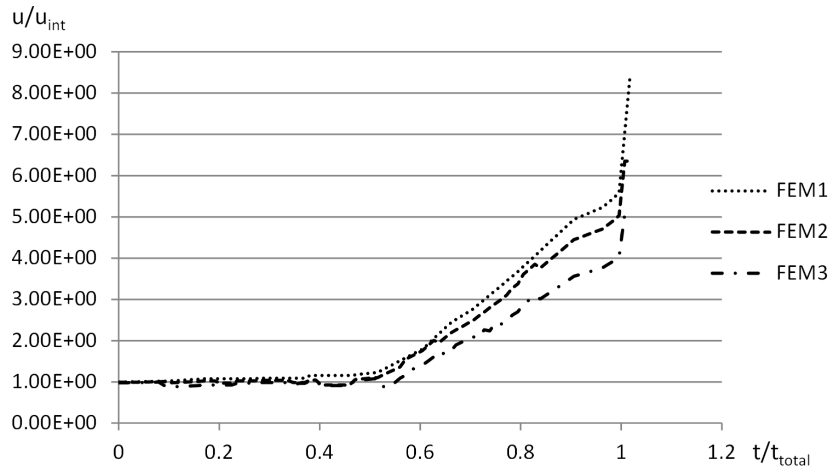

Fig. 20.

a critical time for the structure. Comparable quality solutions for all FE mesh have shown a lack of pathological dependence on the mesh density. Moreover, as the mesh thickens the improvement in the convergence of the results is observed.

4. Final remarks

The numerical research of damage development for different exponent n, defined as a material constant, indicates the existence of qualitatively convergent growth mechanism of damage, but with major differences in the time interval from the moment of structure loads until the first macro crack. The results of numerical calculations show that the qualitative nature of microdamages evolution does not depend on the exponent n, since the difference between the principal values of the damage tensor Ω_1 at critical moment, for each parameter n, does not exceed 5%, which indicates a similar level of material damage. This justifies accepting the evolution equation (2.4) as one parameter description of rupture modelling for a given value of the exponent n [19].

Proper selection of the parameter n, together with the material parameter k, allows for more flexible adaptation of the damage function evolution to experimental results, to achieve compliance of the first critical time and the shape of the evolution course, but the specification of this requires further investigations confirmed by experimental data.

A very high sensitivity of the lifetime of a structure with respect to the value of the exponent n allows to treat this exponent as the second material constant for more precise rupture modelling.

References

1. Lemaitre J., Chaboche J., *Mechanics of solid materials*, Cambridge University Press, New York, 1990.
2. Krajcinovic D., *Damage mechanics*, Elsevier Science, North Holland, 1997.
3. Skrzypek J.J., Ganczarski A., Artur W. [Eds], *Anisotropic behaviour of damaged materials*, Springer Verlag, Berlin, 2003.
4. Kattan P.I., Voyiadjis G.Z., *Damage mechanics with finite elements: practical applications with computer tools*, Springer Verlag, Berlin, 2002.
5. Murakami S., *Continuum damage mechanics: a continuum mechanics approach to the analysis of damage and fracture*, Springer, Dordrecht, Heidelberg, London, New York: Springer, 2012.
6. Lemaitre J., Desmorat R., *Engineering damage mechanics*, Springer, Berlin, 2005.
7. ABAQUS *Theory and users manuals*, ver. 6.11, Dassault Systèmes Simulia Corp., Providence, RI, USA.
8. Ambroziak A., *Identification and validation of damage parameters for elasto-viscoplastic Chaboche model*, Eng. Trans., **55**, 1, 3–28, 2007.
9. Madej J., *Influence of damage on variations of material thermal properties*, Eng. Trans., **51**, 1, 25–46, 2003.
10. Hervé G., Gatuingt F., Ibrahimbegović A., *On numerical implementation of a coupled rate dependent damage-plasticity constitutive model for concrete in application to high-rate dynamics*, Engineering Computations, **22**, 5–6, 583–604, 2005.
11. Rizzi E., Carol I., *Secant stress/strain relations of orthotropic elastic damage with dual properties*, Archives of Mechanics, **59**, 2, 133–171, 2007.
12. Kachanov L.M., *On time to rupture in creep conditions* [in Russian], Izv. AN SSSR, OTN, **8**, 26–31, 1958.
13. Voyiadjis G.Z, Kattan P.I., Yousef M.A., *Some basic issues of isotropic and anisotropic continuum damage mechanics*, [in:] Handbook of damage mechanics – nano to macro scale for materials and structures, pp. 3–42, 2014.
14. Murakami S., Ohno N., *A continuum theory of creep and creep damage*, [in:] Creep in structures, A.R.S. Ponter and D.R. Hayhurst [Eds.], Springer, Berlin, pp. 422–444, 1981.
15. Vakulenko A.A., Kachanov M.L., *Continuum theory of medium with cracks* [in Russian], Izv. A. N. SSSR, MTT, pp. 59–166, 1971.
16. Lemaitre J., *Damage modelling for prediction of plastic or creep fatigue failure in structures*, Trans. 5th Int. Conf. SMiRT, Berlin, North-Holland, Amsterdam, L, L5/1b, pp. 1–8, 1979.
17. Litewka A., *Creep rupture of metals under multi-axial states of stress*, Arch. Mech., **41**, 3–23, 1989.
18. Murakami S., *Notion of continuum damage mechanics and its application to anisotropic creep damage theory*, J. Eng. Mat. Techn., **105**, 2, 99–105, 1983.

19. Litewka A., Hult J., *One parameter CDM model for creep rupture prediction*, Eur. J. Mech., A/Solids, **8**, 185–200, 1989.

20. Mika P., *Modelling of shell structures with damage growth process* [in Polish], Civil and Environmental Engineering, **3**, 58, II, 405–412, 2011.

21. Mika P., *On interaction between damage growth and material stiffness in 3D structures*, J. Theor. App. Mech., **37**, 755–778, 1999.

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